

The Calabi flow on Kähler Surfaces with bounded Sobolev constant (I)

X. X. Chen* and W. Y. He†

Contents

1	Introduction	1
2	Preliminary	4
3	The evolution equations along the Calabi flow	6
4	Formation of singularities along the Calabi flow	10
5	Scalar flat ALE Kähler surface	17
6	Proof of Theorem 1.2	19
7	Appendix	26

1 Introduction

In [3], E. Calabi proposed to deform a Kähler metric in the direction of the Levi-Hessian of its scalar curvature. This is called the Calabi flow and it is a fourth order fully nonlinear parabolic equation aiming to attack the existence of constant scalar curvature metric (cscK metric) in a fixed Kähler class. Note that a cscK metric is a fixed point of the Calabi flow while an extremal Kähler (extK) metric is a soliton-type solution. A Kähler metric is called extremal if the complex gradient vector of its scalar curvature is holomorphic. In particular a Kähler-Einstein metric is an extK metric. The existence of extK metric is one of the central problems in Kähler geometry. The stability conjecture (Yau-Tian-Donaldson conjecture) in Kähler geometry asserts that the existence of extK metric is equivalent to the stability of the underlying Kähler structure in geometric invariant theory. However, a cscK metric (or extK) satisfies a fourth order nonlinear partial differential equation. It is hard to attack its existence

*The author is partially supported by NSF.

†The author is partially supported by a PIMS postdoc fellowship.

problem directly. The study of the Calabi flow seems to be an effective method. In [14], Donaldson described the limit behavior of the Calabi flow in a conceptual level, relevant to solving the stability conjecture in Kähler geometry. The first named author conjectured earlier [8] that the flow exists globally for any smooth initial Kähler metric. However there are rather few analytic techniques which can be applied directly as the Calabi flow is a fourth order parabolic equation. On Riemann surfaces, P. Chrusciel showed that the Calabi flow exists for all time and converges to a constant Gaussian curvature metric, making use of the existence of such a metric and the Bondi mass in general relativity. Without using the Bondi mass, Chen [7], Struwe [32] gave a direct proof of Chrusciel's theorem independently. While in higher dimensions, the nonlinearity of the Calabi flow becomes more acute and the analytic difficulty becomes more daunting. But recently some progress has been made [9, 37, 33] etc.

In an earlier paper [9], the authors proved that the main obstruction to the global existence of the Calabi flow is the bound of Ricci curvature. In particular the curvature blows up if the flow does not exist for all time. It is then natural to study the formation of singularities along the Calabi flow when the curvature blows up. We consider this problem by assuming that the Sobolev constants of the evolved metrics along the Calabi flow are uniformly bounded. The first result of this paper is

Theorem 1.1. *Let $(M, [\omega], J)$ be a compact Kähler manifold. If the Calabi flow on $(M, [\omega], J)$ exists for time $t \in [0, T)$ with uniformly bounded Sobolev constants and the curvature tensors become unbounded when $t \rightarrow \infty$, then there exists a sequence of points $(x_i, t_i) \in (M, [0, T))$ where $\lim_{i \rightarrow \infty} t_i = T$ and $\lim_{i \rightarrow \infty} |Riem(x_i, t_i)| = Q_i \rightarrow \infty$ such that the pointed manifolds*

$$(M, x_i, Q_i g(t_i + t/Q_i^2), J)$$

converge locally smoothly to an ancient solution of the Calabi flow

$$(M_\infty, x_\infty, g_\infty(t), J_\infty), t \in (-\infty, 0].$$

In particular, if (M, J) is a Kähler surface, for any $t \in (-\infty, 0]$, $g_\infty(t) \equiv g_\infty(0)$ and $(M_\infty, g_\infty(0), J_\infty)$ is a scalar flat ALE Kähler surface with finite total energy.

For simplicity, we call (M_∞, g_∞) a *maximal bubble* along the Calabi flow. Theorem 1.1 asserts that a *maximal bubble* along the Calabi flow on a Kähler surface has to be scalar flat with asymptotically locally Euclidean (ALE) structure at infinity. This property provides strong restrictions on the singularities that might form along the Calabi flow. See Section 4 for more details.

Scalar flat ALE Kähler surfaces are studied extensively in literature, for instance, Kronheimer [24], Joyce [23], LeBrun [26, 27] and Calderbank-Singer [6] etc. In particular, Kronheimer [24, 25] gave a complete classification in hyper-Kähler setting. In general the classification of scalar flat ALE Kähler surfaces seems to be hard. But the first named author believes that a scalar flat ALE

Kähler surface is uniquely determined by its underlying geometric structure. However, a proof seems out of reach at this point.

Recently Chen-LeBrun-Weber [10] studied the existence of extK metrics in some Kähler classes on $M \sim \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$; M can be obtained by \mathbb{CP}^2 blown up at two different points. They constructed an Einstein metric on $M \sim \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$ which is conformal to an extK metric in a particular Kähler class. The strategy in [10] is to use continuous deformation of extK metrics and a weak compactness theorem for extK metrics [12]. A possible bubble in the deformation turns out to be a scalar flat ALE Kähler surface. A key step is to eliminate the formation of such a bubble by using the smallness of the Calabi energy. Theorem 1.1 makes it possible for us to adopt the strategy in [10], *albeit* new difficulty arises in our case. For instance the bound of scalar curvature plays a key role in [10] while along the Calabi flow, such a bound is not known. Also we need to bound the Sobolev constants along the Calabi flow. Note that this is a non-collapsing (or local non-collapsing) result. In general it is hard to obtain non-collapsing result (or local non-collapsing) along a geometric evolution equation, such as Perelman's non-local collapsing result along the Ricci flow [30]. On Kähler surfaces with $c_1 > 0$, we can actually bound the Sobolev constants along the Calabi flow geometrically. Combining all these together, we can prove that the Calabi flow exists for all time and converges to an extK metric in many classes on toric surfaces with $c_1 > 0$. For simplicity, we only present a particular example of our results.

We consider $M \sim \mathbb{CP}^2 \# 3\overline{\mathbb{CP}^2}$. M can be obtained by \mathbb{CP}^2 blown up at three generic points. This is known as a del Pezzo surface with tori symmetry. Its automorphism group contains a compact two-torus T^2 . A metric which is invariant under the action of T^2 is called a toric metric. Calabi showed that any extK metric admits the maximal symmetry that the manifold allows [4]; namely an extK metric is invariant under the identity component of a maximal compact subgroup of the automorphism group of the manifold. This implies that any extK metric on M has to be toric. Let H be a hyperplane in \mathbb{CP}^2 . After blown up, we still use H to denote the corresponding hypersurface on M and $E_i, i = 1, 2, 3$ to denote the exceptional divisors. For simplicity, we use $[H], [E_i]$ to denote the homology classes and their Poincaré dual-the cohomology classes. A Kähler class on M can be described as $[\omega]_{x,y,z} = 3[H] - (x[E_1] + y[E_2] + z[E_3])$. When $x = y = z$, the Kähler classes $[\omega]_x = 3[H] - x([E_1] + [E_2] + [E_3])$ are invariant under the action of \mathbb{Z}_3 , which acts on exceptional divisors as a cyclic group. Note that the positivity condition of a Kähler class requires that $x \in (0, 3/2)$ and M degenerates to \mathbb{CP}^2 when $x = 0, 3/2$. We will give a full detailed discussion for $x = 1/2$. We can also prove similar results for any $x \in (0, 3/2)$ using the same method. We will give a rough discussion and state our theorem for $x \in (0, 3/2)$ in appendix.

Theorem 1.2. *Let $[\omega] = 3H - 1/2([E_1] + [E_2] + [E_3])$ on M . Suppose the initial metric $\omega_0 \in [\omega]$ is invariant under the toric action and the action of \mathbb{Z}_3 and*

$$\int_M R^2 dg_0 < 32\pi^2(6 + 25/11)$$

then the Calabi flow exists for all time with uniformly bounded curvature tensor, which converges to a cscK metric in the class $[\omega]$ in the Cheeger-Gromov sense.

Remark 1.3. *A priori lower bound of the Calabi energy is $32\pi^2(75/11)$ for $x = 1/2$. Our result actually implies the existence of a cscK metric in this class which realizes this lower bound.*

The toric structure is only used to classify *maximal bubbles*. It will be interesting to remove this assumption and that will lead to some general existence of extK metrics on Kähler surfaces with $c_1 > 0$.

The existence of extK metric is one of the central problems in Kähler geometry. One can see rapid developments in the last few years. We want to refer readers to [1, 15, 16, 11, 2] etc. for further references in this subject.

The organization of the paper goes as follows: in Section 2 we recall some basic facts in Kähler geometry and the Calabi flow. In Section 3 we calculate the evolution equations for the curvature tensor and derive some integral estimates along the Calabi flow. In Section 4 we discuss the formation of the singularities. We will prove Theorem 1.1 and use it to prove some general properties. In Section 5 we give some detailed description of scalar flat Kähler surfaces, in particular with toric assumption. In Section 6 we prove Theorem 1.2. We exhibit the details, for the particular case, how to bound the Sobolev constants and how to rule out bubbles. In appendix we discuss our results roughly in general case.

Acknowledgement: The first named author wishes to thank G. Tian for insightful discussions on this topic. He is also grateful to S. K. Donaldson and C. LeBrun for many discussions on four dimensional geometry. The second named author would like to thank Song Sun and Brian Weber for insightful discussions. He is also grateful to Zheng Hua, Jeff Viaclovsky and Bing Wang for valuable discussions. The present version is thoroughly rewritten and the authors are grateful to the referee for numerous suggestions.

2 Preliminary

Let $(M, [\omega], J)$ be a compact Kähler manifold of complex dimension n . A Kähler form on M in local coordinates is given by

$$\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge dz^{\bar{j}},$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix. The Kähler condition is $d\omega = 0$. In local coordinates, it says that

$$\frac{\partial g_{i\bar{j}}}{\partial z_{\bar{k}}} = \frac{\partial g_{i\bar{k}}}{\partial z_{\bar{j}}}, \quad \frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}.$$

Note that the Kähler class $[\omega] \in H^2(M, \mathbb{R})$ is a nontrivial cohomology class. By $\partial\bar{\partial}$ -lemma, any other Kähler form in the same cohomology class is of the form

$$\omega_\varphi = \omega + i\partial\bar{\partial}\varphi > 0,$$

for some real valued function φ on M . The corresponding Kähler metric is denoted by $g_\varphi = (g_{i\bar{j}} + \varphi_{,i\bar{j}}) dz^i \otimes dz^{\bar{j}}$. For simplicity, we use both g and ω to denote the Kähler metric. Define the space of Kähler potentials as

$$\mathcal{H}_\omega = \{\varphi | \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \varphi \in C^\infty(M)\}.$$

Given a Kähler metric ω , its volume form is

$$dg := \frac{\omega^n}{n!} = (\sqrt{-1})^n \det(g_{i\bar{j}}) dz^1 \wedge dz^{\bar{1}} \wedge \cdots \wedge dz^n \wedge dz^{\bar{n}}.$$

The Ricci curvature of ω is given by

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{k\bar{l}}).$$

Its Ricci form ρ is of the form

$$\rho = i R_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = -i \partial_i \partial_{\bar{j}} \log \det(g_{k\bar{l}}) dz^i \wedge dz^{\bar{j}}.$$

It is a real, closed $(1, 1)$ form. The cohomology class of the Ricci form is the famous first Chern class $c_1(M)$, independent of the metric.

In 1980s, E. Calabi [3, 4] introduced the Calabi functional as

$$\mathcal{C}(\omega_\varphi) = \int_M R_\varphi^2 dg_\varphi,$$

where R_φ is the scalar curvature of ω_φ . Let \underline{R} be the average of the scalar curvature, which is a constant depending only on the class $[\omega]$ and the underlying complex structure. Usually we use the following modified Calabi energy

$$\tilde{\mathcal{C}}(\omega_\varphi) = \int_M (R_\varphi - \underline{R})^2 dg_\varphi$$

to replace $\mathcal{C}(\omega_\varphi)$ since they only differ by a topological constant. Calabi studied the variational problem to minimize $\int_M |Rm|^2 dg$ in \mathcal{H}_ω , which is equivalent to minimize the Calabi functional $\mathcal{C}(\omega)$. A critical point is either a cscK metric or an extK metric depending on whether the Futaki character [19, 4] vanishes or not. The Futaki character $\mathcal{F} = \mathcal{F}_\varphi : \mathfrak{h}(M) \rightarrow \mathbb{C}$ is defined on the Lie algebra $\mathfrak{h}(M)$ of all holomorphic vector fields of M as follows,

$$\mathcal{F}_\varphi(X) = - \int_M X(F_\varphi) dg_\varphi,$$

where $X \in \mathfrak{h}(M)$ and F_φ is a real valued function defined by

$$\Delta_\varphi F_\varphi = R_\varphi - \underline{R},$$

where Δ_φ is the Laplacian operator of the metric ω_φ . The Calabi flow [3] is defined by

$$\frac{\partial \varphi}{\partial t} = R_\varphi - \underline{R}.$$

Under the Calabi flow, we have

$$\frac{d}{dt} \int_M (R_\varphi - \underline{R})^2 dg_\varphi = -2 \int_M (D_\varphi R_\varphi, R_\varphi) dg_\varphi \leq 0,$$

where D_φ is Lichnerowicz operator with respect to ω_φ . The Lichnerowicz operator D is defined by

$$Df = f_{,\alpha\beta}{}^{\alpha\beta},$$

where the covariant derivative is with respect to ω . The Calabi flow enjoys many wonderful properties. For instance, it is the gradient flow of both the Calabi energy and the Mabuchi energy. It also decreases the distance function in the space of Kähler metrics [5]. Assuming the long time existence, S. Donaldson [14] described the convergence properties of the Calabi flow conceptually and made conjectures regarding the convergence of the Calabi flow, and he also related the convergent behavior of the Calabi flow to the stability conjecture (Yau-Tian-Donaldson conjecture) in Kähler geometry.

3 The evolution equations along the Calabi flow

In this section, we study the evolution equations of the curvature tensor under the Calabi flow and prove some integral estimates. In particular, we give interpolation formulas of curvature tensors under the Calabi flow, similar to the results of R. Hamilton in Ricci flow [21].

Theorem 3.1. *If the curvature is bounded along the Calabi flow, then*

$$\frac{\partial}{\partial t} \int_M |\nabla^k Rm|^2 dg \leq -\frac{1}{2} \int_M |\nabla^{k+2} Rm|^2 dg + C \int_M |Rm|^2 dg,$$

where $C = C(n, k, \max |Rm|)$ is a constant.

The evolution equation of the curvature tensor (Riemannian curvature) is given by

$$\begin{aligned} \frac{\partial}{\partial t} Rm &= -\nabla\bar{\nabla}\nabla\bar{\nabla}R + Rm * \nabla^2 R \\ &= -\Delta^2 Rm + \nabla^2 Rm * Rm + \nabla Rm * \nabla Rm, \end{aligned} \tag{3.1}$$

where we use $*$ to denote possible contractions of tensors for simplicity.

Lemma 3.2. *The evolution for high derivatives of Riemannian curvature is*

$$\frac{\partial}{\partial t} \nabla^k Rm = -\Delta^2 \nabla^k Rm + \sum_{i+j=k+2} \nabla^i Rm * \nabla^j Rm. \tag{3.2}$$

Proof. Note that for any tensor T , by interchanging the derivatives

$$\nabla\Delta T = \Delta\nabla T + \nabla Rm * T + Rm * \nabla T.$$

If $k = 0$, this is true by (3.1). We proceed by induction on n . This gives

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^{k+1} Rm &= \nabla \left(\frac{\partial}{\partial t} \nabla^k Rm \right) + \nabla^2 Rm * \nabla^{k+1} Rm \\ &= \nabla \left(-\Delta^2 \nabla^k Rm + \sum_{i+j=k+2} \nabla^i Rm * \nabla^j Rm \right) + \nabla^2 Rm * \nabla^{k+1} Rm \\ &= -\nabla \Delta^2 \nabla^k Rm + \sum_{i+j=k+3} \nabla^i Rm * \nabla^j Rm \\ &= -\Delta^2 \nabla^{k+1} Rm + \sum_{i+j=k+3} \nabla^i Rm * \nabla^j Rm. \end{aligned}$$

This completes the proof. \square

Lemma 3.3. *In complex setting, for any tensor T , the different norms are relevant as the following,*

$$\int_M |\Delta T|^2 dg = \int_M |\nabla \bar{\nabla} T|^2 dg + \int_M Rm * T * \nabla^2 T \quad (3.3)$$

and

$$\int_M |\Delta T|^2 dg = \frac{1}{2} \int_M |\nabla^2 T|^2 dg + \int_M Rm * T * \nabla^2 T dg + \int_M Rm * \nabla T * \nabla T dg. \quad (3.4)$$

Proof. Integration by parts,

$$\int_M |\Delta T|^2 dg = \int_M T_{,i\bar{i}} T_{,j\bar{j}} dg = \int_M T_{,i\bar{j}} T_{,j\bar{i}} dg = \int_M T_{,i\bar{j}} T_{,\bar{i}j} dg + \int_M Rm * T * \nabla^2 T.$$

This gives (3.3). Similarly,

$$\int_M |\Delta T|^2 dg = \int_M T_{,ij} T_{,\bar{i}\bar{j}} dg + \int_M Rm * T * \nabla T dg + \int_M Rm * \nabla T * \nabla T dg.$$

Combine them,

$$\int_M |\Delta T|^2 dg = \frac{1}{2} \int_M |\nabla^2 T|^2 dg + \int_M Rm * T * \nabla T dg + \int_M Rm * \nabla T * \nabla T dg.$$

This completes the proof. \square

Lemma 3.4. *Along the Calabi flow, we have*

$$\begin{aligned} \frac{\partial}{\partial t} \int_M |\nabla^k Rm|^2 dg &= - \int_M |\nabla^{k+2} Rm|^2 dg + \sum_{i+j=k+2} \int_M \nabla^i Rm * \nabla^j Rm * \nabla^k Rm dg \\ &\quad + \int_M Rm * \nabla^{k+1} Rm * \nabla^{k+1} Rm dg. \end{aligned} \quad (3.5)$$

Proof. We compute

$$\begin{aligned} \frac{\partial}{\partial t} \int_M |\nabla^k Rm|^2 dg &= \int_M \left\langle \frac{\partial}{\partial t} \nabla^k Rm, \overline{\nabla^n Rm} \right\rangle dg + \int_M \left\langle \nabla^k Rm, \frac{\partial}{\partial t} \overline{\nabla^n Rm} \right\rangle dg \\ &\quad + \int_M \nabla^2 Rm * \nabla^k Rm * \nabla^k Rm dg. \end{aligned}$$

By (3.2), we compute

$$\begin{aligned} \frac{\partial}{\partial t} \nabla^k Rm &= -\Delta^2 \nabla^k Rm + \sum_{i+j=k+2} \nabla^i Rm * \nabla^j Rm, \\ \frac{\partial}{\partial t} \overline{\nabla^k Rm} &= \overline{\frac{\partial}{\partial t} \nabla^k Rm} = -\Delta^2 \overline{\nabla^k Rm} + \sum_{i+j=k+2} \nabla^i Rm * \nabla^j Rm. \end{aligned}$$

It gives that

$$\begin{aligned} \frac{\partial}{\partial t} \int_M |\nabla^k Rm|^2 dg &= -2 \int_M |\Delta \nabla^k Rm|^2 dg + \sum_{i+j=k+2} \int_M \nabla^i Rm * \nabla^j Rm * \nabla^k Rm dg \\ &= - \int_M |\nabla^{k+2} Rm|^2 dg + \sum_{i+j=k+2} \int_M \nabla^i Rm * \nabla^j Rm * \nabla^k Rm dg \\ &\quad + \int_M Rm * \nabla^{k+1} Rm * \nabla^{k+1} Rm dg. \end{aligned}$$

□

To derive integral estimates, we need some interpolation inequalities for tensors. These inequalities (3.6) (3.7) and (3.8) are due to R. Hamilton [21] and we list them below for the sake of completeness. Note that the estimates do not depend on the metrics evolved.

Lemma 3.5. (*Interpolation inequalities for tensors*) *Let M be a compact Riemannian manifold of dimension n and T be any tensor on M . Suppose*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \quad \text{with } r \geq 1.$$

Then

$$\left\{ \int_M |\nabla T|^{2r} dg \right\}^{\frac{1}{r}} \leq (2r - 2 + n) \left\{ \int_M |\nabla^2 T|^p dg \right\}^{\frac{1}{p}} \left\{ \int_M |T|^q dg \right\}^{\frac{1}{q}}. \quad (3.6)$$

Corollary 3.6.

$$\int_M |\nabla^i T|^{2k/i} dg \leq C \max_M |T|^{2k/i-2} \int_M |\nabla^k T|^2 dg. \quad (3.7)$$

Corollary 3.7.

$$\int_M |\nabla^i T|^2 dg \leq C \left\{ \int_M |\nabla^k T|^2 dg \right\}^{i/k} \left\{ \int_M |T|^2 dg \right\}^{1-i/k}. \quad (3.8)$$

Now we are ready to to prove Theorem 3.1.

Proof. Recall

$$\begin{aligned} \frac{\partial}{\partial t} \int_M |\nabla^k Rm|^2 dg &= - \int_M |\nabla^{k+2} Rm|^2 dg + \sum_{i+j=k+2} \int_M \nabla^i Rm * \nabla^j Rm * \nabla^k Rm dg \\ &\quad + \int_M Rm * \nabla^{k+1} Rm * \nabla^{k+1} Rm dg. \end{aligned}$$

We know that

$$\left| \int_M Rm * \nabla^{k+1} Rm * \nabla^{k+1} Rm dg \right| \leq C \max_M |Rm| \int_M |\nabla^{k+1} Rm|^2 dg,$$

and

$$\begin{aligned} &\left| \int_M \nabla^i Rm * \nabla^j Rm * \nabla^n Rm dg \right| \\ &\leq \left\{ \int_M |\nabla^i Rm|^{(2k+4)/i} dg \right\}^{i/(2k+4)} \left\{ \int_M |\nabla^j Rm|^{(2k+4)/j} dg \right\}^{j/(2k+4)} \left\{ \int_M |\nabla^k Rm|^2 dg \right\}^{1/2} \end{aligned}$$

with $i + j = k + 2$. By the interpolation result (3.7), we have ($i \neq 0$)

$$\left\{ \int_M |\nabla^i Rm|^{(2k+4)/i} dg \right\}^{i/(2k+4)} \leq C \max_M |Rm|^{1-i/(k+2)} \left\{ \int_M |\nabla^{k+2} Rm|^2 dg \right\}^{i/(2k+4)},$$

and doing the same for j , it follows

$$\left| \int_M \nabla^i Rm * \nabla^j Rm * \nabla^n Rm dg \right| \leq C \max_M |Rm| \left\{ \int_M |\nabla^{k+2} Rm|^2 dg \right\}^{1/2} \left\{ \int_M |\nabla^k Rm|^2 dg \right\}^{1/2}.$$

By the interpolation inequality (3.8) and Young's inequality,

$$\int_M |\nabla^i Rm|^2 dg \leq \epsilon \int_M |\nabla^{k+2} Rm|^2 dg + C \int_M |Rm|^2 dg$$

for any $1 \leq i < k + 2$. It follows that

$$\begin{aligned} \frac{\partial}{\partial t} \int_M |\nabla^k Rm|^2 dg &\leq - \int_M |\nabla^{k+2} Rm|^2 dg + C \int_M |\nabla^{k+1} Rm|^2 dg \\ &\quad + C \left\{ \int_M |\nabla^{k+2} Rm|^2 dg \right\}^{1/2} \left\{ \int_M |\nabla^k Rm|^2 dg \right\}^{1/2} \\ &\leq - \int_M |\nabla^{k+2} Rm|^2 dg + \epsilon \int_M |\nabla^{k+2} Rm|^2 dg + C \int_M |Rm|^2 dg \\ &\leq -\frac{1}{2} \int_M |\nabla^{k+2} Rm|^2 dg + C \int_M |Rm|^2 dg. \end{aligned} \quad (3.9)$$

□

4 Formation of singularities along the Calabi flow

In this section we study the formation of singularities along the Calabi flow with uniformly bounded Sobolev constants. In particular we will prove Theorem 1.1 and use it to prove some general properties. Note that the Calabi flow is a fourth order parabolic equation, the maximum principle is not applicable in general. The tools we use are the integral estimates proved in Section 3 and Sobolev inequalities. First we need a Sobolev inequality for tensors. First recall the Sobolev inequality on a compact Riemannian manifold (M, g) of dimension m ; there exists a constant $C_s = C_s(M, g)$ such that

$$\|f\|_{L^{2m/(m-2)}} \leq C_s \left(\|\nabla f\|_{L^2} + Vol^{-1/m} \|f\|_{L^2} \right), \quad (4.1)$$

where Vol denotes the volume of (M, g) . Note that the Sobolev inequality (4.1) is scaling invariant.

Lemma 4.1 (Sobolev inequality for tensors). *Let (M, g) be a compact Riemannian manifold of dimension m with fixed volume, and let T be any tensor on M . Then the Sobolev inequality holds for T , namely,*

$$\|T\|_{L^{2m/(m-2)}} \leq C_s (\|\nabla T\|_{L^2} + Vol^{-1/m} \|T\|_{L^2}).$$

Proof. Denote $f = |T|$, the Sobolev inequality for functions gives that

$$\|f\|_{L^{2m/m-2}} \leq C_s (\|\nabla f\|_{L^2} + Vol^{-1/m} \|f\|_{L^2}).$$

Since $|\nabla f| = |\nabla |T|| = |\nabla \langle T, T \rangle^{1/2}|$, and when $|T| \neq 0$,

$$|\nabla \langle T, T \rangle^{1/2}| = \frac{\langle \nabla T, T \rangle}{|T|} \leq |\nabla T|.$$

If $|T|_0$ vanishes at some point, take $f = \sqrt{\langle T, T \rangle} + \epsilon$ and apply the Sobolev inequality for f , then let $\epsilon \rightarrow 0$. This completes the proof. □

Now by the integral estimates derived in Section 3, we can prove

Lemma 4.2. *If the curvature tensors and the Sobolev constants evolved are both uniformly bounded in $[0, T)$ ($T \leq \infty$), then all derivatives of curvature are bounded (independent of time T).*

Proof. Recall that

$$\frac{\partial}{\partial t} \int_M |\nabla^k Rm|^2 dg \leq -\frac{1}{2} \int_M |\nabla^{k+2} Rm|^2 dg + C \int_M |Rm|^2 dg,$$

if the curvature tensor is uniformly bounded by Theorem 3.1. For any $t_0 \in [0, T - 1)$ denote

$$F_k(t) = \sum_{i=0}^k t^i \int_M |\nabla^i Rm|^2 dg(t_0 + t),$$

where $t \in [0, 1]$. Then we will have

$$\begin{aligned} \frac{\partial F_k}{\partial t} &= \sum_{i=0}^k i t^{i-1} \int_M |\nabla^i Rm|^2 dg + \sum_{i=1}^k t^i \frac{\partial}{\partial t} \int_M |\nabla^i Rm|^2 dg \\ &\leq \sum_{i=0}^k i t^{i-1} \int_M |\nabla^i Rm|^2 dg + \sum_{i=1}^k t^i \left(-\frac{1}{2} \int_M |\nabla^{i+2} Rm|^2 dg + C \int_M |Rm|^2 dg \right) \\ &\leq \sum_{i=0}^k t^i \left(-\frac{1}{2} \int_M |\nabla^{i+2} Rm|^2 dg + (i+1) \int_M |\nabla^{i+1} Rm|^2 dg \right) + C \int_M |Rm|^2 dg \\ &\leq \sum_{i=0}^k t^i \left(-\frac{1}{4} \int_M |\nabla^{i+2} Rm|^2 dg + C \int_M |Rm|^2 dg \right) + C \int_M |Rm|^2 dg \\ &\leq -\frac{1}{4} \sum_{i=0}^k t^i \int_M |\nabla^{i+2} Rm|^2 dg + C \int_M |Rm|^2 dg. \end{aligned}$$

It follows that

$$\begin{aligned} F_k(1) &= F_k(0) + \int_0^1 \frac{\partial F_k}{\partial t} dt \\ &\leq \int_M |Rm|^2 dg(t_0) + \int_0^1 C \int_M |Rm|^2 dg \\ &\leq C \int_M |Rm|^2 dg(t_0). \end{aligned}$$

Since $F_k(1) = \sum_{i=0}^k \int_M |\nabla^i Rm|^2 dg(t_0 + 1)$, it means L^2 norms of all higher derivatives of curvature are bounded. With uniformly bounded Sobolev constants, it implies that all higher derivatives of the curvature are uniformly bounded independent of time by Sobolev embedding theorem. \square

Now, we are ready to prove Theorem 1.1

Proof. Consider the Calabi flow in $[0, T)$ with uniformly bounded Sobolev constants ($T \leq \infty$). If the curvature blows up when $t \rightarrow T$, pick up a sequence $t_i \rightarrow T$, denote

$$Q_i = \max_{x \in M, t \leq t_i} |Rm|(x, t) = |Rm|(x_i, t_i).$$

Re-scale the metric

$$g_i(t) = Q_i g(t/Q_i^2 + t_i).$$

Note that the Sobolev inequality is scaling invariant. After scaling, the curvature tensors are uniformly bounded. It follows from Lemma 4.2 that all the higher derivatives of the curvature are uniformly bounded for any $g_i(t)$. It is also clear that bounded Sobolev constants with bounded curvature tensors imply that the injectivity radius is bounded away from zero at any point. One can find a nice reference in [38] for all details. Then the pointed manifold $\{M, g_i(t), x_i\}$ converges subsequently smoothly to $\{M_\infty, g_\infty(t), x_\infty\}$ in Cheeger-Gromov sense. The proof is exactly the same as in the Ricci flow shown by R. Hamilton [22]. In particular $\{g_\infty(t), -\infty < t \leq 0\}$ is an ancient solution of the Calabi flow

$$\frac{\partial \tilde{g}}{\partial t} = \nabla \bar{\nabla} \tilde{R}$$

on the complete manifold (non-compact) M_∞ . Now we consider the convergence of the complex structure. By the blowing up process, there is a sequence of compact set K_i , $K_i \subset K_{i+1}$, $\cup K_i = M_\infty$, and a sequence of diffeomorphisms $\Phi_i : K_i \rightarrow \Phi_i(K_i) \subset M$,

$$\Phi_i^*(g_i) \rightarrow g_\infty,$$

where the convergence is smooth in K_{i-1} . Denote $J_i = \Phi_i^{-1} \circ J \circ \Phi_{i*}$. If necessary, by taking a subsequence, $J_i \rightarrow J_\infty$. It is clear that J_∞ is still a complex structure which is compatible with $g_\infty(t)$ for any t .

When (M, J) is a Kähler surface, we can prove that $\{M_\infty, g_\infty(t)\}$ is actually extK since the Calabi energy is scaling invariant on Kähler surface. For any fixed time $t \in (-\infty, 0)$, consider $g_\infty(t)$ as the limit of $g_i(t/Q_i^2 + t_i)$ when $i \rightarrow \infty$. We have

$$\begin{aligned} \int_t^0 \int_{M_\infty} |R_{,\alpha\beta}|^2 dg_\infty dt &\leq \lim_{i \rightarrow \infty} \int_{t/Q_i^2 + t_i}^{t_i} \int_M |R_{,\alpha\beta}|^2 dg_i(s) ds \\ &= \lim_{i \rightarrow \infty} \{\mathcal{C}(t/Q_i^2 + t_i) - \mathcal{C}(t_i)\} \\ &= 0. \end{aligned}$$

It gives that for any $s \in [t, 0]$,

$$\int_{M_\infty} |R_{,\alpha\beta}|^2 dg_\infty = 0.$$

It follows that $g_\infty(s)$ is an extK metric for any s , and so $(M_\infty, g_\infty(s))$ are all biholomorphic. Let $g_\infty = g_\infty(0)$. In particular on (M_∞, g_∞) , for any smooth function f with compact support, the following Sobolev inequality holds for some constant $C_s = C_s(M_\infty, g_\infty)$,

$$\|f\|_{L^{n/n-1}} \leq C_s \|\nabla f\|_{L^2}. \quad (4.2)$$

It is also clear that the L^2 norm of curvature on (M_∞, g_∞) is bounded. By Chen-Weber's work on the moduli of extremal metrics [12] on Kähler surfaces, (M_∞, g_∞) has ALE (asymptotically locally Euclidean) structure at infinity. We then finish the proof by showing the following lemma. \square

Lemma 4.3. *Let $(M_\infty, g_\infty, J_\infty)$ be an ALE extK metric with complex dimension $m > 1$, then $(M_\infty, g_\infty, J_\infty)$ is scalar flat.*

ALE spaces have been studied often in the literature. A complete manifold (X, g) is called asymptotically locally Euclidean (ALE) end of order τ ($\tau > 0$) if there exists a finite subgroup $\Gamma \subset SO(n)$ acting freely on $\mathbb{R}^n \setminus B(0, R)$ and a C^∞ diffeomorphism $\Psi : X \rightarrow (\mathbb{R}^n \setminus B(0, R))/\Gamma$ such that under this identification,

$$\begin{aligned} g_{ij} &= \delta_{ij} + O(r^{-\tau}), \\ \partial^{|k|} g_{ij} &= O(r^{-\tau-k}), \end{aligned}$$

for any partial derivative of order k as $r \rightarrow \infty$. We say an end is ALE is of order 0 if we can find a coordinate system as above with

$$\begin{aligned} g_{ij} &= \delta_{ij} + o(1), \\ \partial^{|k|} g_{ij} &= o(r^{-k}), \end{aligned}$$

as $r \rightarrow \infty$. A complete, non-compact manifold (X, g) is called ALE if X can be written as the disjoint union of a compact set and finitely many ALE ends. When (X, g) is in addition a Kähler manifold, $\Gamma \subset Gl(n/2, \mathbb{C}) \cap SO(n) = U(n/2)$, $n = 2m$. In particular one can show that an ALE Kähler manifold has just one end, as results of Li and Tam [29].

Now we are ready to prove Lemma 4.3.

Proof. Let (M_∞, g_∞) be an ALE Kähler manifold with an extK metric g_∞ . Since $\Gamma \subset U(m)$, there exists an asymptotic coordinate $\mathbb{C}^m \setminus B(0, R)/\Gamma$ on the ALE end. Through $\pi : \mathbb{C}^m \setminus B(0, R) \rightarrow \mathbb{C}^m \setminus B(0, R)/\Gamma$, the metric can be lifted up to $\mathbb{C}^m \setminus B(0, R)$. If g_∞ is extK, then ∇R is the real part of a holomorphic vector, on $\mathbb{C}^m \setminus B(0, R)$, and one can express ∇R as

$$\sum_{i=1}^m f_i(z_1, \dots, z_m) \frac{\partial}{\partial z_i} + \overline{f_i(z_1, \dots, z_m)} \frac{\partial}{\partial \bar{z}_i},$$

where f_i ($1 \leq i \leq m$) are holomorphic functions on $\mathbb{C}^m \setminus B(0, R)$. By Hartogs' theorem, one can extend f_i to \mathbb{C}^m . Since the metric is ALE, $|f_i|$ vanishes at infinity. Now the real part and imaginary part of f_i are both harmonic functions, the maximum principle implies $f_i \equiv 0$. So $\nabla R \equiv 0$. It follows that $R = \text{const}$. The ALE condition implies that R has to be zero. \square

In the following we shall discuss some general properties along the Calabi flow using Theorem 1.1. First we define type I and type II singularities for the Calabi flow.

Definition 4.4. *Suppose the Calabi flow exists in $[0, T)$ for some finite time T and the curvature blows up at T . A singularity is called type I if*

$$\limsup_{t \rightarrow T} |Rm|^2(T - t) < \infty.$$

Otherwise it is called a type II singularity.

Proposition 4.5. *Let T be the maximal existence time of the Calabi flow on a Kähler surface. If $T < \infty$ and the Sobolev constant is uniformly bounded, then there is no type I singularity. Namely,*

$$\limsup_{t \rightarrow T} |Rm|^2(T - t) = \infty.$$

Proof. If not, then

$$\limsup_{t \rightarrow T} |Rm|^2(T - t) = C$$

for some constant C . We can choose a sequence (x_i, t_i) such that

$$Q_i := |Rm|(x_i, t_i) = \max_{0 \leq s \leq t_i, y \in M} |Rm|(y, s)$$

and

$$\lim_{i \rightarrow \infty} Q_i^2(T - t_i) = C.$$

Then

$$\{M, g_i(s) = Q_i g(t_i + sQ_i^{-2})\}$$

converges to one parameter family of *maximal bubble* $(M_\infty, g_\infty(s))$ evolved by the Calabi flow. By Theorem 1.1, $g_\infty(s)$ is scalar flat and $g_\infty(s) \equiv g_\infty(0)$ for any s . In particular, for any $s \in (-\infty, 0]$,

$$\lim_{i \rightarrow \infty} Q_i^{-1} |Rm(x_i, t_i + sQ_i^{-2})| = 1.$$

At time $t_i - Q_i^{-2}(s = -1)$, we have

$$\lim_{i \rightarrow \infty} |Rm(x_i, t_i - Q_i^{-2})|^2(T - (t_i - Q_i^{-2})) = \lim_{i \rightarrow \infty} Q_i^2(T - t_i) + 1 = C + 1.$$

It contradicts

$$\limsup_{t \rightarrow T} |Rm|^2(T - t) = C.$$

It completes the proof. \square

Theorem 1.1 gives strong restrictions of possible singularities along the Calabi flow on a Kähler surface with the uniformly bounded Sobolev constants. We actually expect that the longtime existence holds for the Calabi flow on Kähler surfaces in this case. In this direction we can prove that

Theorem 4.6. *Consider the Calabi flow exists in $[0, T)$ on Kähler surface with bounded Sobolev constants, and in addition if we have*

$$\int_M |Rm|^p dg < \infty$$

for any $p > 2$, then $T = \infty$.

To prove this theorem, first we derive an estimate on $W^{1,2}$ norm of the curvature.

Lemma 4.7. *The Calabi flow exists in $[0, T)$ with uniformly bounded Sobolev constant. We have*

$$\frac{\partial}{\partial t} \int_M |\nabla Rm|^2 dg \leq -\frac{1}{2} \int_M |\nabla^3 Rm|^2 dg + C(p, C_s) |Rm|_{L^p}^{3p/(p-2)} + C(p, C_s) |Rm|_{L^p}^3.$$

for any $p > 2$.

Proof. Recall the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} \int_M |\nabla Rm|^2 dg &= - \int_M |\nabla^3 Rm|^2 dg + \int_M Rm * \nabla^2 Rm * \nabla^2 Rm dg \\ &\quad + \int_M Rm * \nabla Rm * \nabla^3 Rm dg \\ &= - \int_M |\nabla^3 Rm|^2 dg + \int_M Rm * \nabla^2 Rm * \nabla^2 Rm dg \\ &\quad + \int_M \nabla Rm * \nabla Rm * \nabla^2 Rm dg. \end{aligned}$$

By Hölder inequality we know

$$\begin{aligned} \left| \int_M Rm * \nabla^2 Rm * \nabla^2 Rm dg \right| &\leq \int_M |Rm| |\nabla^2 Rm|^2 dg \\ &\leq \left\{ \int_M |Rm|^p dg \right\}^{1/p} \left\{ \int_M |\nabla^2 Rm|^{2q} dg \right\}^{1/q}, \end{aligned}$$

where $1/p + 1/q = 1$. Using Lemma 4.1 (Sobolev inequality for tensors), we obtain

$$|\nabla^2 Rm|_{L^4} \leq C_s (|\nabla^3 Rm|_{L^2} + |\nabla^2 Rm|_{L^2}).$$

Combining the interpolation inequality (3.6) in Section 3, we can get

$$\begin{aligned} \left| \int_M Rm * \nabla^2 Rm * \nabla^2 Rm dg \right| &\leq \int_M |Rm| |\nabla^2 Rm|^2 dg \\ &\leq \left\{ \int_M |Rm|^p dg \right\}^{1/p} \left\{ \int_M |\nabla^2 Rm|^{2q} dg \right\}^{1/q} \\ &\leq |Rm|_{L^p} |\nabla^2 Rm|_{L^4}^{2a} |\nabla^2 Rm|_{L^2}^{2(1-a)} \\ &\leq C |Rm|_{L^p} C_s (|\nabla^3 Rm|_{L^2} + |\nabla^2 Rm|_{L^2})^{2a} |\nabla^3 Rm|_{L^2}^{4(1-a)/3} \\ &\leq C |Rm|_{L^p} |\nabla^3 Rm|_{L^2}^{2(2+a)/3} + C |Rm|_{L^p} |\nabla^3 Rm|_{L^2}^{4/3}, \end{aligned}$$

where $1/2q = a/4 + (1-a)/2$, and so $a = 2 - 2/q$. It is easy to see

$$C |Rm|_{L^p} |\nabla^3 Rm|_{L^2}^{2(2+a)/3} \leq \frac{1}{8} |\nabla^3 Rm|_{L^2}^2 + C |Rm|_{L^p}^{3q/(2-q)},$$

and

$$C |Rm|_{L^p} |\nabla^3 Rm|_{L^2}^{4/3} \leq \frac{1}{8} |\nabla^3 Rm|_{L^2}^2 + C |Rm|_{L^p}^3.$$

It gives us that

$$\left| \int_M \nabla Rm * \nabla Rm * \nabla^2 Rm dg \right| \leq \frac{1}{4} |\nabla^3 Rm|_{L^2}^2 + C |Rm|_{L^p}^{3p/(p-2)} + C |Rm|_{L^p}^3.$$

We can also estimate

$$\begin{aligned} \left| \int_M \nabla Rm * \nabla Rm * \nabla^2 Rm dg \right| &\leq \int_M |\nabla^2 Rm| |\nabla Rm|^2 dg \\ &\leq \left\{ \int_M |\nabla^2 Rm|^q dg \right\}^{1/q} \left\{ \int_M |\nabla Rm|^{2r} dg \right\}^{1/r}. \end{aligned}$$

Recall the interpolation inequality (3.6),

$$\left\{ \int_M |\nabla Rm|^{2r} dg \right\}^{1/r} \leq \left\{ \int_M |\nabla^2 Rm|^s dg \right\}^{1/s} \left\{ \int_M |Rm|^t dg \right\}^{1/t},$$

where $1/r = 1/s + 1/t$. Choose $s = q, t = p$, it follows that

$$\left| \int_M \nabla Rm * \nabla Rm * \nabla^2 Rm dg \right| \leq \left\{ \int_M |\nabla^2 Rm|^q dg \right\}^{2/q} \left\{ \int_M |Rm|^p dg \right\}^{1/p},$$

where $2/q + 1/p = 1$. Similarly we can estimate that

$$\left| \int_M \nabla Rm * \nabla Rm * \nabla^2 Rm dg \right| \leq \frac{1}{4} |\nabla^3 Rm|_{L^2}^2 + C |Rm|_{L^p}^{3p/(p-2)} + C |Rm|_{L^p}^3.$$

Combining these estimates, it follows that

$$\frac{\partial}{\partial t} \int_M |\nabla Rm|^2 dg \leq -\frac{1}{2} \int_M |\nabla^3 Rm|^2 dg + C(p, C_s) |Rm|_{L^p}^{3p/(p-2)} + C(p, C_s) |Rm|_{L^p}^3.$$

for any $p > 2$. □

Now we are in the position to prove Theorem 4.6.

Proof. If $T < \infty$, Lemma 4.7 gives that $W^{1,2}$ norm of the curvature is bounded by L^p norm of curvature for any $p > 2$. If the curvature blows up, we can get a sequence (x_i, t_i) such that

$$Q_i = |Rm|(x_i, t_i) = \max_{x \in M, t \leq t_i} |Rm|(x, t).$$

Then we know that $\{M, g_i(s) = Q_i g(t_i + Q_i^2 s), x_i\}$ converges smoothly (in Cheeger-Gromov sense) to a *maximal bubble* $\{M_\infty, g_\infty, x_\infty\}$. In particular $g_\infty(s) \equiv g_\infty(0)$ for any $s \in (-\infty, 0)$ and $g_\infty(0)$ is scalar flat. Now

$$\begin{aligned} \int_{M_\infty} |\nabla Rm_\infty|^2 dg_\infty &\leq \lim_{i \rightarrow \infty} \int_M |\nabla Rm|_{g_i}^2 dg_i \\ &\leq \lim_{i \rightarrow \infty} Q_i^{-2} \int_M |\nabla Rm|^2 dg = 0. \end{aligned}$$

It gives that

$$\int_{M_\infty} |\nabla Rm_\infty|^2 dg_\infty = 0.$$

It forces that (M_∞, g_∞) is a flat metric. That is a contradiction. \square

5 Scalar flat ALE Kähler surface

Suppose $(M_\infty, g_\infty, J_\infty)$ is a scalar flat ALE Kähler surface in Theorem 1.1. In this section we derive some properties of (M_∞, g_∞) , which will be used crucially in proving Theorem 1.2. Many results in this section follow from [10]. The following lemma is due to [10].

Lemma 5.1. *M_∞ is diffeomorphic to an open subset U of M . If $b_1(M) = 0$, then $b_1(M_\infty) = b_3(M_\infty) = 0$, and $b_2(M_\infty) \leq b_-(M)$.*

Proof. (M_∞, g_∞) is obtained by a pointed Gromov-Hausdorff limit of re-scaled versions of M , and after scaling the curvature is uniformly bounded. Therefore the convergence is smooth (in Cheeger-Gromov sense) in any compact subset of M_∞ , by applying suitable diffeomorphisms. Since M_∞ is a scalar flat ALE Kähler manifold with one end, M_∞ is diffeomorphic to the interior of a compact domain $U \subset M_\infty$ with smooth boundary S^3/Γ . And U can then be mapped diffeomorphically into M , resulting in a decomposition

$$M \approx U \cup_{S^3/\Gamma} V$$

where U and V are manifolds with boundary, $M_\infty \approx \text{Int}(U)$, and $\partial U = \partial V = S^3/\Gamma$. We have the exact sequence

$$0 \rightarrow \Omega^*(M) \rightarrow \Omega^*(U) \oplus \Omega^*(V) \rightarrow \Omega^*(S^3/\Gamma) \rightarrow 0.$$

If $b_1(M) = 0$, the Mayer-Vietories sequence implies that both U and V have $b_1 = b_3 = 0$, while

$$H^2(M, \mathbb{R}) = H^2(U, \mathbb{R}) \oplus H^2(V, \mathbb{R}).$$

Since the analogous statement hold similarly for homology, the intersection form of $M_\infty = \text{Int}(U)$ is just the restriction of the intersection form of $H^2(M)$ to the linear subspace $H^2(U) \subset H^2(M)$. But the intersection form of M_∞ is negative ($W_+ = 0$), hence $b_2(M_\infty) \leq b_-(M)$. \square

First of all (M_∞, J_∞) cannot have trivial topology.

Lemma 5.2 ([10]). *Let (M_∞, g_∞) be a simply connected scalar flat ALE Kähler metric, then $b_2(M_\infty) \neq 0$.*

The following lemma appears also in [10] implicitly.

Lemma 5.3 (Lagrangian condition). *If M_∞ contains some compact 2-sphere S , then M contains a sequence of 2-spheres S_i corresponding to S . Moreover, the homology class of S_i satisfies that*

$$[\phi][S_i] \rightarrow 0,$$

when $i \rightarrow \infty$, which we call *Lagrangian condition* for the sequence of S_i .

Proof. (M_∞, g_∞) is constructed as the pointed limit of (M, g_i) after blowing up. If M_∞ contains a compact 2-sphere S , then (M, g_i) contains a compact 2-sphere for each i , and after blowing up, the image of S_i converges to S . Since the scaling factor goes to infinity as $i \rightarrow \infty$, the area of S_i with respect to g_i must tend to zero. By Wirtinger's inequality, we end up with

$$|[S_i][\phi_i]| \leq \text{Area}_{g_i}(S_i) \rightarrow 0.$$

Hence

$$[\phi][S_i] = [\phi_i][S_i] \rightarrow 0.$$

It means S_i behaves like a Lagrangian class with respect to $[\phi]$ when i is big. \square

The classification of ALE scalar flat Kähler metrics on complete surfaces might be too ambitious. But we know much better if (M_∞, g_∞) admits a toric structure. Following [10] (Proposition 16), we have

Lemma 5.4. *Let $(M, [\omega])$ be a compact toric surface. Suppose the Calabi flow with the initial metric which is invariant under toric action, if the curvature tensor is not uniformly bounded, then we can form a maximal bubble (M_∞, g_∞) . Moreover (M_∞, g_∞) is toric and $H_2(M_\infty, \mathbb{Z})$ is generated by holomorphic $\mathbb{C}\mathbb{P}^1$ s embedded in M_∞ .*

We can understand (M_∞, g_∞) in more detail by its toric structure. If $b_2(M_\infty) = 1$, $H_2(M_\infty, \mathbb{Z})$ is generated by a holomorphic $\mathbb{C}\mathbb{P}^1$ s E with self intersection $(-k)$, then the group $\Gamma = \mathbb{Z}_k$ and

$$c_1 = \frac{k-2}{k}[E].$$

Hence

$$\int_{M_\infty} |\text{Ric}_0|^2 dg_\infty = 8\pi^2 \frac{(k-2)^2}{k}. \quad (5.1)$$

If $b_2(M_\infty) = 2$, then there is a basis for $H_2(M_\infty, \mathbb{Z})$, represented by a pair of totally geodesics and holomorphic $\mathbb{C}\mathbb{P}^1$ s E_1, E_2 , in which the intersection form becomes

$$\begin{pmatrix} -k & 1 \\ 1 & -l \end{pmatrix}$$

for some positive $k \geq 2$ and $l \geq 1$ ($k \geq l$). Moreover $\Gamma = \mathbb{Z}_{kl-1}$ and

$$c_1 = \left(1 - \frac{l+1}{kl-1}\right)[E_1] + \left(1 - \frac{k+1}{kl-1}\right)[E_2].$$

Hence

$$\int_{M_\infty} |Ric_0|^2 dg_\infty = 8\pi^2 \left\{ k \left(1 - \frac{l+1}{kl-1} \right)^2 + l \left(1 - \frac{k+1}{kl-1} \right)^2 \right\} - 16\pi^2 \left(1 - \frac{l+1}{kl-1} \right) \left(1 - \frac{k+1}{kl-1} \right). \quad (5.2)$$

If $b_2(M_\infty) = 3$, there is a basis for $H_2(M_\infty, \mathbb{Z})$, represented by a triple of totally geodesic $\mathbb{C}\mathbb{P}^1$ s E_1, E_2, E_3 , in which the intersection form is given by

$$\begin{pmatrix} -i & 1 & 0 \\ 1 & -j & 1 \\ 0 & 1 & -k \end{pmatrix},$$

for some positive i, j, k . Moreover,

$$c_1 = \left(1 - \frac{jk}{ijk - i - k} \right) [E_1] + \left(1 - \frac{k+i}{ijk - i - k} \right) [E_2] + \left(1 - \frac{ij}{ijk - i - k} \right) [E_3].$$

Hence

$$\begin{aligned} \int_{M_\infty} |Ric_0|^2 dg_\infty = & 8\pi^2 \left\{ \left(1 - \frac{jk}{ijk - i - k} \right)^2 i + \left(1 - \frac{k+i}{ijk - i - k} \right)^2 j + \left(1 - \frac{ij}{ijk - i - k} \right)^2 k \right\} \\ & - 16\pi^2 \left(1 - \frac{k+i}{ijk - i - k} \right) \left(2 - \frac{j(i+k)}{ijk - i - k} \right). \end{aligned} \quad (5.3)$$

6 Proof of Theorem 1.2

It is clear from last section that if a maximal bubble is formed along the Calabi flow, it has to be very restrictive, in particular with toric assumption. One certainly expects that such a bubble cannot be formed along the Calabi flow, in particular for some special cases. This actually leads to the fact that the curvature cannot blow up along the Calabi flow. In the following we prove that the Sobolev constants are uniformly bounded and the bubble cannot be formed for the example in Theorem 1.2. The main tools to rule out bubbles are the Gauss-Bonnet formula and the signature formula. For a compact smooth 4-manifold (M, g) , the Gauss-Bonnet formula is

$$\frac{1}{8\pi^2} \int_M \left(|W_+|^2 + |W_-|^2 + \frac{R^2}{24} - \frac{|Ric_0|^2}{2} \right) dg = \chi(M),$$

and the signature formula is

$$\frac{1}{12\pi^2} \int_M (|W_+|^2 - |W_-|^2) dg = \tau(M).$$

Here R is the scalar curvature and Ric_0 is the traceless Ricci curvature. If M admits an orientation-compatible almost complex structure, then

$$c_1^2 = 2\chi + 3\tau.$$

The Kähler condition implies

$$\int_M |W_+|^2 dg = \int_M \frac{R^2}{24} dg.$$

Suppose $M = \mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$, we get

$$\int_M |Ric_0|^2 dg = \frac{1}{4} \int_M R^2 dg - 8\pi^2 c_1^2 = \frac{1}{4} \int_M R^2 dg - 48\pi^2, \quad (6.1)$$

and

$$\int_M |W_-|^2 dg = \frac{1}{24} \int_M R^2 dg + 24\pi^2. \quad (6.2)$$

If (M_∞, g_∞) is any ALE 4-manifold with finite group $\Gamma \subset SO(4)$ at infinity, then the Gauss-Bonnet formula becomes

$$\frac{1}{8\pi^2} \int_{M_\infty} \left(|W_+|^2 + |W_-|^2 + \frac{R^2}{24} - \frac{|Ric_0|^2}{2} \right) dg_\infty = \chi(M_\infty) - \frac{1}{|\Gamma|}$$

and the signature formula becomes

$$\frac{1}{12\pi^2} \int_{M_\infty} (|W_+|^2 - |W_-|^2) dg_\infty = \tau(M_\infty) + \eta(S^3/\Gamma),$$

where $\chi(M_\infty)$ is the Euler characteristic of non-compact manifold M_∞ and $\eta(S^3/\Gamma)$ is called η invariant. When (M_∞, g_∞) is scalar flat Kähler, the formulas can be reduced to

$$\frac{1}{8\pi^2} \int_{M_\infty} \left(|W_-|^2 - \frac{|Ric_0|^2}{2} \right) dg_\infty = \chi(M_\infty) - \frac{1}{|\Gamma|} \quad (6.3)$$

and

$$- \frac{1}{12\pi^2} \int_{M_\infty} |W_-|^2 d\mu_{g_\infty} = \tau(M_\infty) + \eta(S^3/\Gamma). \quad (6.4)$$

Note that the L^2 norm of curvature is scaling-invariant in 4 dimension. If (M_∞, g_∞) is a maximal bubble of (M, g) , we have

$$\int_M |Ric_0|^2 dg \geq \int_{M_\infty} |Ric_0|^2 dg_\infty, \quad (6.5)$$

and

$$\int_M |W_-|^2 dg \geq \int_{M_\infty} |W_-|^2 dg_\infty. \quad (6.6)$$

To prove Theorem 1.2, we first show that the Sobolev constants are uniformly bounded along the Calabi flow.

Lemma 6.1. *Let $[\omega] = 3H - 1/2([E_1] + [E_2] + [E_3])$ on $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$. Suppose the initial metric $\omega_0 \in [\omega]$ satisfies*

$$\int_M R^2 dg_0 < 32\pi^2(6 + 25/11),$$

then the Sobolev constants of $\omega(t)$ are uniformly bounded from above once the Calabi flow exists.

Proof. The proof is a generalization of Tian's idea for positive constant scalar curvature metrics on Kähler surfaces [34], [35]. Suppose g is a Kähler metric, the Yamabe constant for the conformal class $[g]$ takes the formula

$$Y_{[g]} = \inf_{u>0} \frac{\int_M (6|\nabla u|^2 + R_g u^2) dg}{(\int_M u^4 dg)^{1/2}}.$$

So for any $u > 0$,

$$Y_{[g]} \left(\int_M u^4 dg \right)^{1/2} \leq \int_M (6|\nabla u|^2 + R_g u^2) dg. \quad (6.7)$$

Note that the inequality (6.7) holds also for any u since $|\nabla|u|| \leq |\nabla u|$ at $u \neq 0$. A calculation [35, 10] shows that

$$Y_{[g]}^2 \geq 96\pi^2 c_1^2 - 2 \int_M R^2 dg. \quad (6.8)$$

If the Calabi energy satisfies

$$\int_M R^2 dg < 48\pi^2 c_1^2,$$

one can get a priori positive lower bound for $Y_{[g]}^2$. Once $Y_{[g]} > 0$ and R_g is bounded a priori, the Sobolev constant is bounded from above by (6.7) a priori. These two conditions can be checked directly for extremal metrics (or constant scalar curvature metrics), see examples in [10].

By a careful analysis, one can generalize this idea to the Calabi flow without any bound on scalar curvature. Suppose that $g_0 \in [\omega]_{1/2}$ on $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$ satisfies

$$\int_M R^2 dg_0 < 32\pi^2(6 + 25/11). \quad (6.9)$$

Let \underline{R} be the average of the scalar curvature and let V be the volume of M for any metric in $(M, [\omega])$, then

$$\underline{R}^2 V = 32\pi^2(75/11).$$

It is direct to check that (6.9) is equivalent to

$$\int_M R_0^2 dg_0 < 32\pi^2 c_1^2 + \frac{1}{3} \underline{R}^2 V,$$

which we can rewrite as

$$\int_M (R_{g_0} - \underline{R})^2 dg_0 < 96\pi^2 c_1^2 - 2 \int_M R_{g_0}^2 dg_0.$$

By (6.8), we get that

$$Y_{[g_0]}^2 > \int_M (R_{g_0} - \underline{R})^2 dg_0. \quad (6.10)$$

First we show that $Y_{[g_0]}$ has to be positive. Pick up a minimizing sequence u_i , and there exist positive $\epsilon_i \rightarrow 0$ such that

$$Y_{[g_0]} + \epsilon_i = \frac{\int_M (6|\nabla u_i|^2 + R_{g_0} u_i^2) dg_0}{(\int_M u_i^4 dg_0)^{1/2}}.$$

It follows that

$$(Y_{[g_0]} + \epsilon_i) |u_i|_{L^4}^2 = 6 \int_M |\nabla u_i|^2 dg_0 + \int_M R_{g_0} u_i^2 dg_0.$$

So we can get that

$$(Y_{[g_0]} + \epsilon_i) |u_i|_{L^4}^2 - \int_M (R_{g_0} - \underline{R}) u_i^2 dg_0 = 6 |\nabla u_i|_{L^2}^2 + \underline{R} |u_i|_{L^2}^2. \quad (6.11)$$

If $Y_{[g_0]}$ is negative, by (6.10), the left hand side of (6.11) is negative for some i sufficient large, while the right hand side is positive. Contradiction.

If $Y_{[g_0]}$ is positive, we can rewrite (6.7) as

$$|u|_{L^4}^2 - \frac{1}{Y_{[g_0]}} \int_M (R_{g_0} - \underline{R}) u^2 dg_0 \leq 6 Y_{[g_0]}^{-1} |\nabla u|_{L^2}^2 + \underline{R} Y_{[g_0]}^{-1} |u|_{L^2}^2.$$

By the Cauchy-Schwarz inequality, it follows that

$$|u|_{L^4}^2 \leq \frac{6}{Y_{[g_0]} - |R_{g_0} - \underline{R}|_{L^2}} |\nabla u|_{L^2}^2 + \frac{\underline{R}}{Y_{[g_0]} - |R_{g_0} - \underline{R}|_{L^2}} |u|_{L^2}^2.$$

The sobolev constant on a four dimensional Riemannian manifold is defined to be the smallest constant C_s such that

$$|u|_{L^4}^2 \leq C_s (|\nabla u|_{L^2}^2 + Vol^{-1/2} |u|_{L^2}^2).$$

Note that this definition is scaling invariant. In our case, we can compute that $\underline{R} \sqrt{Vol} = 20\pi \sqrt{6/11}$ for $[\omega]_{1/2}$ and it is scaling invariant. So we can get that

$$C_s \leq 20\pi \sqrt{6/11} (Y_{[g_0]} - |R_{g_0} - \underline{R}|_{L^2})^{-1}.$$

Since the Calabi energy is decreasing, the above estimate certainly holds along the Calabi flow. \square

Now we are ready to show the details to rule out bubbles.

Proof. Suppose a maximal bubble (M_∞, g_∞) is formed. First we examine the group action \mathbb{Z}_3 . Let M be the blown up of $\mathbb{C}\mathbb{P}^2$ at the points $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Then \mathbb{Z}_3 is generated by the permutation

$$P([z_0 : z_1 : z_2]) = [z_1 : z_2 : z_0].$$

Note that (M_∞, g_∞) arises by scaling the interior of a small domain U_j . Now move this domain by the action of \mathbb{Z}_3 to obtain the two domains U'_j, U''_j . If after scaling, the distance for U_j to U'_j (and so the distance U'_j to U''_j) is finite, then Cheeger-Gromov limit includes the limits of U'_j, U''_j . Then (M_∞, g_∞) is invariant under the induced \mathbb{Z}_3 action. Otherwise, three copies of (M_∞, g_∞) will form and therefore we have

$$3 \int_{M_\infty} |W_-|^2 dg_\infty < \int_M |W_-|^2 dg. \quad (6.12)$$

and

$$3 \int_{M_\infty} |Ric_0|^2 dg_\infty < \int_M |Ric_0|^2 dg. \quad (6.13)$$

In other words, if (6.12) or (6.13) is not satisfied, then (M_∞, g_∞) is invariant under the induced \mathbb{Z}_3 action. This symmetry will simplify the calculation a lot.

By Lemma 5.1, $b_2(M_\infty) \leq 3$ and (M_∞, g_∞) satisfies all constraints considered in Section 5. We analyze case by case.

Case 1: $b_2(M_\infty) = 1$. Then (M_∞, g_∞) contains a holomorphic $\mathbb{C}\mathbb{P}^1$ with self-intersection $-k$. Suppose (M_∞, g_∞) is not invariant under the induced action of \mathbb{Z}_3 . By (5.1) and (6.3), we get that

$$\int_{M_\infty} |Ric_0|^2 dg_\infty = 8\pi^2 \frac{(k-2)^2}{k} \quad (6.14)$$

and

$$\int_{M_\infty} |W_-|^2 dg_\infty = 4\pi^2 \frac{k^2 + 2}{k}. \quad (6.15)$$

By (6.2), (6.12) and (6.15), we get that

$$12\pi^2 \frac{k^2 + 2}{k} < \frac{1}{24} \int_M R^2 dg + 24\pi^2. \quad (6.16)$$

However there is no solution of (6.16) if the initial Calabi energy

$$\int_M R^2 dg < 258.9\pi^2.$$

It implies that (M_∞, g_∞) is invariant under the induced \mathbb{Z}_3 action. So (M_∞, g_∞) is diffeomorphic to a domain which is invariant under the action \mathbb{Z}_3 . It follows that M contains a sequence of smoothly embedded 2-sphere S_i of

self-intersection $-k$, which is invariant under \mathbb{Z}_3 . By (6.1), (6.5) and (6.14), we can get that

$$8\pi^2 \frac{(k-2)^2}{k} < \frac{1}{4} \int_M R^2 dg - 48\pi^2.$$

It follows that $1 \leq k \leq 5$. Express the homology class $[S_i]$ as

$$[S_i] = mH - n(E_1 + E_2 + E_3).$$

The Lagrangian condition gives that

$$3m - 3n/2 \rightarrow 0, \tag{6.17}$$

while the self-intersection condition gives that

$$m^2 - 3n^2 = -k.$$

Suppose that m, n are nonnegative, then we have $m < n$. It follows that $2n^2 < k \leq 5$. We get that $n = 1, m = 0$. Contradiction with (6.17).

Case 2: $b_2(M_\infty) = 2$. If $|\Gamma| = 1$, $H_2(M_\infty, \mathbb{Z})$ has intersection form

$$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}.$$

By (5.2) (with $k = 2, l = 1$) and (6.3), we compute

$$\int_{M_\infty} |Ric_0|^2 dg_\infty = 16\pi^2,$$

and

$$\int_{M_\infty} |W_-|^2 dg_\infty = 24\pi^2.$$

If $|\Gamma| \geq 2$, by (6.3), we compute

$$\int_{M_\infty} |W_-|^2 dg_\infty \geq 20\pi^2.$$

In either case, (6.12) does not hold and so (M_∞, g_∞) is invariant under the induced \mathbb{Z}_3 action. So M contains a sequence of two embedded 2 spheres S_i, \tilde{S}_i which are both invariant under the action of \mathbb{Z}_3 with the intersection form

$$\begin{pmatrix} -k & 1 \\ 1 & -l \end{pmatrix}.$$

We can express the homology classes $[S_i], [\tilde{S}_i]$ as

$$[S_i] = m_1H - n_1(E_1 + E_2 + E_3),$$

and

$$[\tilde{S}_i] = m_2 H - n_2(E_1 + E_2 + E_3).$$

The Lagrangian condition gives that (for S_i)

$$3m_1 - 3n_1/2 \rightarrow 0. \quad (6.18)$$

The self-intersection condition gives that (for S_i)

$$m_1^2 - 3n_1^2 = -k. \quad (6.19)$$

Note that if (m_1, n_1) is a solution, $(-m_1, -n_1)$ is also a solution. So we can assume that $m_1 > 0$ since m_1 cannot be zero. By (6.18), $n_1 \geq m_1 + 1$. By (6.19) we get that $k = 3n_1^2 - m_1^2 \geq 3 \times 2^2 - 1 = 11$. Similarly $l \geq 11$.

However, by (6.5) and (5.2), we get that

$$k \left(1 - \frac{l+1}{kl-1}\right)^2 + l \left(1 - \frac{k+1}{kl-1}\right)^2 - 2 \left(1 - \frac{l+1}{kl-1}\right) \left(1 - \frac{k+1}{kl-1}\right) < 2.1. \quad (6.20)$$

For $k, l \geq 11$, (6.20) cannot hold.

Case 3: $b_2(M_\infty) = 3$. By (6.3), we get that

$$\int_{M_\infty} |W_-|^2 dg_\infty \geq 24\pi^2.$$

By the same argument in Case 2, (M_∞, g_∞) is invariant under the induced action of \mathbb{Z}_3 . It implies that M contains a sequence of three embedded 2-sphere S_{1a}, S_{2a}, S_{3a} which are invariant under the action of \mathbb{Z}_3 . Suppose the intersection form is given by

$$\begin{pmatrix} -i & 1 & 0 \\ 1 & -j & 1 \\ 0 & 1 & -k \end{pmatrix},$$

then the same argument in Case 2 implies that $i, j, k \geq 11$. However by (6.5) and (5.3),

$$\begin{aligned} & \left(1 - \frac{jk}{ijk - i - k}\right)^2 i + \left(1 - \frac{k+i}{ijk - i - k}\right)^2 j + \left(1 - \frac{ij}{ijk - i - k}\right)^2 k \\ & - 2 \left(1 - \frac{k+i}{ijk - i - k}\right) \left(2 - \frac{j(i+k)}{ijk - i - k}\right) < 2.1. \end{aligned} \quad (6.21)$$

For $i, j, k \geq 11$, (6.21) cannot hold.

By the case by case analysis, (M_∞, g_∞) cannot be formed. Then the curvature cannot blow up along the Calabi flow. It follows that the flow exists for all time and converges to an extremal metric in Cheeger-Gromov sense.

Namely for any $t_i \rightarrow \infty$, there is a sequence of diffeomorphism Ψ_i such that $(M, \Psi_i^* g_i = g(t_i), \Psi_{i*} J)$ converge smoothly to (M, g_∞, J_∞) , where g_∞ is extremal with respect to J_∞ . But (M, J_∞) might not be bi-holomorphic to (M, J) .

We then finish our proof by showing (M, J) is actually biholomorphic to (M, J_∞) . In general this is a very hard problem. The proof follows from [10] (Theorem 27) by using the toric condition carefully and the classification of complex surface. Theorem 27 in [10] states only for $M \sim \mathbb{C}\mathbb{P}^2 \# 2\overline{\mathbb{C}\mathbb{P}^2}$ but the proof holds for all toric Fano surfaces. The key is that in the limiting process, the torus action converges and (M, g_∞, J_∞) is still toric. Moreover, the 2-torus action for (M, g_∞, J_∞) is holomorphic with respect to J_∞ . The readers can refer to [10] for details. We shall sketch a proof as follows. When $M \sim \mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$, each of holomorphic curves H, E_1, E_2, E_3 is the fixed point set of the isometric action of some circle action of 2-torus, and so each is totally geodesic with respect to the metrics along the Calabi flow. By looking at the corresponding fixed points set of the limit action of circle subgroups, we can find corresponding totally geodesic 2-sphere in (M, g_∞, J_∞) which are the limits of the image of these submanifolds. Moreover, these limit 2-spheres are holomorphic with respect to J_∞ and the homological intersection numbers of these holomorphic spheres do not vary. Namely, we have still three holomorphic $\mathbb{C}\mathbb{P}^1$ s with self-intersection -1 as the images of the original exceptional divisors E_1, E_2, E_3 . Thus, by blowing down the images of E_1, E_2, E_3 and applying the classification of the complex surface, we conclude that (M, J_∞) is biholomorphic to $\mathbb{C}\mathbb{P}^2$ blowup three generic points. So there exists a diffeomorphism Ψ such that $\Psi_* J = J_\infty$. So $\Psi^* g_\infty$ is an extremal metric in the class $[\omega]$ for (M, J) . \square

7 Appendix

If (M, g, J) is an extK metric on a compact complex surface, the Calabi functional takes the value

$$\begin{aligned} \mathcal{C}(g) &= \int_M R^2 dg \\ &= \underline{R}^2 \int_M dg + \int_M (R - \underline{R})^2 dg \\ &= 32\pi^2 \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \|\mathcal{F}\|^2, \end{aligned}$$

where $\|\mathcal{F}\|$ is the norm of the Calabi-Futaki invariant of a Mabuchi-Futaki invariant metric [20], see [8] for the definition of this norm. We define a functional $\mathcal{A}([\omega])$ depending only the Kähler class

$$\mathcal{A}([\omega]) = 32\pi^2 \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} + \|\mathcal{F}\|^2. \quad (7.1)$$

According to Chen [8], Donaldson [17], $\mathcal{A}([\omega])$ provides a natural lower bound of the Calabi functional and it is realized by an extK metric if there exists such a metric in the class $[\omega]$.

A key ingredient of our result is to bound the Sobolev constant along the Calabi flow. In [34], [35], Tian made a shrewd observation that one can bound the Sobolev constants a priori for positive cscK metrics on Kähler surfaces with $c_1 > 0$ if the Kähler class satisfies

$$c_1^2(M) - \frac{2}{3} \frac{(c_1(M) \cdot [\omega])^2}{[\omega]^2} > 0. \quad (7.2)$$

This observation was generalized to extK metrics on Kähler surfaces with $c_1 > 0$ [12] if the Kähler class satisfies the condition

$$48\pi^2 c_1^2(M) - \mathcal{A}([\omega]) > 0. \quad (7.3)$$

Note that if the Futaki invariant is zero ($\|\mathcal{F}\| = 0$), (7.3) is reduced to (7.2). We define Tian's cone to be the set of Kähler classes which satisfy (7.2) and the generalized Tian's cone to be the set of Kähler classes which satisfy (7.3).

To generalize these ideas to the Calabi flow, we need that the Calabi energy is bounded by

$$\int_M R^2 dg < \mathcal{B}([\omega]), \quad (7.4)$$

where the functional $\mathcal{B}([\omega])$ is defined to be

$$\mathcal{B}([\omega]) = 32\pi^2 \left(c_1^2 + \frac{1}{3} \frac{(c_1 \cdot [\omega])^2}{[\omega]^2} \right) + \frac{1}{3} \|\mathcal{F}\|^2. \quad (7.5)$$

Note that the Calabi functional has a natural lower bound $\mathcal{A}([\omega])$. If (7.4) holds, we have

$$\mathcal{A}([\omega]) < \mathcal{B}([\omega]).$$

It is equivalent to

$$48\pi^2 c_1^2 > \mathcal{A}([\omega]),$$

which is exactly the generalized Tian's cone condition.

Consider the Kähler classes $[\omega]_x = 3H - x(E_1 + E_2 + E_3)$ on $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$. It is not hard to check that $[\omega]_x$ satisfies the generalized Tian's cone condition for any $x \in (0, 3/2)$. Also one can compute

$$\mathcal{B}([\omega]_x) = 192\pi^2 + 32\pi^2 \frac{(3-x)^2}{3-x^2}.$$

We can actually prove similar results as in Theorem 1.2.

Theorem 7.1. *Let $[\omega]_x = 3H - x([E_1] + [E_2] + [E_3])$ on $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$. Suppose the initial metric $\omega_0 \in [\omega]$ is invariant under the toric action and the action of \mathbb{Z}_3 and*

$$\int_M R^2 dg_0 < \mathcal{B}([\omega]_x),$$

then the Calabi flow exists for all time with uniformly bounded curvature tensor. Moreover, it converges to a cscK metric in the class $[\omega]_x$ in the Cheeger-Gromov sense.

The strategy is similar as in the proof of Theorem 1.2. All details can be found in the old version of our paper. Note when $x = 1$, the existence of Kähler-Einstein metric was shown by Siu [31] and Tian-Yau [36]. Recently Arezzo-Pacard [2] considered the existence of cscK metrics on blown ups. On $\mathbb{C}\mathbb{P}^2 \# 3\overline{\mathbb{C}\mathbb{P}^2}$, their results imply the existence of cscK metrics in $[\omega]_x$ for x sufficiently close to 0 or $3/2$.

References

- [1] V. Apostolov, D. Calderbank, P. Gauduchon, C. Tonnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry, III Extremal metrics and stability*, arXiv:math/0511118.
- [2] C. Arezzo, F. Pacard, *Blowing up and desingularizing constant scalar curvature Kähler manifolds*, arXiv:math/0411522.
- [3] E. Calabi, *Extremal Kähler metric*, in *Seminar of Differential Geometry*, ed. S. T. Yau, Annals of Mathematics Studies **102**, Princeton University Press (1982), 259-290.
- [4] E. Calabi, *Extremal Kähler metric, II*, in *Differential Geometry and Complex Analysis*, eds. I. Chavel and H. M. Farkas, Springer Verlag (1985), 95-114.
- [5] E. Calabi, X. X. Chen, *The Space of Kähler metrics II*, J. Differential Geom. **61** (2002), no. 2, 173–193.
- [6] D. Calderbank, M. Singer, *Toric self-dual Einstein metrics on compact orbifolds*, Duke Math. J. **133** (2006), no. 2, 237–258.
- [7] X. X. Chen, *Calabi flow in Riemann surfaces revisited*, IMRN, **6**(2001), 275-297.
- [8] X. X. Chen, *Space of Kähler metrics III—On the lower bound of the Calabi energy and geodesic distance*, arXiv:math/0606228.
- [9] X. X. Chen, W. Y. He, *On the Calabi flow*, Amer. J. Math. **130** (2008), no. 2, 539–570.
- [10] X. X. Chen, C. LeBrun, and B. Weber, *On Einstein Conformally Kähler Metrics*, arXiv.0715.0710.
- [11] X. X. Chen, G. Tian, *Geometry of Kähler Metrics and Foliations by Holomorphic Discs*, arXiv:math/0507148.
- [12] X. X. Chen, B. Weber, *Moduli Spaces of critical Riemannian Metrics with $L^{\frac{n}{2}}$ norm curvature bounds*. Preprint, 2006.
- [13] P. T. Chrusciel, *Semi-global existence and convergence of solutions of the Robison-Trautman(2-dimensional Calabi) equation*, Comm.Math.Phys. **137** (1991), 289-313.

- [14] S. Donaldson, *Conjectures in Kähler geometry*, Strings and geometry, 71–78, Clay Math. Proc., **3**, Amer. Math. Soc., Providence, RI, 2004.
- [15] S. Donaldson, Scalar curvature and stability of toric varieties, J. Differential Geom. 62 (2002), no. 2, 289–349.
- [16] S. Donaldson, *Extremal metrics on toric surfaces, I*, arXiv:math/0612120.
- [17] S. Donaldson, *Lower bounds on the Calabi functional*, J. Differential Geom. 70 (2005), no. 3, 453–472.
- [18] J. Fine, *Constant scalar curvature Kähler metrics on fibred complex surfaces*, J. Differential Geom. 68 (2004), no. 3, 397–432.
- [19] A. Futaki, *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. 73 (1983), no. 3, 437–443.
- [20] A. Futaki, T. Mabuchi, *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. 301 (1995), no. 2, 199–210.
- [21] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17(2) (1982).
- [22] R. Hamilton, *A compactness property for solutions of the Ricci flow*. Amer. J. Math. 117 (1995), no. 3, 545–572.
- [23] D. Joyce, *Explicit construction of self-dual 4-manifolds*, Duke Math. J. 77 (1995), no. 3, 519–552.
- [24] P. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differential Geom. 29 (1989), no. 3, 665–683.
- [25] P. Kronheimer, *A Torelli-type theorem for gravitational instantons*, J. Differential Geom. 29 (1989), no. 3, 685–697.
- [26] C. LeBrun, *Twistors, Kähler manifolds, and bimeromorphic geometry I*, J. Amer. Math. Soc. 5 (1992), no. 2, 289–316.
- [27] C. LeBrun, *Explicit self-dual metrics on $CP_2 \# \cdots \# CP_2$* , J. Differential Geom. 34 (1991), no. 1, 223–253.
- [28] C. LeBrun, S. R. Simanca, *Extremal Kähler metrics and Complex Deformation Theory*, Geom. and Fun. Analysis, Vol. 4, No. 3 (1994), 298–336.
- [29] P. Li, L. Tam, *Harmonic functions and the structure of complete manifolds*, J. Differential Geom. 35 (1992), no. 2, 359–383.
- [30] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159.

- [31] Y. T. Siu, *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. of Math. (2) 127 (1988), no. 3, 585–627.
- [32] M. Struwe, *Curvature flows on surfaces*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 1 (2002), no. 2, 247–274.
- [33] G. Székelyhidi, *The Calabi functional on a ruled surface*, arXiv:math/0703562.
- [34] G. Tian, unpublished note.
- [35] G. Tian, J. Viaclovsky, *Moduli spaces of critical metrics in dimension four*, Adv. Math., 196 (2005), 346–372.
- [36] G. Tian, S. T. Yau, *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$* , Comm. Math. Phys. 112 (1987), no. 1, 175–203.
- [37] V. Tosatti, B. Weinkove, *The Calabi flow with small initial energy*, Math. Res. Lett. 14 (2007), no. 6, 1033–1039.
- [38] R. Ye, *The logarithmic Sobolev inequality along the Ricci flow*, arXiv:0707.2424.

Xiuxiong CHEN
 xxchen@math.wisc.edu
 Department of Mathematics
 University of Wisconsin-Madison

Weiyong HE
 whe@math.ubc.ca
 Department of Mathematics
 University of British Columbia
 Current address:
 whe@uoregon.edu
 Department of Mathematics
 University of Oregon