Mean-value property on manifolds with minimal horospheres

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Abstract

Let (M, g) be a non-compact and complete Riemannian manifold with minimal horospheres and infinite injectivity radius. We prove that bounded functions on (M, g) satisfying the mean-value property are constant.
We extend thus a result of the authors in [6] where they proved a similar result for bounded harmonic functions on harmonic manifolds with minimal horospheres.

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1 Introduction

Let (M, g) be a non-compact and complete Riemannian manifold.
A function u defined on (M, g) is said to have the mean-value property if:

$$\forall\, r > 0 \text{ and } \forall\, p \in M\ ,\ u(p) = \frac{1}{V(p,r)} \int_{B(p,r)} u(q)\, d\mu(q)\ ,$$

where $d\mu$ denotes the Riemannian volume element and $V(p, r)$ the volume of the closed ball $B(p, r)$ of centre p and radius r.
Well-known examples of functions satisfying the mean-value property are harmonic functions on harmonic manifolds (see [9]).
In [6] the authors proved that on non-compact harmonic manifolds with minimal horospheres, bounded harmonic functions are constant.
One of the major arguments to obtain this result is the fact that on harmonic manifolds, harmonic functions possess the mean-value property. It thus seems natural to raise the same question by considering the class of functions satisfying the mean-value property and defined on manifolds not necessarily harmonic.
Some analogous of Liouville type results for functions satisfying the mean-value property have been proved by several authors. For example the authors in [8] proved that on certain kinds of homogeneous spaces, the only L^p-function possessing the mean-value property is the zero function. For similar results and related works see also [1, 2, 3, 4, 5, 10] and the references therein.
Our aim is to extend the Liouville type result proved in [6] on bounded functions satysfying the mean-value property and defined on non-compact manifolds with minimal horospheres and infinite injectivity radius.

We refer to [6] and [7] for information and details on the minimality's condition of horospheres in a non-compact manifold.
For a real number $r > 0$, we consider the stability vector field $H(., r)$ defined by:

$$H(p,r) =: \int_{B(p,r)} \exp_p^{-1}(q)\ d\mu(q)\ ,\ \forall\, p \in M\ ,$$

where $\exp_p^{-1}$ denotes the inverse of the exponential map.
Let us note that the vanishing of the stability vector field for any radius $r > 0$ means that any geodesic ball in (M, g) has its Riemannian center of mass (or center of gravity) at the centre of the ball. This is the case for examples for harmonic manifolds, d'Atri spaces or compact locally symmetric spaces.
In the next section we give a result relating the gradient of the volume function and the derivative of the stability vector field that we use in the third section to prove that on non-compact manifolds with minimal horospheres and infinite injectivity radius, bounded functions having the mean-value property are constant.

2 Volume functions and stability vector fields

Let $V : (p,r) \in M\times\ ]0\ ,\ +\infty[\ \longmapsto V(p,r)$ be the function associating to each pair $(p,r) \in M\times\ ]0\ ,\ +\infty[$ the volume $V(p,r)$ of the ball $B(p,r)$.
The volume function V and the stability vector field H are related by the following differential equation:

Lemma 2.1
Let ∇ denotes the gradient operator on (M, g).
For any $r > 0$ and $p \in M$, it holds:

$$\nabla V(p,r) - \frac{1}{r}\frac{\partial}{\partial r}H(p,r) = 0\ .$$

Proof
For $X \in T_pM$, it holds

$$\nabla_X V(p,r) = \int_{S(p,r)} < \eta(q)\ ,\ X(q) >\ d\sigma(q)\ ,$$

where $d\sigma$ denotes the Riemannian measure induced on the sphere $S(p,r)$ with centre p and radius r, $\eta(q)$ is the outward unit normal at q, and $X(q)$ is the parallel transport of X from p to q.
By the Gauss Lemma,

$$< \eta(q)\ ,\ X(q) > = < (d\exp_p^{-1})\eta(q)\ ,\ X >\ .$$

Otherwise

$$(d\exp_p^{-1})\eta(q) = r^{-1}\exp_p^{-1} q\ .$$

It follows then

$$\begin{aligned}
\nabla_X V(p,r) &= \int_{S(p,r)} r^{-1}\exp_p^{-1} q \; d\sigma(q) \\
&= r^{-1}\int_{S(p,r)} \exp_p^{-1} q \; d\sigma(q) \\
&= r^{-1}\frac{\partial}{\partial r}\left(\int_{B(p,r)} \exp_p^{-1} q \; d\mu(q)\right) \\
&= r^{-1}\frac{\partial}{\partial r} H(p,r) \; .
\end{aligned}$$

Hence the result. □

3 A derivative formula

Let $p \in M$ and X a unit vector in T_pM. We consider as in [6] the function:

$$\begin{aligned}
\theta_X : M - \{p\} &\longrightarrow \mathbb{R} \\
q &\longmapsto \theta_X(q) =: \angle_p(X, \dot\gamma_q(0)) \; ,
\end{aligned}$$

where γ_q denote the geodesic defined by $\gamma_q(t) = \exp_p(t\exp_p^{-1} q)$, $\forall\, t \in [0\ ,\ 1]$, and $\angle_p(X,\dot\gamma_q(0))$ the angle at p between the vectors X and $\dot\gamma_q(0)$.
For the geodesic c with $c(0) = p$ and $\dot c(0) = X$, let P_t be the parallel transport along c and f_t the one parameter family of diffeomorphisms of M given by $f_t = \exp_{c(t)} \circ P_t \circ \exp_p^{-1}$.
Let u be a differentiable function on M possessing the mean-value property.
It holds:

Proposition 3.1
For any real number $r > 0$,

$$Xu(p) = \frac{1}{V(p,r)}\int_{S(p,r)} u\cos\theta_X d\sigma - \frac{1}{r}\frac{u(p)}{V(p,r)} < \frac{\partial}{\partial r}H(p,r)\ ,\ X > \; .$$

Proof
Since the function u possesses the mean-value property, we have:

$$u(c(t)) = \frac{1}{V(c(t),r)}\int_{B(c(t),r)} u(q)\; d\mu(q) \; .$$

And then

$$\begin{aligned}
X.u(p) &= \frac{d}{dt}u(c(t))_{|t=0} \\
&= \frac{d}{dt}\left(\frac{1}{V(c(t),r)}\int_{B(c(t),r)} u\; d\mu\right)_{|t=0} \\
&= -\frac{1}{V(p,r)^2} < \nabla V(p,r)\ ,\ X > \int_{B(p,r)} u\; d\mu \\
&\quad + \frac{1}{V(p,r)}\frac{d}{dt}\left(\int_{B(c(t),r)} u\; d\mu\right)_{|t=0} \qquad (i)\; .
\end{aligned}$$

From Lemma 2.1,

$$\nabla V(p,r) = \frac{1}{r}\frac{\partial}{\partial r} H(p,r) \ .$$

Thus

$$\begin{aligned}
\frac{1}{V(p,r)^2} < \nabla V(p,r), X > \int_{B(p,r)} u \ d\mu &= \frac{1}{V(p,r)^2} < \frac{1}{r}\frac{\partial}{\partial r} H(p,r), X > \int_{B(p,r)} u \ d\mu \\
&= \frac{1}{r}\frac{u(p)}{V(p,r)} < \frac{\partial}{\partial r} H(p,r), X > \quad (ii), \\
&\text{since } \ u(p) = \frac{1}{V(p,r)} \int_{B(p,r)} u(q) \ d\mu(q) \ .
\end{aligned}$$

By Theorem 2.1 in [6] we have:

$$\begin{aligned}
\frac{d}{dt}\left(\int_{B(c(t),r)} u \ d\mu \right)_{|t=0} &= \frac{d}{dt}\left(\int_{B(p,r)} f_t^*(u \ d\mu) \right)_{|t=0} \\
&= \int_{B(p,r)} \frac{d}{dt}(f_t^*(u \ d\mu))_{|t=0} \\
&= \int_{S(p,r)} u \cos\theta_X \ d\sigma \quad (iii) \ .
\end{aligned}$$

By replacing (ii) and (iii) in the relation (i) we obtain the result. □

By using the derivative formula given in Proposition 3.1, we get:

Theorem 3.1
Let (M,g) be a non-compact and complete Riemannian manifold with minimal horospheres and infinite injectivity radius.
Any bounded function on (M,g) satisfying the mean-value property is constant.

Proof
Let u be a bounded function on (M,g) satisfying the mean-value property.
By Proposition 3.1,

$$|Xu(p)| \leq \alpha \frac{A(p,r)}{V(p,r)} + \frac{\alpha}{V(p,r)} \| \frac{1}{r}\frac{\partial}{\partial r} H(p,r) \| \ , \ \forall \, p \in M \text{ and } r > 0 \ ,$$

where $A(p,r)$ is the area of the sphere $S(p,r)$, and $\alpha \geq 0$ is such that $|u| \leq \alpha$.
But:

$$\begin{aligned}
\| \frac{1}{r}\frac{\partial}{\partial r} H(p,r) \| &= \| \frac{1}{r}\frac{\partial}{\partial r} \int_{B(p,r)} \exp_p^{-1} q \ d\mu(q) \| \\
&= \| \frac{1}{r} \int_{S(p,r)} \exp_p^{-1} q \ d\sigma(q) \| \\
&\leq \frac{1}{r} \int_{S(p,r)} \| \exp_p^{-1} q \| \ d\sigma(q) \\
&= A(p,r) \ , \ \text{since} \ \| \exp_p^{-1} q \| = r \ , \ \forall \, q \in S(p,r) \ .
\end{aligned}$$

Thus we get:

$$|Xu(p)| \leq 2\alpha \frac{A(p,r)}{V(p,r)} \ .$$

Due to the minimality of horospheres (see [6] for details)

$$\lim_{r \to +\infty} \frac{A(p,r)}{V(p,r)} = K_\infty = 0 \ .$$

By taking the limit of the previous inequality as $r \to \infty$, it follows then:

$$|Xu(p)| = 0 \ , \quad \text{for any} \ \ p \in M \text{ and any unit vector } X \in T_pM \ .$$

Hence u is a constant function. □

Remark: From the proof of Theorem 3.1 it is easy to see that the same result can be obtained by assuming the horospheres with bounded mean-value and not necessarily minimal.

References

[1] W. Hansen, N. Nadirashvili, *Liouville's theorem and the restricted mean-value property*, J. Math. Pures Appl. (9) 74 (1995), no.2, 185-198.

[2] D. Heath, *Functions possessing the mean value properties*, Proc. Amer. Math. Soc. 41 (1973), 588-595.

[3] M. El Kadiri, *Théorème de Liouville et propriété de la moyenne biharmonique restreinte*, C.R. Acad. Sci. Paris, Ser. I 1340 (2005) 563-566.

[4] M. El Kadiri, *Liouville's theorem and restricted mean value property on the real line*, Rend. Accad. Naz. Sci. Xli Mem. Mat. Appl. 5, 27 (2003) 89-94.

[5] J. Kim, M. W. Wang, *Invariant mean value property and harmonic functions*, Complex Var. Theory Appl. 50 (2005), no.14, 1049-1059.

[6] A. Ranjan, H. Shah, *Harmonic manifolds with minimal horospheres*, *The Journal of Geometric Analysis*, Vol. 12, No. 4, (2002), 683-694.

[7] A. Ranjan, H. Shah, *Busemann functions in a harmonic manifold*, *Geometrica Dedicata*, 101, (2003), 167-183.

[8] A. Sitaram, G. A. Willis, *L^p-functions satisfying the mean value property on homogeneous spaces*, J. Austral. Math. Soc. Ser. A, 56 (1994), no.3, 384-390.

[9] T.J. Willmore, *Mean value theorems in harmonic spaces*, *J. London Math. Soc.*, 25, (1950), 54-57.

[10] F. Zucca, *Strong and weak mean value property on trees*, Int. J. Pures Appl. Math. 6 (2003), no.1, 33-49.