Kähler Ricci Flow on Fano Surfaces (I)

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Abstract

We show the properties of the blowup limits of Kähler Ricci flow solutions on Fano surfaces if Riemannian curvature is unbounded. As an application, on every toric Fano surface, we prove that Kähler Ricci flow converges to a Kähler Ricci soliton metric if the initial metric has toric symmetry. Therefore we give a new Ricci flow proof of existence of Kähler Ricci soliton metrics on toric surfaces.

1 Introduction

This is the first part of our study of Kähler Ricci flow on Fano surfaces. In this note, we study the convergence of Kähler Ricci flow on Fano surfaces if Riemannian curvature is uniformly bounded and we discuss the methods to obtain the Riemannian curvature bound.

In [Ha1], Hamilton defined Ricci flow and applied it to prove that every simply connected 3-manifold with positive Ricci curvature metric admits a constant curvature metric, hence it is diffeomorphic to S^3. From then on, Ricci flow became a powerful tool to search Einstein metrics on manifolds. If the underlying manifold is a Kähler manifold whose first Chern class has definite sign, then the normalized Ricci flow is called the Kähler Ricci flow. It was proved by Cao [Cao85], who followed Yau's fundamental estimates, that Kähler Ricci flow always exists globally. If the first Chern class of the underlying manifold is negative or null, Cao showed that Kähler Ricci flow will converge to a Kähler Einstein (KE) metric. If the first Chern class of the underlying manifold is positive, then the convergence is much harder and still not clearly now. The first breakthrough in this direction is the work of [CT1] and [CT2]. There Chen and Tian showed that the Kähler Ricci flow converges to a KE metric if the initial metric has positive bisectional curvature. Around 2002, Perelman made some fundamental estimates along Kähler Ricci flow, he showed that scalar curvatures, diameters and normalized Ricci potentials are all uniformly bounded along every Kähler Ricci flow. Together with his no-local-collapsing theorem, these estimates give us a lot of information. After Perelman's fundamental estimates, there are numerous works concerning the convergence of Kähler Ricci flow. Due

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to the limited knowledge of authors, we'll not give a complete list of all the contributors.

In this note, we only consider the convergence of Kähler Ricci flow on Fano surfaces. If we assume Riemannian curvature is uniformly bounded along the flow, then we can show that this flow converges to a Kähler Ricci soliton (KRS) metric in the same complex structure.

Theorem 1. *Suppose $\{(M, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution on a Fano surface M, J is the complex structure compatible with $(M, g(t))$. If Riemannian curvature is uniformly bounded along this flow, then for every sequence $t_i \to \infty$, there is a subsequence of times t_{i_k} and diffeomorphisms $\Phi_{i_k} : M \to M$ such that*

$$\Phi_{i_k}^* g(t_{i_k}) \xrightarrow{C^\infty} h, \quad (\Phi_{i_k})_*^{-1} \circ J \circ (\Phi_{i_k})_* \xrightarrow{C^\infty} J.$$

under a fixed gauge. Here h is a Kähler metric compatible with complex structure J and (M, h) is a Kähler Ricci soliton metric, i.e., there exists a smooth function f on M such that

$$Ric_h - h = \mathscr{L}_{\nabla f} h.$$

In general dimension Kähler Ricci flow, even if Riemannian curvature is uniformly bounded, we can only obtain $(\Phi_{i_k})_*^{-1} \circ J \circ (\Phi_{i_k})_* \xrightarrow{C^\infty} \tilde{J}$ for some complex structure $\tilde{J}$ compatible with h. $\tilde{J}$ may be different from J. The reason that we can obtain the same J here is that we are dealing with Fano surfaces now. The classification of Fano surfaces gives us a lot of information about the limit complex structures.

In order to set up a uniform Riemannian curvature bound, we observe that there are actually some topological and geometric obstructions for Riemannian curvature to be unbounded along the flow. Because if the Riemannian curvature is not uniformly bounded, we can blowup the flow at maximal Riemannian curvature points. We denote such blowup limits as "deepest bubbles" and find that they satisfy strong topological and geometric conditions.

Theorem 2. *Suppose $\{(M, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution on a Fano surface M. If Riemannian curvature is not uniformly bounded, then every deepest bubble X_∞ is a finite quotient of a hyper Kähler ALE manifold. Moreover, X_∞ doesn't contain any compact divisor.*

Part of this theorem was obtained independently in [RZZ].

On one hand, X_∞ has particular topology and geometry. On the other hand, all the topology and geometry of X_∞ come out from the underlying manifolds. Therefore, on a Fano surface with simple topology and geometry, it's plausible that the Riemannian curvature is uniformly bounded. Actually, we can prove the following theorem.

Theorem 3. *Suppose* $\{(M, g(t)), 0 \le t < \infty\}$ *is a Kähler Ricci flow solution on a toric Fano surface. If the initial metric is toric symmetric, then the Riemannian curvature is uniformly bounded along this flow.*

This result is not new. In [Zhu], Zhu showed that staring from any toric symmetric metric, Kähler Ricci flow will converge to a KRS metric on every toric manifold. He used complex Monge-Ampere equation theory and reduce the convergence of Kähler Ricci flow to the control of C^0-norm of Ricci potential functions on M. Using the toric condition, he is able to obtain the required C^0-estimates. Since Theorem 3 only concerns toric Fano surfaces, it's not surprising that our proof is simpler and more geometric. In fact, toric Fano surfaces with toric symmetric metrics admit almost the simplest topology and geometry one can imagine for Fano surfaces. Because of this simplicity, the formation of deepest bubbles is prevent, therefore the Riemannian curvature must be bounded along the flow. This ruling-out-bubble idea originates from [CLW]. However, in [CLW], explicit energy bounds are calculated to exclude deepest bubbles. This calculation is avoided in our cases and we can exclude deepest bubbles directly by topological and geometric obstructions.

As the combination of Theorem 1 and Theorem 3, we know every toric Fano surface admits a KRS metric. Recall that a KRS metric becomes a KE metric if and only if the Futaki invariant of the underlying manifold vanishes. By the classification of Fano surfaces, every toric Fano surface must be one of the following types: $\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^2, \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 (1 \le k \le 3)$. Only $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ have nonvanishing Futaki invariants. Therefore, we have

Corollary 1. *There exist nontrivial KRS metrics on* $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ *and* $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$. *There exist KE metrics on* $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$ *and* $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$.

This result is well known although our proof is new. The existence of KRS on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ was first proved by Koiso ([Ko]). The existence of KE metric on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ was first proved by Tian and Yau([TY]). For a general toric manifold, the existence of KRS metric was proved by X. Wang and Zhu ([WZ]).

The organization of this note is as follows: In section 2, we setup the basic notations. Then we prove Theorem 1, Theorem 2 and Theorem 3 in section 3, section 4 and section 5 respectively.

2 Setup of notations

Let (M, g, J) be an n-dimensional compact complex manifold with Riemannian metric g, ω be the form compatible with g and J. (M, g, J) is Kähler if and only if $\nabla J \equiv 0$. This holds if and only if ω is a positive closed $(1,1)$-form. We'll only study Kähler manifold in this note. In local complex coordinates $\{z_1, \cdots, z_n\}$, the metric form ω is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} d z^i \wedge d z^{\bar{j}} > 0,$$

where $\{g_{i\bar{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω is a closed positive $(1,1)$-form. Given a Kähler metric ω, the curvature tensor is

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial z^{\bar{l}}} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}, \qquad \forall\, i, j, k, l = 1, 2, \cdots n.$$

The Ricci curvature form is

$$\mathrm{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{i\bar{j}}(\omega) d z^i \wedge d z^{\bar{j}} = -\sqrt{-1} \partial \bar{\partial} \log \, \det(g_{k\bar{l}}).$$

It is a real, closed $(1,1)$-form. Recall that $[\omega]$ is called a canonical Kähler class if this Ricci form is cohomologous to ω, , i.e., $[Ric] = [\omega]$.

Now we assume that the first Chern class $c_1(M)$ is positive. The normalized Ricci flow equation(c.f. [Cao85]) in the canonical class of M is

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \qquad \forall\, i,\, j = 1, 2, \cdots, n. \tag{1}$$

It follows that on the level of Kähler potentials, the flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi{}^n}{\omega^n} + \varphi - h_\omega, \tag{2}$$
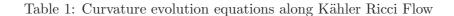
where h_ω is defined by

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_\omega, \text{ and } \int_X (e^{h_\omega} - 1) \omega^n = 0.$$

Along Kähler Ricci Flow , the evolution equations of curvatures are listed in Table 1.

One shall note that the Laplacian operator appears in the above formulae is the Laplacian-Beltrami operator on functions. As usual, the flow equation (1) or (2) is referred as the Kähler Ricci flow on M. It is proved by Cao [Cao85], who followed Yau's celebrated work [Yau78], that the Kähler Ricci flow exists globally for any smooth initial Kähler metric.

In his unpublished work, Perelman obtained some deep estimates along Kähler Ricci flow on Fano manifolds. The detailed proof can be found in Sesum and Tian's note (**author?**) [ST].

$$
\begin{aligned}
\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} =& \triangle R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}} R_{q\bar{p}k\bar{j}} + R_{i\bar{j}k\bar{l}} \\
& - \frac{1}{2} \left(R_{i\bar{p}} R_{p\bar{j}k\bar{l}} + R_{p\bar{j}} R_{i\bar{p}k\bar{l}} + R_{k\bar{p}} R_{i\bar{j}p\bar{l}} + R_{p\bar{l}} R_{i\bar{j}k\bar{p}} \right). \\
\frac{\partial}{\partial t} R_{i\bar{j}} =& \triangle R_{i\bar{j}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}} - R_{i\bar{p}} R_{p\bar{j}}. \\
\frac{\partial}{\partial t} R =& \triangle R + R_{i\bar{j}} R_{j\bar{i}} - R.
\end{aligned}
$$

Table 1: Curvature evolution equations along Kähler Ricci Flow

Proposition 2.1 (Perelman, c.f. [ST]). *Suppose $\{(M^n, g(t)), 0 \leq t < \infty\}$ is a Kähler Ricci flow solution. There are two positive constants D, κ depending only on this flow such that the following two estimates hold.*

1. *Let R_{g_t} be the scalar curvature under metric g_t, $h_{\omega_{\varphi(t)}}$ be the Ricci potential of form $\omega_{\varphi(t)}$ satisfying $\frac{1}{V} \int_M e^{h_{\omega_{\varphi(t)}}} \omega_{\varphi(t)}^n = 1$. Then we have*

$$
\|R_{g_t}\|_{C^0} + \operatorname{diam}_{g_t} M + \|h_{\omega_{\varphi(t)}}\|_{C^0} + \|\nabla h_{\omega_{\varphi(t)}}\|_{C^0} < D.
$$

2. $\dfrac{\operatorname{Vol}(B_{g_t}(x,r))}{r^{2n}} > \kappa$ *for every $r \in (0,1)$, $(x,t) \in M \times [0, \infty)$.*

These fundamental estimates will be used essentially in our arguments.

3 Convergence of Kähler Ricci flow on Fano Surfaces if Riemannian Curvature is Uniformly Bounded

In this section, we will prove Theorem 1.

Proof. Since the Riemannian curvature is uniformly bounded, diameter is uniformly bounded. By the compactness theorem of Ricci flow, we know that for every sequence of times $t_i \to \infty$, there is a subsequence $t_{i_k} \to \infty$ such that

$$
\{(M, g(t + t_{i_k})), -t_{i_k} < t \leq 0\} \longrightarrow \{(\hat{M}, \hat{g}(t)), -\infty < t \leq 0\}
$$

in Cheeger-Gromov sense. In particular, $(M, g(t_{i_k}))$ converges to $(\hat{M}, \hat{g}(0))$ in Cheeger-Gromov sense, i.e., there are diffeomorphisms $\varphi_{i_k} : \hat{M} \to M$ such that

$$
\varphi_{i_k}^*(g(t_{i_k})) \xrightarrow{C^\infty} \hat{g}(0).
$$

For the simplicity of notations, we will rewrite this as $\varphi_i^*(g(t_i)) \xrightarrow{C^\infty} \hat{g}$. Note that on $\hat{M}$, complex structure $J_i = (\varphi_i)_*^{-1} \circ J \circ (\varphi_i)_*$ is compatible with $\varphi_i^* g(t_i)$ and it satisfies

$\nabla_{\varphi_i^* g(t_i)} J_i \equiv 0$. Taking limit, we have $J_i \to \hat{J}$ and $\nabla_{\hat{g}} \hat{J} \equiv 0$. This means that $(\hat{M}, \hat{g}, \hat{J})$ is a Kähler manifold. By the monotonicity of Perelman's μ-functional, we can argue that $(\hat{M}, \hat{g})$ is a KRS as Natasa Sesum did in [Se]. On a KRS, $Ric_{\hat{g}} - \hat{g} = \mathscr{L}_{\nabla f} \hat{g}$ for some smooth function f. Clearly, $[\hat{R}_{i\bar{j}}] = [\hat{g}_{i\bar{j}}] > 0$ and it implies that

$$2\pi c_1(\hat{M})^2 = \text{Vol}_{\hat{g}}(\hat{M}) = \lim_{i \to \infty} \text{Vol}_{g(t_i)}(M) = 2\pi c_1^2(M).$$

It follows that $(\hat{M}, \hat{g})$ is a Fano surface satisfying $c_1^2(\hat{M}) = c_1^2(M)$ and $\hat{M}$ is diffeomorphic to M.

By classification of Fano surfaces, every Fano surface M must satisfy $1 \leq c_1^2(M) \leq 9$. Now we discuss cases by the values of $c_1^2(M)$.

1. $9 \geq c_1^2(M) = c_1^2(\hat{M}) \geq 5$. In this case, M is diffeomorphic to one of $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{CP}^2$ or $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 (1 \leq k \leq 4)$. By classification of Fano surfaces, under each of such diffeomorphic structure, there is only one complex structure such that M becomes Fano. Let ψ be a diffeomorphism map $\psi : M \to \hat{M}$. Then both (M, J) and $(M, (\psi)_*^{-1} \circ \hat{J} \circ \psi_*)$ are Fano manifolds with $9 \geq c_1^2(M) \geq 5$. Therefore, we have $J = (\psi)_*^{-1} \circ \hat{J} \circ \psi_*$. Recall that we have

$$\varphi_i^*(g(t_i)) \xrightarrow{C^\infty} \hat{g}, \quad (\varphi_i)_*^{-1} \circ J \circ (\varphi_i)_* \xrightarrow{C^\infty} \hat{J}.$$

It follows that

$$(\varphi_i \circ \psi)^*(g(t_i)) \xrightarrow{C^\infty} h = \psi^* \hat{g}, \quad (\varphi_i \circ \psi)_*^{-1} \circ J \circ (\varphi_i \circ \psi) \xrightarrow{C^\infty} J = (\psi)_*^{-1} \circ \hat{J} \circ (\psi)_*.$$

Let $\Psi_i = \varphi_i \circ \psi$, $h = \psi^* \hat{g}$, we finish the proof of Theorem 1 whenever $9 \geq c_1^2(M) \geq 5$.

2. $4 \geq c_1^2(M) = c_1^2(\hat{M}) \geq 1$. In this case, M is diffeomorphic to one of $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 (5 \leq k \leq 8)$. Under each of such diffeomorphic structure, there are a lot of complex structures $\tilde{J}$ such that every $(M, \tilde{J})$ is a Fano surface. Therefore it is possible that $J \neq (\psi)_*^{-1} \circ \hat{J} \circ (\psi)_*$ and the previous argument fails. However, in this case, $\hat{M}$ doesn't admit any holomorphic vector fields. This forces the KRS metric on $\hat{M}$ to be a KE metric. Moreover, the first eigenvalue of $(\hat{M}, \hat{g})$ is strictly greater than 1 (c.f. Lemma6.3 of [CT1]). Note that first eigenvalues converge as the Riemannian manifolds converge in Cheeeger-Gromov sense, a contradiction argument shows that the first eigenvalue of $(M, g(t))$ is strictly greater than 1, i.e., there is a positive constant δ such that

$$\lambda_1(-\triangle_{g(t)}) \geq 1 + \delta, \quad \forall \, t \in [0, \infty).$$

Then using the first part of the proof of Proposition 10.2 of [CT1], we are able to prove that

$$\frac{1}{V} \int_M (\dot{\varphi} - c(t))^2 \omega_\varphi^2 < C_\alpha e^{-\alpha t}$$

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where $c(t) = \frac{1}{V}\int_M \dot\varphi \omega_\varphi^2$, α is some positive number. As Riemannian curvature is uniformly bounded, we have a uniform Sobolev constant along this flow. As in [CT1], a parabolic Moser iteration then implies that $\dot\varphi$ is exponentially decaying, i.e.,

$$\dot\varphi \le C_1 e^{-\alpha t}.$$

The right hand side is an integrable function on $[0, \infty)$. It follows that φ is uniformly bounded along the flow. By virtue of Yau's estimate, we can show that all higher derivatives of φ are uniformly bounded in a fixed gauge. In particular, for every $t_i \to \infty$, we have subsequence t_{i_k} such that

$$\omega + \sqrt{-1}\varphi_{\alpha\bar\beta}(t_{i_k}) \to \omega + \sqrt{-1}\varphi_{\alpha\bar\beta}(t_\infty)$$

for some smooth function $\varphi(t_\infty)$ and $(M, \omega + \sqrt{-1}\varphi_{\alpha\bar\beta}(t_\infty))$ is a KE metric. This argument is standard. Readers are referred to [CT1], [CT2], [PSSW1], [PSSW2], [CW] for more details.

Let $\Psi_i \equiv id, h = \hat g$. We prove Theorem 1 whenever $4 \ge c_1^2(M) \ge 1$.

$\square$

4 Properties of Deepest Bubbles

In this section, we discuss the behavior of Kähler Ricci flow on Fano surfaces at maximal curvature points whenever the Riemannian curvature is not uniformly bounded along the flow. As a corollary, we prove Theorem 2.

Let $\{(M, g(t)), 0 \le t < \infty\}$ be a Kähler Ricci flow solution on a Fano surface M. If the Riemannian curvature is not uniformly bounded, then there is a sequence of points (x_i, t_i) satisfying $Q_i = |Rm|_{g(t_i)}(x_i) \to \infty$ and

$$|Rm|_{g(t)}(x) \le Q_i, \ \forall \ (x, t) \in M \times [0, t_i].$$

After rescaling, we have

$$|Rm|_{g_i(t)}(x) \le 1, \ \forall \ (x, t) \in M \times [-Q_i t_i, 0],$$

where $g_i(t) = Q_i g(Q_i^{-1} t + t_i)$. Shi's estimate([Shi1], [Shi2]) along Ricci flow implies that all the derivatives of Riemannian curvature are uniformly bounded. Therefore, we have the convergence

$$\{(M, x_i, g_i(t)), -\infty < t \le 0\} \overset{C^\infty}{\to} \{(X, x_\infty, g_\infty(t)), -\infty < t \le 0\}.$$

Here C^∞ means that this convergence is in the Cheeger-Gromov sense. In particular, we have $(M, x_i, g_i(0)) \overset{C^\infty}{\to} (X, x_\infty, g_\infty(0))$.

Every $g_i(t)$ satisfies the following conditions

$$\frac{\partial g_i}{\partial t} = Q_i^{-1} g_i(t) - Ric_{g_i(t)}, \quad \sup_{M \times [-Q_i t_i, 0]} |R_{g_i(t)}(x)| \leq CQ_i^{-1}.$$

So the limit flow $\{(X, x_\infty, g_\infty(t)), -\infty < t \leq 0\}$ satisfies the equations

$$\frac{\partial g_\infty}{\partial t} = -Ric_{g_\infty(t)}, \quad R_{g_\infty(t)} \equiv 0.$$

It is an unnormalized Ricci flow solution, so the scalar curvature satisfies the equation $\triangle R = \frac{1}{2}\triangle R + |Ric|^2$. It forces that the limit solution is Ricci flat.

As every manifold $(M, g_i(0))$ is a Kähler manifold with complex structure J satisfying $\nabla_{g_i(0)} J = 0$, there is a limit complex structure J_∞ such that $\nabla_{g_\infty(0)} J_\infty = 0$. Therefore, $(X, g_\infty(0), J_\infty)$ is a Ricci flat Kähler manifold.

For simplicity of notation, we denote g_∞ as $g_\infty(0)$.

Definition 4.1. *We call such a limit (X, g_∞, J_∞) as a deepest bubble.*

Since Riemannian curvature's L^2 norm is a rescaling invariant, we have

$$\begin{aligned}
\int_X |Rm|^2_{g_\infty} d\mu_{g_\infty} &\leq \limsup_{i\to\infty} \int_M |Rm|^2_{g_i(0)} d\mu_{g_i(0)} \\
&= \limsup_{i\to\infty} (\int_M |R|^2_{g_i(0)} d\mu_{g_i(0)} + C(M)) \\
&= \limsup_{i\to\infty} (\int_M |R|^2_{g(t_i)} d\mu_{g(t_i)} + C(M)) \\
&\leq C_0.
\end{aligned}$$

In the last step, we use Perelman's estimate that scalar curvature is uniformly bounded. From the noncollapsing property of this flow (Proposition 2.1), the limit manifold (X, g_∞) must be κ-noncollapsing on all scales, i.e.,

$$\frac{\text{Vol}(B(p, r))}{r^4} \geq \kappa > 0, \quad \forall\, p \in X, \, r > 0.$$

As Ricci curvature is bounded, this inequality implies that (X, g_∞) has uniform Sobolev constant. Therefore (X, g_∞) is a Ricci flat manifold with bounded energy and uniform Sobolev constant. Such a manifold must be an an Asymptotically Locally Euclidean (ALE) space. The detailed proof can be found in either [An89] or [BKN], [Tian90]. So we have the following property.

Proposition 4.1. *Every deepest bubble is a Kähler, Ricci flat Asymptotically Locally Euclidean (ALE) space.*

Moreover, the fundamental group of every deepest bubble must be finite.

Proposition 4.2. *If Y^4 is a non flat ALE space with flat Ricci curvature, then $\pi_1(Y)$ is finite.*

This is proved in [Ant]. For the simplicity of readers, we write down a simple proof here.

Proof. Fix $p \in Y$ and let $\pi : \tilde{Y} \to Y$ be the universal covering map.

We argue by contradiction. Suppose $\pi_1(Y)$ is a infinite group, then we can find a sequence of points $p_0, p_1, p_2, p_3, \cdots \in \pi^{-1}(p)$. Let γ_i be the shortest geodesic connecting p_0 and p_i, q_i be the center point of γ_i, v_i be the tangent direction represented by γ_i in $T_{q_i}\tilde{Y}$.

Since Y is an ALE space, there is a constant $R > 0$ such that Y is diffeomorphic to $B(p, R)$. Therefore, every loop $\pi(\gamma_i)$ can be smoothly deformed into a loop in $B(p, R)$. Consequently the shortest distance property of γ_i assures that $\pi(\gamma_i)$ must locate in $\overline{B(p, 2^m R)}$. As $\overline{B(p, 2^m R)}$ is a compact space, we may assume that $(\pi(q_i), \pi_*(v_i))$ converges to a point $(q, v) \in TX$. Remember $q \in \overline{B(p, 2^m R)}$ and v is a unit vector in $T_q Y$.

Let $(x, u) \in T\tilde{Y}$ such that $(\pi(x), \pi_*(u)) = (q, v)$. Then there is deck transformation $\sigma_i \in \pi_1(Y)$ such that $(\sigma_i(q_i), (\sigma_i)_*(v_i)) \to (x, u)$. $\sigma_i(\gamma_i)$ is clearly a shortest geodesic connecting $\sigma_i(p_0)$ and $\sigma_i(p_i)$ and $\sigma_i(q_i)$ is the center of $\sigma_i(\gamma_i)$. Now there are only two possibilities.

case1. $\limsup\limits_{i \to \infty} |\gamma_i| = |\sigma_i(\gamma_i)| = \infty$

In this case, by taking subsequence if necessary, we can assume that $\sigma_i(\gamma_i)$ tends to a line passing through q_0. As $\tilde{Y}$ has flat Ricci curvature, it splits a line. So $\tilde{Y} = N^3 \times \mathbb{R}$. N^3 must be Ricci flat and therefore Riemannian flat. This implies $\tilde{Y}$ and Y are flat. According to the assumption of Y, this is impossible.

case2. $|\gamma_i| < C$ *uniformly.*

Since all p_i's locate in $\overline{B(p_0, C)} \subset \tilde{Y}$ which is a compact set. Therefore for every small $\epsilon > 0$, there exists i, j such that $d(p_i, p_j) < \epsilon$. It follows that Y contains a geodesic lasso passing through p and its length is less than ϵ no matter how small ϵ is. Of course this will not happen on a smooth manifold Y.

Since both cases will not happen, our assumption must be wrong. Therefore $\pi_1(Y)$ is finite. $\qquad\square$

Let $\tilde{X} \xrightarrow{\pi} X$ be the universal covering. Then $(\tilde{X}, \tilde{g}, \tilde{J})$ is a Kähler Ricci flat manifold, where $\tilde{g} = \pi^*(g_\infty)$, $\tilde{G} = \pi_* \circ J \circ (\pi_*)^{-1}$. So its holonomy group is $SU(2) = Sp(1)$. Therefore, $\tilde{X}$ has a hyper-Kähler structure by Berger's classification. Since $\pi_1(X)$ is finite, we have

$$\int_{\tilde{X}} |Rm|^2_{\tilde{g}} d\mu_{\tilde{g}} = |\pi_1(X)| \int_X |Rm|^2_{g_\infty} d\mu_{g_\infty} < \infty.$$

It follows that $\tilde{X}$ is an ALE space as we argued before. This means that $\tilde{X}$ is a hyper-Kähler ALE space. However, all these hyper Kähler ALE spaces has been classified by Kroheimer in [Kr89].

Proposition 4.3 (Kroheimer)**.** *Let Γ be a finite subgroup of $SU(2)$ and $\pi : M \to \mathbb{C}^2/\Gamma$ be the minimal resolution of the quotient space $\mathbb{C}^2/\Gamma$ as a complex variety. Suppose that three cohomology classes $\alpha_I, \alpha_J, \alpha_K \in H^2(M; \mathbb{R})$ satisfy the non-degeneracy condition for each $\Sigma \in H_2(M; \mathbb{Z})$ with $\sigma \cdot \sigma = -2$, there exists $A \in \{I, J, K\}$ with $\alpha_A(\Sigma) \neq 0$.*

Then there exists an ALE Riemannian metric g on M of order 4 together with a hyper Kähler structure (I, J, K) for which the cohomology class of the Kähler form $[\omega_A]$ determined by the complex structure A is given by α_A for all $A \in I, J, K$. Conversely every hyper Kähler ALE 4-manifold of order 4 can be obtained as above.

As a corollary of this property, we know $\tilde{X}$ must be diffeomorphic to a minimal resolution of $\mathbb{C}^2/\Gamma$ for some finite group $\Gamma \subset SU(2)$.

Now let's consider the property of X by its submanifolds. On the deepest bubble (X, g_∞, J_∞), let ω_∞ be the metric form determined by g_∞ and J_∞.

Proposition 4.4. *For every closed 2-dimensional submanifold C of X, we have*

$$\int_C \omega_\infty = 0.$$

In particular, (X, g_∞, J_∞) doesn't contain any compact holomorphic curve.

Proof. Fix a closed 2-dimensional submanifold $C \subset X$. By the smooth convergence property, we know there is a sequence of closed 2-dimensional smooth manifolds $C_i \subset M$ such that

$$(C_i, g_i(0)|_{C_i}) \overset{C^\infty}{\to} (C, g_\infty|_C);$$
$$(C_i, \omega_i(0)|_{C_i}) \overset{C^\infty}{\to} (C, \omega_\infty|_C). \tag{3}$$

Therefore,

$$Area_{g_\infty}(C) = \lim_{i \to \infty} Area_{g_i(0)}(C_i).$$

Use Wirtinger's inequality, we get

$$\begin{aligned}
|\int_{C_i} \omega_i(0)| = |\int_{C_i} \cos \alpha d\mu_{g_i(0)}| \\
\leq \int_{C_i} |\cos \alpha| d\mu_{g_i(0)} \\
\leq \int_{C_i} d\mu_{g_i(0)} = Area_{g_i(0)}(C_i)
\end{aligned}$$

where α is the Kähler angle. Consequently, for large i, the following inequality hold

$$|\int_{C_i} \omega_i(0)| \leq 2 Area_{g_\infty}(C).$$

On the other hand, we know $[\omega_i(0)] = Q_i c_1(M)$, this tells us that

$$\int_{C_i} \omega_i(0) = Q_i \int_{C_i} c_1(M) = Q_i a_i \tag{4}$$

where we denote $a_i = \int_{C_i} c_1(M) \in \mathbb{Z}$. So we have inequality

$$Q_i |a_i| \leq 2 Area_{g_\infty} C$$

or

$$|a_i| \leq \frac{2 Area_{g_\infty} C}{Q_i}.$$

Note that $Area_{g_\infty} C$ is a fixed number and $Q_i \to \infty$, so $|a_i| \to 0$. However, a_i are integers. This forces that for large i, $a_i \equiv 0$. Now we go back to equality (4) and see for large i, we have

$$\int_{C_i} \omega_i \equiv 0.$$

Using smooth convergence property, equation (3) yields

$$\int_C \omega_\infty = \lim_{i \to \infty} \int_{C_i} \omega_i(0) = 0. \tag{5}$$

$\square$

Remark 4.1. *The reason for no compact divisors in X is that Ricci flow evolves metric forms continuously, so it cannot change the class which is discrete. This should be some common phenomenon in geometric flow. For example, in [Sj], Jeff Streets proved that on a nontrivial bundle, the base manifold of a blowup limit along a renormalization group flow must be noncompact.*

Combining the previous propositions, we can conclude this section by Theorem 2.

5 Bound Riemannian Curvature of Flow on Toric Fano Surfaces

This section is devoted to the proof of Theorem 3.

Lemma 5.1. *If (X, g_∞, J_∞) is a bubble coming out of a Kähler Ricci Flow solution with toric symmetric metrics, then (X, g_∞, J_∞) is also a toric surface. Moreover, X is simply connected, $b_2(X) > 0$ and $H_2(X)$ is generated by holomorphically embedded $\mathbb{CP}_1$'s in X.*

Proof. According to the construction of X, we have the following convergence in Cheeger-Gromov sense

$$(M, x_i, g_i(0)) \xrightarrow{C^\infty} (X, x_\infty, g_\infty).$$

Since the toric symmetry property will be preserved under Kähler Ricci Flow , we see that every metric $g_i(0) = Q_i g(t_i)$ is a toric symmetric metric. Using exactly the statement of Proposition 16 of [CLW], we know that X is a toric surface with nontrivial $H_2(X)$. Moreover, $H_2(X)$ are generated by holomorphic $\mathbb{CP}^1$'s in X. Actually, according to this proof, there is a Morse function defined on X. Furthermore, every critical point of X has even indices. Therefore X is homotopic to a CW-complex with only cells of even dimension. Consequently, X must be simply connected by considering the Euler characteristic number of X. $\square$

Now we are able to prove Theorem 3.

Proof. We only need to show Riemannian curvature is uniformly bounded. If Riemannian curvature is not bounded, then we can blowup a toric deepest bubble X. According to Proposition 4.4, it doesn't contain any compact divisor. On the other hand, Lemma 5.1 implies that X must contain a $\mathbb{CP}^1$ as a compact divisor. Contradiction! $\square$

Remark 5.1. *If $M \sim \mathbb{CP}^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, Theorem 3 can be proved in a different way. Suppose Riemannian curvature is unbounded. Then we can obtain a bubble (X, g_∞, J_∞) which is simply connected. Therefore itself is a hyper Kähler ALE space. By Kroheimer's classification (Proposition 4.3), X must contain a compact smooth 2-dimensional submanifold C whose self intersection number is -2. By the smooth convergence property, we can get a sequence of closed smooth 2-dimensional smooth manifolds $C_i \subset M$ such that $(C_i, g_i(0)|_{C_i}) \xrightarrow{C^\infty} (C, h|_C)$. In particular C_i is diffeomorphic to C and a small tubular neighborhood of C_i is diffeomorphic to a small tubular neighborhood of C. Therefore, $[C_i] \cdot [C_i] = [C] \cdot [C] = -2$. Note that $[C_i] \in H_2(M, \mathbb{Z})$. However, $H_2(\mathbb{CP}^2, \mathbb{Z})$ and $H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2, \mathbb{Z})$ doesn't contain any element of self intersection number -2. So we obtain a contradiction!*

Remark 5.2. *Theorem 3 needs the condition that initial metric $g(0)$ is toric symmetric. Natasa Sesum conjectured that the toric symmetry condition is not necessary. In our proof, the symmetry is only a technical condition, we believe that it can be dropped. Actually, we believe that starting from any metric in canonical class, Kähler Ricci flow will evolve the metrics into a KRS on every toric manifold. It will be discussed in a subsequent paper.*

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