A NEW PROOF OF GROMOV'S THEOREM ON GROUPS OF POLYNOMIAL GROWTH

BRUCE KLEINER

Abstract. We give a proof of Gromov's theorem that any finitely generated group of polynomial growth has a finite index nilpotent subgroup. The proof does not rely on the Montgomery-Zippin-Yamabe structure theory of locally compact groups.

Contents

1. Introduction

1.1. Statement of results.

Definition 1.1. Let G be a finitely generated group, and let $B_G(r) \subset G$ denote the ball centered at $e \in G$ with respect to some fixed word norm on G. The group G has **polynomial growth** if for some $d \in (0, \infty)$

$$\limsup_{r \to \infty} \frac{|B_G(r)|}{r^d} < \infty, \tag{1.2}$$

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and has **weakly polynomial growth** if for some $d \in (0,\infty)$

$$\liminf_{r\to\infty} \frac{|B_G(r)|}{r^d} < \infty, \tag{1.3}$$

We give a proof of the following special case of a theorem of Colding-Minicozzi, without using Gromov's theorem on groups of polynomial growth:

Theorem 1.4 ([CM97])**.** *Let Γ be a Cayley graph of a group G of weakly polynomial growth, and $d \in [0,\infty)$. Then the space of harmonic functions on Γ with polynomial growth at most d is finite dimensional.*

Note that although [CM97] stated the result for groups of polynomial growth, their proof also works for groups of weakly polynomial growth, in view of [vdDW84].

We then use this to derive the following corollaries:

Corollary 1.5. *If G is an infinite group of weakly polynomial growth, then G admits a finite dimensional linear representation $G \to GL(n,\mathbb{R})$ with infinite image.*

Corollary 1.6 ([Gro81, vdDW84])**.** *If G is a group with weakly polynomial growth, then G is virtually nilpotent.*

We emphasize that our proof of Corollary 1.6 yields a new proof of Gromov's theorem on groups of polynomial growth, which does not involve the Montgomery-Zippin-Yamabe structure theory of locally compact groups [MZ74]; however, it still relies on Tits' alternative for linear groups [Tit72] (or the easier theorem of Shalom that amenable linear groups are virtually solvable [Sha98]).

Remark 1.7. There are several important applications of the Wilkie-Van Den Dries refinement [vdDW84] of Gromov's theorem [Gro81] that do not follow from the original statement; for instance [Pap05], or the theorem of Varopoulos that a group satisfies a d-dimensional Euclidean isoperimetric inequality unless it is virtually nilpotent of growth exponent $< d$.

1.2. **Sketch of the proofs.** The proof of Theorem 1.4 is based on a new Poincare inequality which holds for any Cayley graph Γ of any finitely generated group G:

$$\int_{B(R)} |f - f_R|^2 \leq 8\,|S|^2\,R^2\,\frac{|B(2R)|}{|B(R)|}\,\int_{B(3R)} |\nabla f|^2, \tag{1.8}$$

Here f is a piecewise smooth function on $B(3R)$, f_R is the average of u over the ball $B(R)$, and S is the generating set for G.

The remainder of the proof has the same rough outline as [CM97], though the details are different. Note that [CM97] assumes a uniform doubling condition as well as a uniform Poincare inequality. In our context, we may not appeal to such uniform bounds as their proof depends on Gromov's theorem. Instead, the idea is to use (1.8) to show that one has uniform bounds at certain scales, and that this is sufficient to deduce that the space of harmonic functions in question is finite dimensional.

The proof of Corollary 1.5 invokes a Theorem of [Mok95, KS97] to produce a fixed point free isometric G-action $G \curvearrowright \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, and a G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$ from the Cayley graph of G to $\mathcal{H}$. Theorem 1.4 then implies that f takes values in a finite dimensional subspace of $\mathcal{H}$, and this implies Corollary 1.5. See Section 4.

Corollary 1.6 follows from Corollary 1.5 by induction on the degree of growth, as in the original proof of Gromov; see Section 5.

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2. A Poincare inequality for finitely generated groups

Let G be a group, with a finite generating set $S \subset G$. We denote the associated word norm of $g \in G$ by $|g|$. For $R \in [0, \infty) \cap \mathbb{Z}$, let $V(R) = |B_G(R)| = |B_G(e, R)|$. We will denote the R-ball in the associated Cayley graph by $B(R) = B(e, R)$.

Remark 2.1. We are viewing the Cayley graph as (the geometric realization of a) 1-dimensional simplicial complex, not as a discrete space. Thus $B_G(R)$ is a finite set, whereas $B(R)$ is typically 1-dimensional.

Theorem 2.2. *For every $R \in [0,\infty)\cap\mathbb{Z}$ and every smooth function $f : B(3R) \to \mathbb{R}$,*

$$\int_{B(R)} |f - f_R|^2 \le 8\,|S|^2\,R^2\,\frac{V(2R)}{V(R)}\int_{B(3R)} |\nabla f|^2, \tag{2.3}$$

where f_R is the average of f over $B(R)$.

Proof. Fix $R \in [0,\infty)\cap\mathbb{Z}$.

Let $\delta f : B_G(3R-1) \to \mathbb{R}$ be given by

$$\delta f(x) = \int_{B(x,1)} |\nabla f|^2.$$

For every $y \in G$, we choose a shortest vertex path $\gamma_y : \{0,\ldots,|y|\} \to G$ from $e \in G$ to y. If $y \in B_G(2R-2)$, then

$$\sum_{x\in B(R-1)}\ \sum_{i=0}^{|y|} (\delta f)(x\,\gamma_y(i)) \le 2R \sum_{z\in B(3R-1)} (\delta f)(z), \tag{2.4}$$

since the map $B(R-1)\times\{0,\ldots,|y|\} \to B(3R-1)$ given by $(x,i)\mapsto x\,\gamma_y(i)$ is at most $2R$-to-1.

For every ordered pair (e_1,e_2) of edges contained in $B(R)$, let $x_i \in e_i \cap G$ be elements such that $d(x_1,x_2) \le 2R-2$, and let $y = x_1^{-1}x_2$. By the Cauchy-Schwarz inequality,

$$\int_{(p_1,p_2)\in e_1\times e_2} |f(p_1)-f(p_2)|^2\,dp_1dp_2 \ \le\ 2R\sum_{i=0}^{|y|} (\delta f)(x_1\,\gamma_y(i)). \tag{2.5}$$

Now

$$\int_{B(R)} |f - f_R|^2 \le \frac{1}{V(R)}\int_{B(R)\times B(R)} |f(p_1)-f(p_2)|^2\,dp_1dp_2$$

$$= \frac{1}{V(R)} \sum_{(e_1,e_2)\subset B(R)\times B(R)} \int_{(p_1,p_2)\in e_1\times e_2} |f(p_1)-f(p_2)|^2\,dp_1dp_2$$

$$\le \frac{1}{V(R)} \sum_{(e_1,e_2)\subset B(R)\times B(R)} 2R\sum_{i=0}^{|y|} (\delta f)(x_1\,\gamma_y(i)),$$

where x_1 and y are as defined above. The map $(e_1,e_2)\mapsto(x_1,y)$ is at most $|S|^2$-to-one, so

$$\int_{B(R)} |f-f_R|^2 \ \le\ 2R\,|S|^2\,\frac{1}{V(R)} \sum_{x_1\in B(R-1)}\ \sum_{y\in B(2R-2)}\ \sum_{i=0}^{|y|} (\delta f)(x_1\,\gamma_y(i))$$

$$\leq\ 4\,R^2\,|S|^2\frac{1}{V(R)}\sum_{y\in B(2R-2)}\sum_{z\in B(3R-1)}(\delta f)(z)\quad\text{(by (2.4))}$$

$$=4\,R^2\,|S|^2\,\frac{V(2R)}{V(R)}\sum_{z\in B(3R-1)}(\delta f)(z)\ \leq\ 8\,R^2\,|S|^2\,\frac{V(2R)}{V(R)}\int_{B(3R)}|\nabla f|^2.$$

□

Remark 2.6. Although the theorem above is not in the literature, the proof is virtually contained in [CSC93, pp.308-310]. When hearing of my more complicated Poincare inequality, Laurent Saloff-Coste's immediate response was to state and prove Theorem 2.2.

3. The proof of Theorem 1.4

In this section G will be a finitely generated group with a fixed finite generating set S, and the associated Cayley graph and word norm will be denoted Γ and $\|\cdot\|$, respectively. For $R\in\mathbb{Z}_+$ we let $B(R):=B(e,R)\subset\Gamma$ and $V(R):=|B_G(R)|=|B(R)\cap G|$.

Let $\mathcal{V}$ be a $2k$-dimensional vector space of harmonic functions on Γ. We equip $\mathcal{V}$ with the family of quadratic forms $\{Q_R\}_{R\in[0,\infty)}$, where

$$Q_R(u,u):=\int_{B(R)}u^2.$$

The remainder of this section is devoted to proving the following statement, which clearly implies Theorem 1.4:

Theorem 3.1. *For every $d\in(0,\infty)$ there is a $C=C(d)\in(0,\infty)$ such that if*

$$\liminf_{R\to\infty}\frac{V(R)\,(\det Q_R)^{\frac{1}{\dim\mathcal{V}}}}{R^d}\ <\ \infty,\tag{3.2}$$

then $\dim\mathcal{V}<C$.

The overall structure of the proof is similar to that of Colding-Minicozzi [CM97].

3.1. **Finding good scales.** We begin by using the polynomial growth assumption to select a pair of comparable scales $R_1<R_2$ at which both the growth function V and the determinant $(\det Q_R)^{\frac{1}{\dim\mathcal{V}}}$ have doubling behavior. Later we will use this to find many functions in $\mathcal{V}$ which have doubling behavior at scale R_2. Similar scale selection arguments appear in both [Gro81] and [CM97]; the one here is a hybrid of the two.

Observe that the family of quadratic forms $\{Q_R\}_{R\in[0,\infty)}$ is nondecreasing in R, in the sense that $Q_{R'}-Q_R$ is positive semi-definite when $R'\geq R$. Also, note that Q_R is positive definite for sufficiently large R, since $Q_R(u,u)=0$ for all R only if $u\equiv 0$. Choose $i_0\in\mathbb{N}$ such that $Q_R>0$ whenever $R\geq 16^{i_0}$.

We define $f:\mathbb{Z}_+\to\mathbb{R}$ and $h:\mathbb{Z}\cap[i_0,\infty)\to\mathbb{R}$ by

$$f(R)=V(R)\,(\det Q_R)^{\frac{1}{\dim\mathcal{V}}}\,,\quad\text{and}\quad h(i)=\log f(16^i).$$

Note that since Q_R is a nondecreasing function of R, both f and h are nondecreasing functions, and (3.2) translates to

$$\liminf_{i\to\infty}\ (h(i)-di\log 16)<\infty. \tag{3.3}$$

Put $a=4d\log 16$, and pick $w\in\mathbb{N}$.

Lemma 3.4. *There are integers $i_1,i_2\in[i_0,\infty)$ such that*

$$i_2-i_1\in(w,3w), \tag{3.5}$$

$$h(i_2+1)-h(i_1)<wa, \tag{3.6}$$

and

$$h(i_1+1)-h(i_1)<a,\quad h(i_2+1)-h(i_2)<a. \tag{3.7}$$

Proof. There is a nonnegative integer j_0 such that

$$h(i_0+3w(j_0+1))-h(i_0+3wj_0)<wa. \tag{3.8}$$

Otherwise, for all $l\in\mathbb{N}$ we would get

$$h(i_0+3wl)=h(i_0)+\sum_{j=0}^{l-1}\left(h(i_0+3w(j+1))-h(i_0+3wj)\right)$$

$$\geq h(i_0)+wal=h(i_0)+\left(\frac{4}{3}d\log 16\right)(3wl)\,,$$

which contradicts (3.3) for large l.

Let $m:=i_0+3wj_0$.

Then there are integers $i_1\in[m,m+w)$ and $i_2\in[m+2w,m+3w)$ such that (3.7) holds, for otherwise we would have either $h(m+w)-h(m)\geq wa$ or $h(m+3w)-h(m+2w)\geq wa$, contradicting (3.8).

These i_1 and i_2 satisfy the conditions of the lemma, because

$$h(i_2+1)-h(i_1)\leq h(m+3w)-h(m)<wa.$$

□

3.2. **A controlled cover.** Let $R_1 = 2 \cdot 16^{i_1}$ and $R_2 = 16^{i_2}$. Choose a maximal R_1-separated subset $\{x_j\}_{j\in J}$ of $B(R_2) \cap G$, and let $B_j := B(x_j, R_1)$. Then the collection $\mathcal{B} := \{B_j\}_{j\in J}$ covers $B(R_2)$, and $\frac{1}{2}\mathcal{B} := \{\frac{1}{2}B_j\}_{j\in J}$ is a disjoint collection.

Lemma 3.9. (1) *The covers $\mathcal{B}$ and $3\mathcal{B} := \{3B_j\}_{j\in J}$ have intersection multiplicity $< e^a$.*
(2) *$\mathcal{B}$ has cardinality $|J| < e^{wa}$.*
(3) *There is a $C \in (0,\infty)$ depending only on $|S|$ such that for every $j \in J$ and every smooth function $v : 3B_i \to \mathbb{R}$,*

$$\int_{B_i} |v - v_{B_i}|^2 \le C\, e^a\, R_1^2 \int_{3B_i} |\nabla v|^2. \tag{3.10}$$

Proof. (1) If $z \in 3B_{j_1} \cap \ldots \cap 3B_{j_l}$, then $x_{j_m} \in B(x_{j_1}, 6R_1)$ for every $m \in \{1, \ldots, l\}$, so $\{B(x_{j_m}, \frac{R_1}{2})\}_{m=1}^{l}$ are disjoint balls lying in $B(x_{j_1}, 8R_1)$, and hence

$$\log l \le \log \frac{V(3R_1)}{V(\frac{R_1}{2})} = \log V(3R_1) - \log V\left(\frac{R_1}{2}\right) \le h(i_1+1) - h(i_1) < a.$$

This shows that the multiplicity of $3\mathcal{B}$ is at most e^a. This implies (1), since the multiplicity of $\mathcal{B}$ is not greater than that of $3\mathcal{B}$.

(2) The balls $\{B(x_j, \frac{R_1}{2})\}_{j\in J}$ are disjoint, and are contained in $B(R_2 + \frac{R_1}{2}) \subset B(2R_2)$, so

$$|J| \le \frac{V(2R_2)}{V(\frac{R_1}{2})} \le \frac{V(16^{i_2+1})}{V(16^{i_1})} < e^{wa},$$

by (3.6).

(3) By Theorem 2.2 and the translation invariance of the inequality,

$$\int_{B_i} |v - v_{B_i}|^2 \;\le\; 8\,|S|^2\, R_1^2 \frac{V(2R_1)}{V(R_1)} \int_{3B_i} |\nabla v|^2$$

$$\le 8\,|S|^2\, R_1^2\, e^a \int_{3B_i} |\nabla v|^2.$$

□

3.3. **Estimating functions relative to the cover $\mathcal{B}$.** We now estimate the size of a harmonic function in terms of its averages over the B_j's, and its size on a larger ball.

We define a linear map $\Phi : \mathcal{V} \to \mathbb{R}^J$ by

$$\Phi_j(v) := \frac{1}{|B_j|} \int_{B_j} v.$$

Lemma 3.11 (cf. [CM97, Prop. 2.5]). *There is a constant $C \in (0, \infty)$ depending only on the size of the generating set S, with the following property.*

(1) *If u is a smooth functions on $B(16R_2)$, then*

$$Q_{R_2}(u,u) \quad \leq \quad C\,V(R_1)\,|\Phi(u)|^2 + \; C\,e^{2a} R_1^2 \int_{B(2R_2)} |\nabla u|^2. \tag{3.12}$$

(2) *If u is harmonic on $B(16R_2)$, then*

$$Q_{R_2}(u,u) \leq C\,V(R_1)|\Phi(u)|^2 + C\,e^{2a} \left(\frac{R_1}{R_2}\right)^2 Q_{16R_2}(u,u). \tag{3.13}$$

Proof. We will use C to denote a constant which depends only on $|S|$; however, its value may vary from equation to equation.

We have

$$Q_{R_2}(u,u) = \int_{B(R_2)} u^2 \leq \sum_{j\in J} \int_{B_j} u^2$$

$$\leq 2 \sum_{j \in J} \int_{B_j} \left(|\Phi_j(u)|^2 + |u - \Phi_j(u)|^2\right). \tag{3.14}$$

We estimate each of the terms in (3.14) in turn.

For the first term we get:

$$\sum_{j\in J} \int_{B_j} |\Phi_j(u)|^2 = \sum_{j \in J} |B_j|\,|\Phi_j(u)|^2 \leq C\,V(R_1)\;|\Phi(u)|^2. \tag{3.15}$$

For the second term we have:

$$\sum_{j\in J} \int_{B_j} |u - \Phi_j(u)|^2$$

$$\leq C\,e^a\,R_1^2 \sum_{j\in J} \int_{3B_j} |\nabla u|^2 \quad \text{(by (3) of Lemma 3.9)}$$

$$\leq Ce^a R_1^2 \left(e^a \int_{B(2R_2)} |\nabla u|^2 \right) \quad \text{(by (1) of Lemma 3.9)}$$
$$= Ce^{2a}\, R_1^2 \int_{B(2R_2)} |\nabla u|^2.$$

Combining this with (3.15) yields (1).

Inequality (3.13) follows from (3.12) by applying the reverse Poincare inequality, which holds for any harmonic function v defined on $B(16R_2)$:

$$R_2^2 \int_{B(2R_2)} |\nabla v|^2 \leq C\, Q_{16R_2}(v,v).$$

(For the proof, see [SY95, Lemma 6.3], and note that for harmonic functions their condition $u \geq 0$ may be dropped.) □

3.4. **Selecting functions from $\mathcal{V}$ with controlled growth.** Our next step is to select functions in $\mathcal{V}$ which have doubling behavior at scale R_2.

Lemma 3.16 (cf. [CM97, Prop. 4.16])**.** *There is a subspace $\mathcal{U} \subset \mathcal{V}$ of dimension at least $k = \frac{\dim \mathcal{V}}{2}$ such that for every $u \in \mathcal{U}$*

$$Q_{16R_2}(u,u) \leq e^{2a}\, Q_{R_2}(u,u). \tag{3.17}$$

Proof. Since $R_2 = 16^{i_2} > 16^{i_0}$, the quadratic form Q_{R_2} is positive definite. Therefore there is a Q_{R_2}-orthonormal basis $\beta = \{v_1, \ldots, v_{2k}\}$ for $\mathcal{V}$ which is orthogonal with respect to Q_{16R_2}.

Suppose there are at least l distinct elements $v \in \beta$ such that $Q_{16R_2}(v,v) \geq e^{2a}$. Then since β is Q_{R_2}-orthonormal and Q_{16R_2}-orthogonal,

$$\log \left(\frac{\det Q_{16R_2}}{\det Q_{R_2}} \right)^{\frac{1}{2k}} = \log \left(\prod_{j=1}^{2k} \frac{Q_{16R_2}(v_j,v_j)}{Q_{R_2}(v_j,v_j)} \right)^{\frac{1}{2k}} = \log \left(\prod_{j=1}^{2k} Q_{16R_i}(v_j,v_j) \right)^{\frac{1}{2k}}$$
$$\geq \log \left(e^{2al} \right)^{\frac{1}{2k}} = \frac{l}{k} a.$$

On the other hand,

$$a > h(i_2+1) - h(i_2) \geq \log \left(\det Q_{16R_2} \right)^{\frac{1}{2k}} - \log \left(\det Q_{R_2} \right)^{\frac{1}{2k}}.$$

So we have a contradiction if $l \geq k$.

Therefore we may choose a k element subset $\{u_1, \ldots, u_k\} \subset \{v_1, \ldots, v_{2k}\}$ such that $Q_{16R_2}(u_j,u_j) < e^{2a}$ for every $j \in \{1, \ldots, k\}$. Then every element of $\mathcal{U} := \operatorname{span}\{u_1, \ldots, u_k\}$ satisfies (3.17). □

3.5. **Bounding the dimension of $\mathcal{V}$.** We now assume that w is the smallest integer such that

$$\left(\frac{R_1}{R_2}\right)^2 = 2\cdot 16^{i_1-i_2} < 2\cdot 16^{-w} < \frac{1}{2Ce^{4a}}, \tag{3.18}$$

where C is the constant in (3.13). Therefore $2\cdot 16^{-(w-1)} \geq \frac{1}{2Ce^{4a}}$, and this implies

$$e^{wa} \leq 64\, C\, e^{64d^2 \log 16}. \tag{3.19}$$

If $u \in \mathcal{U}$ lies in the kernel of Φ, then

$$\begin{aligned} Q_{R_2}(u,u) &\leq Ce^{2a}\left(\frac{R_1}{R_2}\right)^2 Q_{16R_2}(u,u) \quad \text{(by (3.13))} \\ &\leq Ce^{2a}\left(\frac{R_1}{R_2}\right)^2 \left(e^{2a}\, Q_{R_2}(u,u)\right) \quad \text{(by Lemma 3.16)} \\ &\leq \frac{1}{2} Q_{R_2}(u,u) \quad \text{(by (3.18))}. \end{aligned}$$

Therefore $u = 0$, and we conclude that $\Phi\big|_{\mathcal{U}}$ is injective. Hence by Lemma 3.9 and (3.19),

$$\dim \mathcal{V} = 2 \dim \mathcal{U} \leq 2|J| \leq 2e^{wa} \leq 128\, C\, e^{64d^2 \log 16}.$$

□

4. Proof of Corollary 1.5 using Theorem 1.4

Let G be as in the statement of the Corollary, and let Γ denote some Cayley graph of G with respect to a symmetric finite generating set S.

Note that G is amenable, for if $R_k \to \infty$ and $V(R_k) < AR_k^d$ for all k, then for every k there must be an $r_k \in [\frac{R_k}{2}, R_k]$ such that the ball $B_G(r_k)$ satisfies

$$|\partial B_G(r_k)| = |S_G(r_k)| < 3A\, R_k^{d-1};$$

this means that the sequence of balls $\{B_G(r_k)\}$ is a Folner sequence for G.

Hence G does not have Property (T). Therefore by a result of Mok [Mok95] and Korevaar-Schoen [KS97, Theorem 4.1.2], there is an isometric action $G \curvearrowright \mathcal{H}$ of G on a Hilbert space $\mathcal{H}$ which has no fixed points, and a nonconstant G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$. In the case of Cayley graphs, the Mok/Korevaar-Schoen result is quite elementary, so we give a short proof in Appendix A.

Since f is G-equivariant, it is Lipschitz.

Each bounded linear functional $\phi \in \mathcal{H}^*$ gives rise to a Lipschitz harmonic function $\phi \circ f$, and hence we have a linear map $\Phi : \mathcal{H}^* \to \mathcal{V}$, where $\mathcal{V}$ is the space of Lipschitz harmonic functions on Γ. Since the target is finite dimensional by Theorem 1.4, the kernel of Φ has finite codimension, and its annihilator $\ker(\Phi)^\perp \subset \mathcal{H}$ is a finite dimensional subspace containing the image of f. It follows that the affine hull A of the image of f is finite dimensional and G-invariant. Therefore we have an induced isometric G-action $G \curvearrowright A$. This action cannot factor through a finite group, because it would then have fixed points, contradicting the fact that the original representation is fixed point free. The associated homomorphism $G \to \mathrm{Isom}(A)$ yields the desired finite dimensional representation of G.

□

5. Proof of Corollary 1.6 using Corollary 1.5

We prove Gromov's theorem using Corollary 1.5. The proof is a recapitulation of Gromov's argument, which reproduce here for the convenience of the reader.

The proof is by induction on the degree of growth.

Definition 5.1. Let G be a finitely generated group. The **degree (of growth) of** G is the minimum $\deg(G)$ of the nonnegative integers d such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^d} < \infty.$$

A group whose degree of growth is 0 is finite, and hence Corollary 1.6 holds for such a group.

Assume inductively that for some $d \in \mathbb{N}$ that every group of degree at most $d-1$ is virtually nilpotent, and suppose $\deg(G) = d$. Then G is infinite, and by Corollary 1.5 there is a finite dimensional linear representation $G \to GL(n)$ with infinite image $H \subset GL(n)$. Since H has polynomial growth, by [Tit72] (see [Sha98] for an easier proof) it is virtually solvable, and by [Wol68, Mil68] it must be virtually nilpotent.

After passing to finite index subgroups, we may assume H is nilpotent, and that its abelianization is torsion-free. It follows that there is a short exact sequence

$$1 \longrightarrow K \to G \stackrel{\alpha}{\to} \mathbb{Z} \longrightarrow 1.$$

By [vdDW84, Lemma (2.1)], the normal subgroup K is finitely generated, and $\deg(K) \leq \deg(G) - 1$.

By the induction hypothesis, K is virtually nilpotent. Let K' be a finite index nilpotent subgroup of K which is normal in G, and let $L \subset G$ be an infinite cyclic subgroup which is mapped isomorphically by α onto $\mathbb{Z}$. Then $K'L \subset G$ is a finite index solvable subgroup of G. As it has polynomial growth, by [Wol68, Mil68] it is virtually nilpotent.

□

Appendix A. Property (T) and equivariant harmonic maps

In this expository section, we will give a simple proof of the special case of the Korevaar-Schoen/Mok existence result needed in the proof of Corollary 1.6.

Suppose G is a finitely generated group, $S = S^{-1} \subset G$ is a symmetric finite generating set, and Γ is the associated Cayley graph.

Given an action $G \curvearrowright X$ on a metric space X, we define the **energy function** $E : X \to \mathbb{R}$ by

$$E(x) = \sum_{s \in S} d^2(sx, x).$$

We recall that a G has Property (T) iff every isometric action of G on a Hilbert space has a fixed point.

The following theorem is a very weak version of some results in [FM05], see also [Gro03, pp.115-116]:

Theorem A.1. *The following are equivalent:*

(1) *G has Property (T).*
(2) *There is a constant $D \in (0, \infty)$ such that if $G \curvearrowright \mathcal{H}$ is an isometric action on a Hilbert space and $x \in \mathcal{H}$, then G fixes a point in $B(x, D\sqrt{E(x)})$.*
(3) *There are constants $D \in (0, \infty)$, $\lambda \in (0, 1)$ such that if $G \curvearrowright \mathcal{H}$ is an isometric action on a Hilbert space and $x \in \mathcal{H}$, then there is a point $x' \in B(x, D\sqrt{E(x)})$ such that $E(x') \leq \lambda E(x)$.*
(4) *There is no isometric action $G \curvearrowright \mathcal{H}$ on a Hilbert space such that the energy function $E : \mathcal{H} \to \mathbb{R}$ attains a positive minimum.*

Proof. Clearly (2) $\implies$ (1). Also, (1) $\implies$ (4) since the energy function E is zero at a fixed point.

(3) $\implies$ (2). Suppose (3) holds. Let $G \curvearrowright \mathcal{H}$ be an isometric action, and pick $x_0 \in \mathcal{H}$. Define a sequence $\{x_k\} \subset \mathcal{H}$ inductively, by choosing $x_{k+1} \in B(x_k, D\sqrt{E(x_k)})$ such that $E(x_{k+1}) \leq \lambda E(x_k)$. Then $E(x_k) \leq \lambda^k E(x_0)$ and $d(x_{k+1}, x_k) \leq D\sqrt{E(x_k)} \leq D\lambda^{\frac{k}{2}}\sqrt{E(x_0)}$. Therefore $\{x_k\}$ is Cauchy, with limit x_∞ satisfying

$$d(x_\infty, x_0) \leq \frac{D\sqrt{E(x_0)}}{1-\lambda^{\frac{1}{2}}}.$$

Then $E(x_\infty) = \lim_{k\to\infty} E(x_k) = 0$, and x_∞ is fixed by G. Therefore (2) holds.

(4) $\implies$ (3). We prove the contrapositive. Assume that (3) fails. Then for every $k \in \mathbb{N}$, we can find an isometric action $G \curvearrowright \mathcal{H}_k$ on a Hilbert space, and a point $x_k \in \mathcal{H}_k$ such that

$$E(y) > \left(1 - \frac{1}{k}\right) E(x_k) \tag{A.2}$$

for every $y \in B(x_k, k\sqrt{E(x_k)})$. Note that in particular, $E(x_k) > \left(1-\frac{1}{k}\right) E(x_k)$, forcing $E(x_k) > 0$.

Let $\mathcal{H}'_k$ be the result of rescaling the metric on $\mathcal{H}_k$ by $\frac{1}{\sqrt{E(x_k)}}$. Then (A.2) implies that the induced isometric action $G \curvearrowright \mathcal{H}'_k$ satisfies $E(x_k) = 1$ and

$$E(y) \geq 1 - \frac{1}{k} \tag{A.3}$$

for all $y \in B(x_k, k)$. Then any ultralimit (see [Gro93, KL97]) of the sequence $(\mathcal{H}_k, x_k)$ of pointed Hilbert spaces is a pointed Hilbert space $(\mathcal{H}_\omega, x_\omega)$ with an isometric action $G \curvearrowright \mathcal{H}_\omega$ such that

$$E(x_\omega) = 1 = \inf_{y\in\mathcal{H}_\omega} E(y).$$

Therefore (4) fails.

□

Before proceeding we recall some facts about harmonic maps on graphs. Suppose $\mathcal{G}$ is a locally finite metric graph, where all edges have length 1. If $f : \mathcal{G} \to \mathcal{H}$ is a piecewise smooth map to a Hilbert space, then the following are equivalent:

- f is harmonic.
- The Dirichlet energy of f (on any finite subgraph) is stationary with respect to compactly supported variations of f.
- The restriction of f to each edge of $\mathcal{G}$ has constant derivative, and for every vertex $v \in \mathcal{G}$,

$$\sum_{d(w,v)=1} (f(w) - f(v)) = 0.$$

Note that if $G \curvearrowright \mathcal{H}$ is an isometric action on a Hilbert space, then E is a smooth convex function, and its derivative is

$$DE(x)(v) = 2\left(\sum_{s\in S} \langle sx - x, (Ds)(v)\rangle - \sum_{s\in S} \langle sx - x, v\rangle\right)$$

$$= 2\left(\sum_{s\in S} \langle x - s^{-1}x, v\rangle + \sum_{s\in S} \langle x - sx, v\rangle\right) = 4\sum_{s\in S} \langle x - sx, v\rangle.$$

Therefore

$$x \in \mathcal{H} \text{ is a critical point of } E$$
$$\iff x \text{ is a minimum of } E$$
$$\iff \sum_{s\in S} (x - sx) = 0. \tag{A.4}$$

Therefore the G-equivariant map $f_0 : G \to \mathcal{H}$ given by $f_0(g) := gx$ extends to a G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$ if and only if

$$\sum_{s\in S} (f_0(se)) - f_0(e)) = \sum_{s\in S} (sx - x) = 0$$
$$\iff x \text{ is a minimum of } E.$$

The next result is a very special case of a theorem from [Mok95, KS97].

Lemma A.5. *The following are equivalent:*

1. *G does not have Property (T).*
2. *There is an isometric action $G \curvearrowright \mathcal{H}$ on a Hilbert space $\mathcal{H}$ and a nonconstant G-equivariant harmonic map $f : \Gamma \to \mathcal{H}$.*

Proof. (1) $\implies$ (2). If G does not have Property (T), then by Theorem A.1 there is an isometric action $G \curvearrowright \mathcal{H}$ on a Hilbert space, and a point $x \in \mathcal{H}$ with $E(x) = \inf_{y\in\mathcal{H}} E(y) > 0$. Let $f : \Gamma \to \mathcal{H}$ be the G-equivariant map with $f(g) = gx$ for every $g \in G \subset \Gamma$, and whose restriction to each edge e of Γ has constant derivative. Then f is harmonic, and obviously nonconstant.

(2) $\Longrightarrow$ (1). Suppose (2) holds, and $f : \Gamma \to \mathcal{H}$ is the G-equivariant harmonic map. Then $f(e)$ is a positive minimum of $E : \mathcal{H} \to \mathbb{R}$; in particular the action $G \curvearrowright \mathcal{H}$ has no fixed points. Therefore G does not have Property (T).

□

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