A REMARK ON ODD DIMENSIONAL NORMALIZED RICCI FLOW

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ABSTRACT. Let (M^n, g_0) (n odd) be a compact Riemannian manifold with $\lambda(g_0) > 0$, where $\lambda(g_0)$ is the first eigenvalue of the operator $-4\Delta_{g_0} + R(g_0)$, and $R(g_0)$ is the scalar curvature of (M^n, g_0). Assume the maximal solution $g(t)$ to the normalized Ricci flow with initial data (M^n, g_0) satisfies $|R(g(t))| \leq C$ and $\int_M |Rm(g(t))|^{n/2} d\mu_t \leq C$ uniformly for a constant C. Then we show that the solution sub-converges to a shrinking Ricci soliton. Moreover,when $n = 3$, the condition $\int_M |Rm(g(t))|^{n/2} d\mu_t \leq C$ can be removed.

Since Hamilton's seminal work [H1] Ricci flow has been an important tool used extensively in geometry and topology. In particular, there is the recent breakthrough of Perelman [P1],[P2].

In this short note we prove a convergence result for odd dimensional volume-normalized Ricci flow,

$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \frac{2}{n} r g_{ij}$,

where $r = \frac{\int_M R d\mu_t}{\int_M d\mu_t}$ is the average scalar curvature of $(M^n, g(t))$. More precisely, we have the following

Theorem Let (M^n, g_0) (n odd) be a compact Riemannian manifold with $\lambda(g_0) > 0$, where $\lambda(g_0)$ is the first eigenvalue of the operator $-4\Delta_{g_0} + R(g_0)$. Assume the maximal solution $g(t)$ to the volume-normalized Ricci flow with initial data (M^n, g_0) satisfies $|R(g(t))| \leq C$ and $\int_M |Rm(g(t))|^{n/2} d\mu_t \leq C$ uniformly for a constant C. Then the solution sub-converges to a shrinking Ricci soliton. Moreover,when $n = 3$, the condition $\int_M |Rm(g(t))|^{n/2} d\mu_t \leq C$ can be removed.

(Here, $R(g_0)$ is the scalar curvature of (M^n, g_0), and $Rm(g(t))$ is the curvature tensor of $g(t)$.)

Proof Let $[0, T)$ be the maximal time interval of existence of $g(t)$. First we show that $|Rm(g(t))| \leq C'$ uniformly on $[0, T)$ for a constant C'. Suppose this is not the case, then there exist a sequence of times $t_i \to T$ and points $x_i \in M$ such that $Q_i = |Rm(g(t_i))|(x_i) = max_{x \in M} |Rm(g(t_i))|(x) \to \infty$. Note that the condition $\lambda(g_0) > 0$ implies the corresponding un-normalized solution blows up in finite time, since we have $\frac{d}{d\tilde{t}} \lambda(\tilde{t}) \geq \frac{2}{n} \lambda^2(\tilde{t})$ by Perelman [P1], where $\lambda(\tilde{t})$ is the first eigenvalue of the operator $-4\Delta + R$ for the un-normalized Ricci flow. By Perelman's no local collapsing theorem [P1] and Hamilton's compactness theorem [H2], a subsequence of the rescaled solutions $(M, Q_i g(Q_i{}^{-1} t + t_i), x_i)$ converges smoothly to a pointed complete ("normalized"-)Ricci flow $(M_\infty, g_\infty(t), x_\infty)$, such

1991 *Mathematics Subject Classification.* 53C44.

Key words and phrases. Ricci flow, no local collapse, non-singular solution.

Partially supported by NSFC no.10671018.

that $g_\infty(t)$ is κ-noncollapsed relative to upper bounds of the scalar curvature on all scales, where κ is certain positive constant depending only on n and the initial data g_0. Clearly M_∞ is non-compact. We have $R(g_\infty(t)) = 0$, hence $g_\infty(t)$ is Ricci flat since it is a solution to the Ricci flow. Moreover, the conditions (ii) and (iii) in (3.14) of [A] are also fulfilled for $(M_\infty, g_\infty(0))$. Combined with the odd dimensional assumption, it follows from [A, Theorem 3.5](see also [BKN,Theorem(1.5)]) that (the double cover of) $(M_\infty, g_\infty(0))$ is the n-dimensional Euclidean space, which contradicts the fact $|Rm(g_\infty(0))|(x_\infty) = 1$.

When $n = 3$ and the condition $\int_M |Rm(g(t))|^{n/2} d\mu_t \leq C$ is removed, then if the curvature tensors of $g(t)$ are not bounded uniformly on $[0, T)$, the $(M_\infty, g_\infty(0))$ constructed above is Ricci flat and hence flat, and again we get a contradiction.

Then it follows that $T = \infty$ and $g(t)$ is nonsingular. By [FZZ, Proposition 2.2] for any sequence of times $t_k \to \infty$, there is a subsequence t_{k_i} such that $g(t + t_{k_i})$ converges to a shrinking Ricci soliton. Moreover,when $n = 3$, we must have that M is diffeomorphic to a spherical space form (cf. [H3]).

Remark 1 A similar argument was used by Ruan, Zhang and Zhang in [RZZ], where they proved a related result (see Proposition 1.3 in [RZZ]).

Remark 2 Of course, compact three manifolds (M^3, g_0) with $\lambda(g_0) > 0$ have been classified by Perelman [P2].

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