SOME CLASSIFICATIONS OF ∞-HARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS

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ABSTRACT

∞-Harmonic maps are a generalization of ∞-harmonic functions. They can be viewed as the limiting cases of p-harmonic maps as p goes to infinity. In this paper, we give complete classifications of linear and quadratic ∞-harmonic maps from and into a sphere, quadratic ∞-harmonic maps between Euclidean spaces. We describe all linear and quadratic ∞-harmonic maps between Nil and Euclidean spaces, between Sol and Euclidean spaces. We also study holomorphic ∞-harmonic maps between complex Euclidean spaces.

1. INTRODUCTION

In this paper, we work in the category of smooth objects so that all manifolds, vector fields, and maps are assumed to be smooth unless there is an otherwise statement.

The infinity Laplace equation

$$\Delta_\infty u := \frac{1}{2}\langle \nabla\, u, \nabla\, |\nabla u|^2 \rangle = \sum_{i,j=1}^{m} u_{ij}u_iu_j = 0, \tag{1}$$

where $u : \Omega \subset \mathbb{R}^m \longrightarrow \mathbb{R}$, $u_i = \frac{\partial u}{\partial x^i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$, was first discovered and studied by G. Aronsson in his study of "optimal" Lipschitz extension of functions in the late 1960s ([Ar1], [Ar2]).

To see why this nonlinear and highly degenerate elliptic PDE has been so fascinating, we recall that the famous minimal surface equation can be written

1991 *Mathematics Subject Classification.* 58E20, 53C12.

Key words and phrases. infinity-harmonic maps, Euclidean spaces, spheres, Nil space, Sol space.

as

$$(1+|\nabla u|^2)\Delta u+\sum_{i,j=1}^{m}u_iu_ju_{ij}=0,$$

from which we see that the ∞-Laplace equation can be obtained as **harmonic minimal surface equation** meaning the equation for harmonic functions with minimal graphs.

The solutions of the ∞-Laplace equation are called ∞-harmonic functions which have the following interpretations:

Lemma 1.1. *(see* [Ou1]*) Let $u:(M^m,g)\longrightarrow \mathbb{R}$ be a function. Then the following conditions are equivalent:*

1. *u is an ∞-harmonic function, i.e., $\Delta_\infty u=0$,*
2. *u is horizontally homothetic;*
3. *∇u is perpendicular to $\nabla|\nabla u|^2$;*
4. $\mathrm{Hess}_u(\nabla u,\nabla u)=0$*;*
5. *$|\nabla u|^2$ is constant along any integral curve of ∇u.*

Also, the ∞-Laplace equation can be viewed (see [Ar1]) as the formal limit, as $p\to\infty$, of p-Laplace equation

$$\Delta_p\, u:=|\nabla\, u|^{p-2}\left(\Delta\, u+\frac{p-2}{|\nabla\, u|^2}\Delta_\infty\, u\right)=0.$$

Finally, the ∞-Laplace equation can be viewed as the Euler-Lagrange equation of the L^∞ variational problem of minimizing

$$E_\infty(u)=\operatorname{ess\,sup}_\Omega|\mathrm{d}\, u|$$

among all Lipschitz continuous functions u with given boundary values on $\partial\Omega$ (see [ACJ], [Ba], and [BEJ] and the references therein for more detailed background).

Recently, a great deal of research work has been done in the study of the ∞-Laplace equation after the work of Crandall and Lions (see e.g. [CIL]) on the theory of viscosity solutions for fully nonlinear problems. Many important results have been achieved and published in, e.g., [ACJ], [BB], [Ba], [BLW1], [BLW2], [BEJ], [Bh], [CE], [CEG], [CIL], [CY], [EG], [EY], [J], [JK], [JLM1], [JLM2], [LM1], [LM2], [Ob].

On the other hand, the ∞-Laplace equation has been found to have some very interesting applications in areas such as image processing (see e.g. [CMS], [Sa]), mass transfer problems (see e.g. [EG]), and the study of shape metamorphism (see e.g. [CEPB]).

The generalization from harmonic functions to harmonic maps between Riemannian manifolds was so fruitful that it has not only opened new fields of study in differential geometry, analysis, and topology but also brought important applications to many branches in mathematics and theoretical physics. It would be interesting to study maps between Riemannian manifolds that generalize ∞-harmonic functions. This was initiated in [OTW] where the notion of ∞-harmonic maps between Riemannain manifolds was introduced as a natural generalization of ∞-harmonic functions and as the limit case of p-harmonic maps as $p \to \infty$.

Definition 1.2. ([OTW]) *A map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is called an ∞-harmonic map if the gradient of its energy density belongs to the kernel of its tangent map, i.e., φ is a solution of the PDEs*

$$\tau_\infty(\varphi) := \frac{1}{2}{\rm d}\varphi({\rm grad}\,|{\rm d}\varphi|^2) = 0, \tag{2}$$

where $|{\rm d}\varphi|^2 = {\rm Trace}_g \varphi^* h$ *is the energy density of* φ.

A direct computation using local coordinates yields (see also [OTW])

Corollary 1.3. *In local coordinates, a map $\varphi : (M, g) \longrightarrow (N, h)$ with $\varphi(x) = (\varphi^1(x), \ldots, \varphi^n(x))$ is ∞-harmonic if and only if*

$$g({\rm grad}\,\varphi^\alpha, {\rm grad}\,|{\rm d}\varphi|^2) = 0, \qquad \alpha = 1, 2, ..., n. \tag{3}$$

Clearly, any ∞-harmonic function is an ∞-harmonic map by Definition 1.2. It also follows from the definition that any map between Riemannian manifolds with constant energy density, i.e., $|{\rm d}\varphi|^2 = {\rm Trace}_g \varphi^* h = $ constant is an ∞-harmonic map. Thus, the following important and familiar families are all ∞-harmonic maps:

- totally geodesic maps,
- isometric immersions,
- Riemannian submersions,
- eigenmaps between spheres.

Examples of ∞-harmonic maps with nonconstant energy density include the following classes:

- projections of multiply warped products (e.g., the projection of the generalized Kasner spacetimes),
- equator maps, and
- radial projections.

We refer the readers to [OTW] for details of these and many other examples and other results including methods of constructing ∞-harmonic maps into Euclidean spaces and spheres, characterizations of ∞-harmonic immersions and submersions, study of ∞-harmonic morphisms which can be characterized as horizontally homothetic submersions, and the transformation ∞-Laplacians under the the conformal change of metrics.

In this paper, we study the classification of ∞-harmonic maps between certain model spaces. We give complete classifications of linear and quadratic ∞-harmonic maps from and into a sphere, quadratic ∞-harmonic maps between Euclidean spaces. We describe all linear and quadratic ∞-harmonic maps between Nil and Euclidean spaces and between Sol and Euclidean spaces. We also study holomorphic ∞-harmonic maps complex Euclidean spaces.

2. QUADRATIC ∞-HARMONIC MAPS BETWEEN EUCLIDEAN SPACES

As we mentioned in Section 1 that any map with constant energy density is ∞-harmonic. It follows that any affine map $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ with $\varphi(X) = AX + b$, where A is an $n \times m$ matrix and $b \in R^n$ is a constant, is an ∞-harmonic map because of its constant energy density. Note that there are also globally defined ∞-harmonic maps between Euclidean spaces which are not affine maps. For example, one can check that $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $\varphi(x, y, z) = (\cos x + \cos y + \cos z,\ \sin x + \sin y + \sin z)$ is a map with constant energy density $|{\rm d}\varphi|^2 = {\rm Trace}_g \varphi^* h = 3$ and hence an ∞-harmonic maps. In this section, we give a complete classification of ∞-harmonic maps between Euclidean spaces defined by quadratic polynomials. First, we prove the following lemma which will be used frequently in this paper.

Lemma 2.1. *Let A_i, $i = 1, 2, \ldots, n$, be symmetric matrices of $m \times m$. Then, $(\sum_{j=1}^n A_j^2)A_i + A_i(\sum_{j=1}^n A_j^2) = 0$ for all $i = 1, 2 \ldots, n$ if and only if $A_i = 0$ for $i = 1, 2 \ldots, n$*

Proof. Suppose otherwise, i.e., one of A_i is not zero, without loss of generality, we may assume $A_1 \neq 0$. Then $rank(A_1) = K$ with $1 \leq K \leq m$. Without loss of generality, we can choose a suitable orthogonal matrix T such that $T^{-1}A_1T$ takes the diagonal form

$$T^{-1}A_1T = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} \tag{4}$$

where $\lambda_{i_k} \neq 0, k = 1, 2, \ldots, K$
Note that

$$T^{-1}\sum_{j=1}^{n}(A_j^2)T = \sum_{j=1}^{n}T^{-1}(A_j^2)T = \sum_{j=1}^{n}(T^{-1}A_jT)^2 \tag{5}$$

with each $(T^{-1}A_jT)^2$ being symmetric matrix. It follows that

$$\begin{aligned} 0 = T^{-1}0T &= T^{-1}\sum_{j=1}^{n}(A_j^2A_1 + A_1A_j^2)T \\ &= \sum_{j=1}^{n}(T^{-1}A_jT)^2T^{-1}A_1T + T^{-1}A_1T(T^{-1}A_jT)^2). \end{aligned} \tag{6}$$

This is impossible because the i-th entry in the main diagonal of the matrix on the right-hand side of Equation (6) takes the form

$$2\lambda_i(\lambda_i^2 + \sum_{j\geq 2}^{n}|(T^{-1}A_jT)^i|^2), \tag{7}$$

where $(T^{-1}A_jT)^i$ denotes the i-th row vector in $(T^{-1}A_jT)$, and we know that at least one λ_i is not zero. The contradiction proves the Lemma. □

Theorem 2.2. *Let $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a quadratic map with $\varphi(X) = (X^tA_1X, \ldots, X^tA_nX)$, where $X^t = (x^1, \ldots, x^m) \in \mathbb{R}^m$. Then, φ is an ∞-harmonic map if and only if φ is a constant map.*

Proof. A straightforward computation gives:

$$\begin{aligned} &\nabla\varphi^i = 2X^tA_i, \\ &|\mathrm{d}\varphi|^2 = \delta^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j\delta_{ij} = \sum_{i=1}^{n}g(\nabla\varphi^i, \nabla\varphi^i) \\ &= \sum_{i=1}^{n}\langle 2X^tA_i, 2X^tA_i\rangle = 4\sum_{i=1}^{n}X^tA_i^2X, \text{ and} \\ &\nabla\,|\mathrm{d}\varphi|^2 = 8\sum_{i=1}^{n}X^tA_i^2. \end{aligned}$$

It follows from Corollary 1.3 that φ is ∞-harmonic if and only if

$$g(\nabla\,\varphi^i, \nabla\,|\mathrm{d}\varphi|^2) = 0, \qquad i = 1, 2, \ldots, n,$$

which is equivalent to

$$X^tA_i(\sum_{j=1}^n A_j^2)X=0,\ i=1,2,\ldots,n. \tag{8}$$

As the coefficient matrix $A_i(\sum_{j=1}^n A_j^2)$ of the quadratic form on the left-hand side of (8) is not symmetric in general we can rewrite (8) as

$$X^t(A_i(\sum_{j=1}^n A_j^2)+(\sum_{j=1}^n A_j^2)A_i)X=0,\ i=1,2,\ldots,n. \tag{9}$$

Since $A_i(\sum_{j=1}^n A_j^2)+(\sum_{j=1}^n A_j^2)A_i$ is a symmetric matrix of $m\times m$ we conclude that φ is ∞-harmonic if and only if

$$A_i(\sum_{j=1}^n A_j^2)+(\sum_{j=1}^n A_j^2)A_i=0, \qquad i=1,2,\ldots,n. \tag{10}$$

It follows from this and Lemma 2.1 that $A_i=0$ for $i=1,2,\ldots,n$, and hence $\varphi(X)=0$, a constant map, from which we obtain the Theorem.

□

Theorem 2.3. *Let $\varphi:\mathbb{R}^m\longrightarrow\mathbb{R}^n$, $\varphi(X)=(X^tA_1X,\ldots,X^tA_nX)+(AX)^t+b$ be a polynomial map, where A_i is an $m\times m$ symmetric matrix for $i=1,2,\ldots,n$, A an $n\times m$ matrix, and $b\in\mathbb{R}^n$. Then, φ is an ∞-harmonic map if and only if φ is an affine map with $\varphi(X)=(AX)^t+b$.*

Proof. Let $\alpha_i\in\mathbb{R}^m,\ \ i=1,2,\ldots,n$, denote the i-th row vector of the matrix A. Then,

$$\nabla\varphi^i=2X^tA_i+\alpha_i,\ i=1,2,...,n, \tag{11}$$

$$\begin{aligned}|\mathrm{d}\varphi|^2&=g^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j\delta_{ij}\\&=\sum_{i=1}^n g(\nabla\varphi^i,\nabla\varphi^i)\\&=\sum_{i=1}^n\langle 2X^tA_i+\alpha_i,2X^tA_i+\alpha_i\rangle\\&=4\sum_{i=1}^n X^tA_i^2X+\sum_{i=1}^n g\langle\alpha_i,\alpha_i\rangle+4\sum_{i=1}^n\alpha_iA_iX,\end{aligned}$$

and

$$\nabla\,|\mathrm{d}\varphi|^2=8\sum_{i=1}^n X^tA_i^2+4\sum_{i=1}^n\alpha_iA_i. \tag{12}$$

Substituting (11) and (12) the ∞-harmonic map equation (3) we conclude that φ is ∞-harmonic if and only if

$$\begin{aligned} 0 &= g(\nabla \varphi^i, \nabla |\mathrm{d}\varphi|^2) \\ &= \langle 2X^tA_i + \alpha_i, 8\sum_{j=1}^n X^tA_j^2 + 4\sum_{j=1}^n \alpha_j A_j\rangle \\ &= 16\sum_{j=1}^n X^tA_iA_j^2X + 8\sum_{j=1}^n X^tA_iA_j(\alpha_j)^t + 8\sum_{j=1}^n(\alpha_iA_j^2)X + 4\sum_{j=1}^n \alpha_iA_j(\alpha_j)^t. \end{aligned} \tag{13}$$

Since Equation (13) if true for arbitrary X, it is actually an identity of polynomial in X. By comparing the coefficients of the leading terms of the polynomials of at both sides we have that, if φ is ∞-harmonic, then

$$16X^tA_i\sum_{j=1}^n A_j^2X = 0, \quad i = 1, 2, \ldots, n, \tag{14}$$

which is the same as Equation (8). Now we can use Lemma 2.1 to conclude that if φ is ∞-harmonic, then $A_i = 0$ for $i = 1, 2, \ldots, n$ and hence $\varphi(X) = (AX)^t + b$ is an affine map. The converse statement clearly true because an affine map has constant energy density. Therefore, we obtain the theorem. □

Remark 1. (A) It would be interesting to know if there is any ∞-harmonic maps $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ defined by homogeneous polynomials of degree greater than 2.

(B) We also remark that the situation for the ∞-harmonic maps between semi-Euclidean spaces is quite different in that there are many examples of non-constant ∞-harmonic maps between semi-Euclidean spaces defined by quadratic polynomials, for example, let $\mathbb{R}^2_1$ denote the 2-dimensional semi-Euclidean space with semi-Euclidean metric $ds^2 = -dx^2 + dy^2$, then one can check that the quadratic map $\varphi : \mathbb{R}^2_1 \longrightarrow \mathbb{R}^2_1$ defined by $\varphi(x, y) = (12x^2+12y^2, 13x^2+10xy+13y^2)$ is an map with energy density $|\mathrm{d}\varphi|^2 = \mathrm{Trace}_g\varphi^*h = (\varphi^1_1)^2-(\varphi^1_2)^2-(\varphi^2_1)^2+(\varphi^2_2)^2 = 0$, hence it is an ∞-harmonic map. For more examples and study of ∞-harmonic maps between Semi-Euclidean spaces see [Zh].

3. Linear ∞-harmonic maps from and into a sphere

In this section, we first derive an equation for linear ∞-harmonic map between conformally flat spaces. We then use it to give a complete classification of linear ∞-harmonic maps between a Euclidean space and a sphere.

Lemma 3.1. *Let* $\varphi : (\mathbb{R}^m, g = F^{-2}\delta_{ij}) \longrightarrow (\mathbb{R}^n, h = \lambda^{-2}\delta_{\alpha\beta})$ *with*

$$\varphi(X) = AX = \left(A^1X, \cdots, A^nX\right),$$

where A^i is the i-th row vector of A, be a linear map between conformally flat spaces. Then, φ is ∞-harmonic if and only if $A = 0$, i.e., $\varphi(X) = AX = 0$ is a constant map, or

$$\langle A^\alpha, \nabla(\frac{F}{\lambda\circ\varphi})\rangle = 0, \quad \alpha = 1, 2, \ldots, n. \tag{15}$$

where $\langle,\rangle$ is the Euclidean inner product and ∇f denotes the gradient of f taken with respect to the Euclidean metric on $\mathbb{R}^m$.

Proof. It is easy to check that for the linear map $\varphi : (\mathbb{R}^m, g = F^{-2}\delta_{ij}) \longrightarrow (\mathbb{R}^n, h = \lambda^{-2}\delta_{\alpha\beta})$ with $\varphi(X) = AX = (A^1X, \cdots, A^nX)$ we have:

$$|\mathrm{d}\varphi|^2 = F^2\delta^{ij}{\varphi^\alpha}_i{\varphi^\beta}_j(\lambda^{-2}\delta_{\alpha\beta})\circ\varphi = (\frac{F}{\lambda\circ\varphi})^2\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}^2 = (\frac{F}{\lambda\circ\varphi})^2|A|^2.$$

By Corollary 1.3, φ is ∞-harmonic if and only if

$$\begin{aligned}
&g(\mathrm{grad}\,\varphi^\alpha, \mathrm{grad}\,|\mathrm{d}\varphi|^2) = g^{ij}\varphi_i^\alpha(|\mathrm{d}\varphi|^2)_j\\
&= F^2\delta^{ij}\varphi_i^\alpha(|\mathrm{d}\varphi|^2)_j = F^2\langle A^\alpha, \nabla(\frac{F}{\lambda\circ\varphi})^2|A|^2\rangle\\
&= \frac{2F^3|A|^2}{\lambda\circ\varphi}\langle A^\alpha, \nabla(\frac{F}{\lambda\circ\varphi})\rangle = 0, \qquad \alpha = 1, 2, ..., n,
\end{aligned}$$

from which the Lemma follows. □

Let (S^n, g_{can}) be the n-dimensional sphere with the standard metric. It is well known that we can identify $(S^n \setminus \{N\}, g_{can})$ with $(\mathbb{R}^n, \lambda^{-2}\delta_{ij})$, where $\lambda = \frac{1+|x|^2}{2}$. Using coordinate $\{x_i\}$ we can write the components of g_U as:

$$\bar{g}_{ij} = \lambda^{-2}\delta_{ij}, \;\; \bar{g}^{ij} = \lambda^2\delta_{ij}.$$

As an application of Lemma 3.1 we give the following classification of ∞-harmonic maps between spheres.

Theorem 3.2. *A linear map $\varphi : (\mathbb{R}^m, F^{-2}\delta_{ij}) \equiv (S^m\backslash\{N\}, g_{can}) \longrightarrow (\mathbb{R}^n, \lambda^{-2}\delta_{ij}) \equiv (S^n\backslash\{N\}, g_{can})$ between two spheres with $\varphi(X) = (A^1X, \cdots, A^nX)$ is ∞-harmonic if and only if $A = 0$, i.e., φ is a constant map, or, $A^tA = I_{m\times m}$, i.e., φ is an isometric immersion.*

Proof. To prove the theorem, we applying Lemma 3.1 with $F = \frac{1+|X|^2}{2}$ and $\lambda = \frac{1+|Y|^2}{2}$ we conclude that φ is ∞-harmonic if and only if $A = 0$, $\varphi(X) = AX = 0$

is a constant map, or

$$\langle A^{\alpha}, \nabla(\frac{F}{\lambda\circ\varphi})\rangle = \frac{1}{(\lambda\circ\varphi)^2}\langle A^{\alpha}, (\lambda\circ\varphi)\nabla F - F\nabla(\lambda\circ\varphi)\rangle$$
$$= \frac{1}{(\lambda\circ\varphi)^2}\langle A^{\alpha}, (\lambda\circ\varphi)\nabla[\frac{1}{2}(1+|X|^2)] - F\nabla[\frac{1}{2}(1+|AX|^2)]\rangle$$
$$= \frac{1}{2(\lambda\circ\varphi)^2}\langle A^{\alpha}, (1+|AX|^2)X - (1+|X|^2)\nabla(X^tA^tAX)\rangle$$
$$= \frac{1}{2(\lambda\circ\varphi)^2}\langle A^{\alpha}, (1+|AX|^2)X - (1+|X|^2)A^tAX\rangle = 0, \quad \alpha = 1,2,\ldots,n,$$

which is equivalent to

$$(1+|AX|^2)AX - (1+|X|^2)AA^tAX = 0, \tag{16}$$

for any $X\in\mathbb{R}^m$. It follows that Equation (16) is an identity of polynomials. By comparing coefficients we have

$$\begin{cases} AX - AA^tAX = 0 \\ |AX|^2AX - |X|^2AA^tAX = 0. \end{cases} \tag{17}$$

for any $X\in\mathbb{R}^m$. It is easy to see that Equation (17) implies that $A=0$, or, $A^tA = I_{m\times m}$ and $|\varphi(X)|^2 = |AX|^2 = |X|^2$, from which we obtain the theorem. □

For linear maps between a Euclidean space and a sphere we have

Theorem 3.3. (1) *A linear map* $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^n, \lambda^{-2}\delta_{ij}) \equiv (S^n\setminus\{N\}, g_{can})$ *from a Euclidean space into a sphere with* $\varphi(X) = (A^1X,\cdots,A^nX)$ *is ∞-harmonic if and only if* $A=0$, *i.e.,* φ *is a constant map.*
(2) *A linear map* $\varphi : (\mathbb{R}^m, \lambda^{-2}\delta_{ij}) \equiv (S^m\setminus\{N\}, g_{can}) \longrightarrow \mathbb{R}^n$ *from a sphere into a Euclidean space with* $\varphi(X) = (A^1X,\cdots,A^nX)$ *is ∞-harmonic if and only if* $A=0$, *i.e.,* φ *is a constant map.*

Proof. To prove the first Statement, we applying Lemma 3.1 with $F=1$ and $\lambda = \frac{1+|Y|^2}{2}$ we conclude that φ is ∞-harmonic if and only if $A=0$, $\varphi(X) = AX = 0$ is a constant map, or

$$\langle A^{\alpha}, \nabla(\frac{1}{\lambda\circ\varphi})\rangle = -\frac{1}{(\lambda\circ\varphi)^2}\langle A^{\alpha}, \nabla(\lambda\circ\varphi)\rangle$$
$$= -\frac{1}{(\lambda\circ\varphi)^2}\langle A^{\alpha}, \nabla[\frac{1}{2}(1+|AX|^2)]\rangle$$
$$= -\frac{1}{2(\lambda\circ\varphi)^2}\langle A^{\alpha}, \nabla(X^tA^tAX)\rangle$$
$$= -\frac{1}{(\lambda\circ\varphi)^2}\langle A^{\alpha}, A^tAX\rangle = 0, \quad \alpha = 1,2,\ldots,n,$$

which is equivalent to

$$AA^tAX = 0. \tag{18}$$

for any $X \in \mathbb{R}^m$. By letting $X = (A^i)^t,\ i = 1, \ldots, n$ in Equation (18) we conclude that φ is ∞-harmonic if and only if $AA^tAA^t = 0$. Note that $AA^tAA^t = (AA^t)(AA^t)^t = 0$ implies that $\mathrm{Trace}(AA^t) = \sum_{i=1}^{n} |A^i|^2 = 0$. It follows that $|A| = 0$, i.e., φ is a constant map. This gives the first Statement of the theorem.

For the second Statement, we apply Lemma 3.1 with $\lambda = 1$ and $F = \frac{1+|X|^2}{2}$ to conclude that φ is ∞-harmonic if and only if $A = 0$, $\varphi(X) = AX = 0$ is a constant map, or

$$\langle A^\alpha, \nabla F\rangle = \langle A^\alpha, X\rangle = 0 \tag{19}$$

for $\alpha = 1, 2, \ldots, n$ and for all $X \in \mathbb{R}^m$. It is easy to see that Equation (19) implies that $A^\alpha = 0$ for for $\alpha = 1, 2, \ldots, n$ and hence $A = 0$, i.e., φ is a constant. This completes the proof of the Theorem. □

4. Quadratic ∞-harmonic maps from and into a sphere

Again, we identify $(S^n \setminus \{N\}, g_{can})$ with $(\mathbb{R}^n, \lambda^{-2}\delta_{ij})$, where $\lambda = \frac{1+|X|^2}{2}$.

Theorem 4.1. (1) *A quadratic map* $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^n, \lambda^{-2}\delta_{ij}) \equiv (S^n \setminus \{N\}, g_{can})$ *into sphere with* $\varphi(X) = (X^tA_1X, X^tA_2X, \ldots, X^tA_nX)$ *is* ∞*-harmonic if and only it is a constant map.*
(2) *A quadratic map from a sphere into a Euclidean space* $\varphi : (\mathbb{R}^m, \lambda^{-2}\delta_{ij}) \longrightarrow \mathbb{R}^n$ *with* $\varphi(X) = (X^tA_1X, X^tA_2X, \ldots, X^tA_nX)$ *is* ∞*-harmonic if and only it is a constant map.*

Proof. For the Statement (1), we compute:

$$\nabla\varphi^\alpha = 2X^tA_\alpha,$$
$$|\mathrm{d}\varphi|^2 = \delta^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j\delta_{ij}(\lambda\circ\varphi)^{-2} = 4\sigma^2\sum_{j=1}^{n} X^tA_j^2X,$$

where $\sigma = \frac{1}{\lambda\circ\varphi}$. A further computation gives

$$\nabla\,|\mathrm{d}\varphi|^2 = 8\sigma(X^t\sum_{j=1}^{n} A_j^2X)\nabla\sigma + 8\sigma^2X^t\sum_{j=1}^{n} A_j^2. \tag{20}$$

The ∞-harmonic map equation for φ reads

$$0 = 16\sigma(X^t\sum_{j=1}^{n} A_j^2X)\langle X^tA_\alpha, \nabla\sigma\rangle + 16\sigma^2X^tA_\alpha\sum_{j=1}^{n} A_j^2X. \tag{21}$$

A direct computation yields $\nabla\sigma = -2\sigma^2(X^tA_1y_1 + X^tA_2y_2 + \ldots + X^tA_ny_n)$. where $y_\alpha = X^tA_\alpha X$ and for $\alpha = 1, 2, \ldots, n$. Substituting this into Equation (21) we have

$$\begin{aligned} 0 = \quad & -32\sigma^3(X^t\sum_{j=1}^{n}A_j^2X)(X^tA_\alpha A_1Xy_1 + \ldots + X^tA_\alpha A_nXy_n) \\ & +16\sigma^2X^tA_\alpha\sum_{j=1}^{n}A_j^2X \end{aligned} \tag{22}$$

which is equivalent to

$$\begin{aligned} & 0 = -32(X^t\sum_{j=1}^{n}A_j^2X)(X^tA_\alpha A_1Xy_1 + \ldots + X^tA_\alpha A_nXy_n) \\ & +\tfrac{16}{\sigma}X^tA_\alpha\sum_{j=1}^{n}A_j^2X \end{aligned} \tag{23}$$

i.e.,

$$0 = -P(X) + 8X^tA_\alpha\sum_{j=1}^{n}A_j^2X \tag{24}$$

where $P(X)$ denotes a polynomial in X of degree greater than 2. Noting that the equation is an identity of polynomials we conclude that if φ is ∞-harmonic, then

$$X^tA_\alpha\sum_{j=1}^{3}A_j^2X = 0, \qquad \alpha = 1, 2, \ldots, n, \tag{25}$$

which is exactly the Equation (8) and the same arguments used in the proof of Theorem 2.2 apply to give the required results.

To prove the second statement, let $\nabla f = (f_1, \ldots, f_m)$ denotes the Euclidean gradient of function f. Then, a straightforward computation gives:

$$\begin{aligned} & \nabla\varphi^\alpha = 2X^tA_\alpha, \\ & |\mathrm{d}\varphi|^2 = \lambda^2\delta^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j\delta_{ij} = 4\lambda^2\sum_{j=1}^{n}X^tA_j^2X, \end{aligned}$$

and

$$\nabla\,|\mathrm{d}\varphi|^2 = 8\lambda(X^t\sum_{j=1}^{n}A_j^2X)X + 8\lambda^2X^t\sum_{j=1}^{n}A_j^2, \tag{26}$$

where we have used the fact that $\nabla\lambda = X$. It follows that φ is ∞-harmonic if and only if

$$g(\mathrm{grad}\,\varphi^i, \mathrm{grad}\,|\mathrm{d}\varphi|^2) = 0, \qquad i = 1, 2, \ldots, n,$$

which is equivalent to

$$
\begin{aligned}
0 &= g^{ij}\varphi_i^\alpha \,|\mathrm{d}\varphi|_j^2 = \lambda^2\langle \nabla\varphi^\alpha, \nabla\,|\mathrm{d}\varphi|^2\rangle \\
&= \lambda^2\langle 2X^tA_\alpha, 8\lambda(X^t\sum_{j=1}^n A_j^2X)X + 8\lambda^2 X^t\sum_{j=1}^n A_j^2\rangle \\
&= 16\lambda^3(X^t\sum_{j=1}^n A_j^2X)X^tA_\alpha X + 16\lambda^4X^tA_\alpha\sum_{j=1}^n A_j^2X
\end{aligned} \tag{27}
$$

for all $X\in\mathbb{R}^m$ and for $\alpha=1,2,\ldots,n$. Note that Equation (27) is an identity of polynomials since λ is also a polynomial. Comparing the coefficients of the polynomials we conclude that φ is ∞-harmonic implies that $X^tA_\alpha\sum_{j=1}^n A_j^2X=0$ for $\alpha=1,2,\ldots,n$, which is exactly the Equation (8) and the same arguments used in the proof of Theorem 2.2 apply to give the required results. □

Remark 2. It is well known that any eigenmap between spheres is of constant energy density, so any eigenmap is ∞-harmonic. It would be interesting to know if there is any ∞-harmonic maps between spheres which is not an eigenmap.

5. Linear ∞-harmonic maps from and into Nil space

In this section we will give a complete classification of linear ∞-harmonic maps between Euclidean spaces and Nil space.

Theorem 5.1. *Let $(\mathbb{R}^3, g_{Nil})$ denote Nil space, where the metric with respect to the standard coordinates (x,y,z) in $\mathbb{R}^3$ is given by $g_{Nil}=\mathrm{d}x^2+\mathrm{d}y^2+(\mathrm{d}z-x\mathrm{d}y)^2$. Then*

1. *A linear function $f:(\mathbb{R}^3,g_{Nil})\longrightarrow\mathbb{R},\ f(x,y,z)=Ax+By+Cz$ is an ∞-harmonic function if and only if $A=0$ or $C=0$.*
2. *A linear map $\varphi:(\mathbb{R}^3,g_{Nil})\longrightarrow\mathbb{R}^n$ $(n\geq 2)$ is ∞-harmonic if and only if φ is a composition of the projection $\pi_1:(\mathbb{R}^3,g_{Nil})\longrightarrow\mathbb{R}^2,\ \pi_1(x,y,z)=(x,y)$ followed by a linear map $\mathbb{R}^2\longrightarrow\mathbb{R}^n$, or, φ is a composition of the projection $\pi_2:(\mathbb{R}^3,g_{Nil})\longrightarrow\mathbb{R}^2,\ \pi_2(x,y,z)=(y,z)$ followed by a linear map $\mathbb{R}^2\longrightarrow\mathbb{R}^n$.*

Proof. For Statement (1), we note that it has been proved in [Ou1] that a linear function $f:(\mathbb{R}^3,g_{Nil})\longrightarrow\mathbb{R},\ f(x,y,z)=Ax+By+Cz$ is an 1-harmonic if and only if it is horizontally homothetic which is equivalent to f being ∞-harmonic. It was further shown that this is equivalent to $A=0$ or $C=0$. To prove Statement (2), one can easily compute the following components of Nil metric:

$$
\begin{cases}
g_{11}=1,\ g_{12}=g_{13}=0,\ g_{22}=1+x^2,\ g_{23}=-x,\ g_{33}=1;\\
g^{11}=1,\ g^{12}=g^{13}=0,\ g^{22}=1,\ g^{23}=x,\ g^{33}=1+x^2.
\end{cases} \tag{28}
$$

Let $\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}^n$ $(n \geq 2)$ be a linear map with

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{29}$$

A straightforward computation gives the energy density of φ as:

$$\begin{aligned} |{\rm d}\varphi|^2 &= g^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j\delta_{ij} \\ &= \sum_{i=1}^n (g^{11}(\tfrac{\partial\varphi^i}{\partial x_1})^2 + g^{22}(\tfrac{\partial\varphi^i}{\partial x_2})^2 + g^{33}(\tfrac{\partial\varphi^i}{\partial x_3})^2 + g^{23}\tfrac{\partial\varphi^i}{\partial x_2}\tfrac{\partial\varphi^i}{\partial x_3} + g^{32}\tfrac{\partial\varphi^i}{\partial x_3}\tfrac{\partial\varphi^i}{\partial x_2}) \\ &= \sum_{i=1}^n a_{i3}^2x^2 + 2\sum_{i=1}^n a_{i2}a_{i3}x + \sum_{j=1}^3\sum_{i=1}^n a_{ij}^2, \end{aligned} \tag{30}$$

and

$$\begin{aligned} \tfrac{\partial|{\rm d}\varphi|^2}{\partial x_1} &= \tfrac{\partial|{\rm d}\varphi|^2}{\partial x} = 2\sum_{i=1}^n a_{i3}^2x + 2\sum_{i=1}^n a_{i2}a_{i3}, \\ \tfrac{\partial|{\rm d}\varphi|^2}{\partial x_2} &= \tfrac{\partial|{\rm d}\varphi|^2}{\partial y} = 0, \\ \tfrac{\partial|{\rm d}\varphi|^2}{\partial x_3} &= \tfrac{\partial|{\rm d}\varphi|^2}{\partial z} = 0. \end{aligned} \tag{31}$$

It follows from Corollary 1.3 and (31) that φ is ∞-harmonic if and only if

$$\begin{aligned} & g(\nabla\,\varphi^i, \nabla\,|{\rm d}\varphi|^2) \\ &= g^{\alpha\beta}{\varphi_\alpha}^i|{\rm d}\varphi|^2_\beta \\ &= g^{11}\tfrac{\partial\varphi^i}{\partial x_1}\tfrac{\partial|{\rm d}\varphi|^2}{\partial x_1} + g^{22}\tfrac{\partial\varphi^i}{\partial x_2}\tfrac{\partial|{\rm d}\varphi|^2}{\partial x_2} + g^{33}\tfrac{\partial\varphi^i}{\partial x_3}\tfrac{\partial|{\rm d}\varphi|^2}{\partial x_3} + g^{23}\tfrac{\partial\varphi^i}{\partial x_2}\tfrac{\partial|{\rm d}\varphi|^2}{\partial x_3} + g^{32}\tfrac{\partial\varphi^i}{\partial x_3}\tfrac{\partial|{\rm d}\varphi|^2}{\partial x_2} \\ &= 2a_{i1}(\sum_{i=1}^n a_{i3}^2x + \sum_{i=1}^n a_{i2}a_{i3}) = 0, \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{32}$$

Solving Equation (32) we have $a_{i1} = 0$, for $i = 1, 2, \ldots, n$, or $a_{i3} = 0$, for $i = 1, 2, \ldots, n$, from which we conclude that the linear map $\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}^n$ $(n \geq 2)$ defined by (29) is ∞-harmonic if and only if

$$\varphi(X) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ \cdots & \cdots & \cdots \\ 0 & a_{n2} & a_{n3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{33}$$

or

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{34}$$

Thus, we obtain the theorem. □

Remark 3. (*i*) We remark that in both cases, the maximum possible rank of the linear ∞-harmonic map φ is 2.
(*ii*) Using the energy density formula (30) we can check that in case of (33) the linear ∞-harmonic map φ has non-constant energy density given by a quadratic polynomial whilst in case of (34) the linear ∞-harmonic map φ has constant energy density.
(*iii*) It follows from our Theorem that we can choose to have submersion $\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}^2$ with

$$\varphi(X) = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{35}$$

so that φ has non-constant energy density. Clearly, φ cannot be a Riemannian submersion because the energy density is not constant.

Theorem 5.2. *A linear map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Nil})$ into Nil space is ∞-harmonic if and only if φ is a composition of a linear map $\mathbb{R}^m \longrightarrow \mathbb{R}^2$ followed by the inclusion map $i_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $i_1(y, z) = (0, y, z)$, or, φ is a composition of a linear map $\mathbb{R}^m \longrightarrow \mathbb{R}^2$ followed by the inclusion map $i_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $i_2(x, z) = (x, 0, z)$.*

Proof. Let $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Nil})$ be a linear map with

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ a_{31} & a_{32} & \ldots & a_{3m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}. \tag{36}$$

We can check that the energy density of φ is given by:

$$\begin{aligned} |\mathrm{d}\varphi|^2 &= \delta^{ij}\varphi_i^\alpha \varphi_j{}^\beta g_{\alpha\beta} \circ \varphi \\ &= \sum_{j=1}^m \left(\sum_{\alpha=1}^3 (\tfrac{\partial \varphi^\alpha}{\partial x_j})^2 g_{\alpha\alpha} \circ \varphi + \tfrac{\partial \varphi^2}{\partial x_j}\tfrac{\partial \varphi^3}{\partial x_j} g_{23} \circ \varphi + \tfrac{\partial \varphi^3}{\partial x_j}\tfrac{\partial \varphi^2}{\partial x_j} g_{32} \circ \varphi \right) \\ &= \sum_{j=1}^m \left(a_{1j}^2 + a_{2j}^2(1 + x^2) + a_{3j}^2 - a_{2j}a_{3j}x - a_{3j}a_{2j}x \right) \\ &= \sum_{j=1}^m a_{2j}^2 x^2 - 2\sum_{j=1}^m a_{2j}a_{3j}x + \sum_{i=1}^2 \sum_{j=1}^3 a_{ij}^2, \end{aligned} \tag{37}$$

where $x = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m$. Noting that the domain manifold is Euclidean space we have

$$\nabla \varphi^i = (a_{i1},\, a_{i2}, \ldots, a_{im}),\ i = 1, 2, 3,$$

and

$$\nabla |\mathrm{d}\varphi|^2 = \frac{\partial |\mathrm{d}\varphi|^2}{\partial x_i}\frac{\partial}{\partial x_i} = (2a_{11}(\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j}), \ldots, 2a_{1m}(\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j})).$$

By Corollary 1.3, the ∞-harmonic map equation for φ becomes

$$\langle \nabla \varphi^i, \nabla |\mathrm{d}\varphi|^2 \rangle = 0, \qquad i = 1, 2, 3, \tag{38}$$

which is equivalent to

$$\begin{aligned}
&2a_{i1}a_{11}(\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j}) + 2a_{i2}a_{12}(\textstyle\sum_{j=1}^{m} a_{2j}^2 x - \displaystyle\sum_{j=1}^{m} a_{2j}a_{3j})\\
&+ \ldots + 2a_{im}a_{1m}(\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j})\\
&= 2\sum_{k=1}^{m} a_{ik}a_{1k}(\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j}) = 0, \qquad i = 1, 2, 3,
\end{aligned}$$

or

$$\begin{cases} 2\sum_{k=1}^{m} a_{1k}^2(\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j}) &= 0;\\ 2\sum_{k=1}^{m} a_{1k}a_{2k}(\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j}) &= 0;\\ 2\sum_{k=1}^{m} a_{1k}a_{3k}(\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j}) &= 0. \end{cases} \tag{39}$$

Solving (39) we obtain

$$\begin{cases} 2\sum_{k=1}^{m} a_{1k}^2 = 0;\\ 2\sum_{k=1}^{m} a_{1k}a_{2k} = 0;\\ 2\sum_{k=1}^{m} a_{1k}a_{3k} = 0. \end{cases} \tag{40}$$

or

$$\sum_{j=1}^{m} a_{2j}^2 x - \sum_{j=1}^{m} a_{2j}a_{3j} = 0. \tag{41}$$

From Equation (40) we have

$$a_{1k} = 0, \text{ for } k = 1, 2, \ldots, m. \tag{42}$$

Note that if $a_{1k} \neq 0$, for some $k = 1, 2, \ldots, m$, then $x = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m : \mathbb{R}^m \longrightarrow \mathbb{R}$ is an onto map and it follows that Equation (41) is true for any x as a polynomial in x. Therefore, we have

$$a_{2k} = 0, \ for\ k = 1, 2, \ldots, m, \tag{43}$$

from which we conclude that the linear map $\varphi : (\mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Nil})$ defined by (36) is ∞-harmonic if and only if

$$\varphi(X) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \tag{44}$$

or

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & 0 & \dots & 0 \\ a_{31} & a_{32} & \dots & a_{3m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \tag{45}$$

Thus, we obtain the theorem. □

Remark 4. (i) We remark that in both cases, the maximum possible rank of the linear ∞-harmonic map φ is 2.
(ii) Using the energy density formula (37) can check that in both cases of (44) and (45), the linear ∞-harmonic map φ has constant energy density.

6. Linear ∞-harmonic maps from and into Sol space

In this section we give a complete classification of linear ∞-harmonic maps between Euclidean spaces and Sol space.

Theorem 6.1. *Let $(\mathbb{R}^3, g_{Sol})$ denote Sol space, where the metric with respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$ is given by $g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2$. Then*

1. *A linear function $f : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R},\ f(x, y, z) = Ax + By + Cz$ is an ∞-harmonic function if and only if $C = 0$ or $A = B = 0$.*
2. *A linear map $\varphi : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^n$ $(n \geq 2)$ is ∞-harmonic if and only if φ is a composition of the projection $\pi_1 : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^2,\ \pi_1(x, y, z) = (x, y)$ followed by a linear map $\mathbb{R}^2 \longrightarrow \mathbb{R}^n$, or, φ is a composition of the projection $\pi_2 : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^2,\ \pi_2(x, y, z) = (z)$ followed by a linear map $\mathbb{R} \longrightarrow \mathbb{R}^n$.*

Proof. The Statement (1) is proved in [Ou1]. To prove Statement (2), one can easily compute the following components of Sol metric:

$$\begin{aligned} &g_{11} = e^{2z},\ g_{22} = e^{-2z},\ g_{33} = 1,\ \text{all other},\ g_{ij} = 0; \\ &g^{11} = e^{-2z},\ g^{22} = e^{2z},\ g^{33} = 1,\ \text{all other},\ g^{ij} = 0. \end{aligned} \tag{46}$$

Let $\varphi : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^n$ $(n \geq 2)$ be a linear map with

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{47}$$

A straightforward computation gives the energy density of φ as:

$$\begin{aligned} &|\mathrm{d}\varphi|^2 = g^{\alpha\beta}{\varphi_\alpha}^i {\varphi_\beta}^j \delta_{ij} \\ &= \sum_{i=1}^n g^{11}(\tfrac{\partial \varphi^i}{\partial x_1})^2 + g^{22}(\tfrac{\partial \varphi^i}{\partial x_2})^2 + g^{33}(\tfrac{\partial \varphi^i}{\partial x_3})^2 \\ &= \sum_{i=1}^n a_{i1}^2 e^{-2z} + \sum_{i=1}^n a_{i2}^2 e^{2z} + \sum_{i=1}^n a_{i3}^2, \end{aligned}$$

and

$$\begin{aligned} &\tfrac{\partial |\mathrm{d}\varphi|^2}{\partial x_1} = \tfrac{\partial |\mathrm{d}\varphi|^2}{\partial x} = 0, \\ &\tfrac{\partial |\mathrm{d}\varphi|^2}{\partial x_2} = \tfrac{\partial |\mathrm{d}\varphi|^2}{\partial y} = 0, \\ &\tfrac{\partial |\mathrm{d}\varphi|^2}{\partial x_3} = \tfrac{\partial |\mathrm{d}\varphi|^2}{\partial z} = -2\sum_{i=1}^n a_{i1}^2 e^{-2z} + 2\sum_{i=1}^n a_{i2}^2 e^{2z}. \end{aligned} \tag{48}$$

It follows from Corollary 1.3 and (48) that φ is ∞-harmonic if and only if

$$\begin{aligned} &g(\nabla \varphi^i, \nabla |\mathrm{d}\varphi|^2) \\ &= g^{\alpha\beta}{\varphi_\alpha}^i |\mathrm{d}\varphi|^2_\beta \\ &= g^{11}\tfrac{\partial \varphi^i}{\partial x_1}\tfrac{\partial |\mathrm{d}\varphi|^2}{\partial x_1} + g^{22}\tfrac{\partial \varphi^i}{\partial x_2}\tfrac{\partial |\mathrm{d}\varphi|^2}{\partial x_2} + g^{33}\tfrac{\partial \varphi^i}{\partial x_3}\tfrac{\partial |\mathrm{d}\varphi|^2}{\partial x_3} \\ &= -2a_{i3}(\sum_{i=1}^n a_{i1}^2 e^{-2z} - \sum_{i=1}^n a_{i2} e^{2z}) = 0, \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{49}$$

Solving Equation (49) we have $a_{i3} = 0$, for $i = 1, 2, \ldots, n$, or $a_{i1} = a_{i2} = 0$, for $i = 1, 2, \ldots, n$, from which we conclude that the linear map $\varphi : (\mathbb{R}^3, g_{Sol}) \longrightarrow \mathbb{R}^n$ $(n \geq 2)$ defined by (47) is ∞-harmonic if and only if

$$\varphi(X) = \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ \cdots & \cdots & \cdots \\ 0 & 0 & a_{n3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{50}$$

or

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{51}$$

Thus, we obtain the theorem. □

Remark 5. It follows from Theorem 6.1 that the maximum rank of the linear ∞-harmonic maps from Sol space into Euclidean space is 2. In case of (51) the linear ∞-harmonic map has non-constant energy density.

Theorem 6.2. *A linear map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Sol})$ is ∞-harmonic if and only if it is a composition of a linear map $\mathbb{R}^m \longrightarrow \mathbb{R}^2$ followed by the inclusion map $i_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $i_1(x,y) = (x,y,0)$, or, φ is a composition of a linear map $\mathbb{R}^m \longrightarrow \mathbb{R}$ followed by an inclusion map $i_2 : \mathbb{R} \longrightarrow \mathbb{R}^3$, $i_2(z) = (0,0,z)$.*

Proof. For the linear map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Sol})$ with

$$\varphi(X) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \tag{52}$$

we have:

$$\nabla \varphi^i = (a_{i1},\, a_{i2}, \dots, a_{im}),\ i = 1,2,3,$$

and the energy density

$$\begin{aligned} |\mathrm{d}\varphi|^2 &= \delta^{ij} \varphi_i^\alpha {\varphi_j}^\beta g_{\alpha\beta} \circ \varphi \\ &= \sum_{j=1}^m \sum_{\alpha=1}^3 (\tfrac{\partial \varphi^\alpha}{\partial x_j})^2 g_{\alpha\alpha} \circ \varphi \\ &= \sum_{j=1}^m a_{1j}^2 e^{2z} + \sum_{j=1}^m a_{2j}^2 e^{-2z} + \sum_{j=1}^m a_{3j}^2, \end{aligned}$$

where $z = a_{31}x_1 + a_{32}x_2 + \ldots + a_{3m}x_m$. Since the domain manifold in a Euclidean space, tt follows from Corollary 1.3 that the ∞-harmonic map equation for φ becomes

$$\langle \nabla \varphi^i, \nabla \left|\mathrm{d}\varphi\right|^2 \rangle = 0, \qquad i = 1,2,3, \tag{53}$$

which is equivalent to which is equivalent to

$$\begin{aligned} &2a_{i1}a_{31}(\sum_{j=1}^m a_{1j}^2 e^{2z} - \sum_{j=1}^m a_{2j}^2 e^{-2z}) + 2a_{i2}a_{32}(\sum_{j=1}^m a_{1j}^2 e^{2z} - \sum_{j=1}^m a_{2j}^2 e^{-2z}) \\ &+ \ldots + 2a_{im}a_{3m}(\sum_{j=1}^m a_{1j}^2 e^{2z} - \sum_{j=1}^m a_{2j}^2 e^{-2z}) \\ &= 2\sum_{k=1}^m a_{ik}a_{3k}(\sum_{j=1}^m a_{1j}^2 e^{2z} - \sum_{j=1}^m a_{2j}^2 e^{-2z}) = 0, \qquad i = 1,2,3, \end{aligned} \tag{54}$$

or

$$\begin{cases} 2\sum_{k=1}^{m} a_{1k}a_{3k}(\sum_{j=1}^{m} a_{1j}^2 e^{2z} - \sum_{j=1}^{m} a_{2j}^2 e^{-2z}) &= 0 \\ 2\sum_{k=1}^{m} a_{2k}a_{3k}(\sum_{j=1}^{m} a_{1j}^2 e^{2z} - \sum_{j=1}^{m} a_{2j}^2 e^{-2z}) &= 0 \\ 2\sum_{k=1}^{m} a_{3k}^2(\sum_{j=1}^{m} a_{1j}^2 e^{2z} - \sum_{j=1}^{m} a_{2j}^2 e^{-2z}) &= 0. \end{cases} \tag{55}$$

From this we have either

$$\begin{cases} 2\sum_{k=1}^{m} a_{1k}a_{3k} = 0; \\ 2\sum_{k=1}^{m} a_{2k}a_{3k} = 0; \\ 2\sum_{k=1}^{m} a_{3k}^2 = 0, \end{cases} \tag{56}$$

or

$$(\sum_{j=1}^{m} a_{1j}^2 e^{2z} - \sum_{j=1}^{m} a_{2j}^2 e^{-2z}) = 0. \tag{57}$$

Solving Equation (56) we have

$$a_{3k} = 0,\ for\ k = 1, 2, \ldots, m. \tag{58}$$

Solving Equation (57) we have

$$a_{1k} = a_{2k} = 0,\ for\ k = 1, 2, \ldots, m. \tag{59}$$

The Theorem follows from (58) and (59). □

Remark 6. It follows from Theorem 6.2 that the maximum rank of the linear ∞-harmonic maps from Euclidean space into Sol space is 2, and any linear ∞-harmonic map into Sol space has constant energy density.

7. Quadratic ∞-harmonic maps into Sol and Nil spaces

Theorem 7.1. *Let $(\mathbb{R}^3, g_{Sol})$ denote Sol space, where the metric with respect to the standard coordinates (x, y, z) in $\mathbb{R}^3$ is given by $g_{Sol} = e^{2z}\mathrm{d}x^2 + e^{-2z}\mathrm{d}y^2 + \mathrm{d}z^2$. Then, a quadratic map $\varphi : \mathbb{R}^m \longrightarrow (\mathbb{R}^3, g_{Sol})$ with $\varphi(X) = (X^tA_1X, X^tA_2X, X^tA_3X)$ is an ∞-harmonic map if and only if it is a constant map.*

Proof. One can easily compute the following components of Sol metric:

$$h_{11} = e^{2z},\ h_{22} = e^{-2z},\ h_{33} = 1,\ \text{all other } h_{ij} = 0;$$
$$h^{11} = e^{-2z},\ h^{22} = e^{2z},\ h^{33} = 1,\ \text{all other } h^{ij} = 0.$$

A straightforward computation gives:

$$\nabla\varphi^i = 2X^tA_i,$$
$$|\mathrm{d}\varphi|^2 = \delta^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j h_{ij}\circ\varphi = \langle\nabla\varphi^i,\nabla\varphi^j\rangle h_{ij}\circ\varphi$$
$$= |\nabla\,\varphi^i|^2\, h_{ii}\circ\varphi = (4X^tA_1^2X)e^{2z} + (4X^tA_2^2X)e^{-2z} + 4X^tA_3^2X,$$

where $z = X^tA_3X$.

Furthermore,

$$\begin{aligned}&\nabla\,|\mathrm{d}\varphi|^2 = 8X^tA_1^2e^{2z} + 8X^tA_1^2Xe^{2z}\nabla z\\ &+8X^tA_2^2e^{-2z} - 8X^tA_2^2Xe^{-2z}\nabla z\\ &+8X^tA_3^2\\ &= 8X^tA_1^2e^{2z} + (16X^tA_1^2Xe^{2z})X^tA_3^2\\ &+8X^tA_2^2e^{-2z} - (16X^tA_2^2Xe^{-2z})X^tA_3^2\\ &+8X^tA_3^2,\end{aligned}$$

and

$$\begin{aligned}&g(\nabla\,\varphi^i,\nabla\,|\mathrm{d}\varphi|^2)\\ &= \langle 2X^tA_i, 8X^tA_1^2e^{2z} + (16X^tA_1^2Xe^{2z})X^tA_3^2 + 8X^tA_2^2e^{-2z} - (16X^tA_2^2Xe^{-2z})X^tA_3^2 + 8X^tA_3^2\rangle\\ &= 16(X^tA_iA_1^2X + 2X^tA_1^2XX^tA_iA_3X)e^{2z}\\ &+16(X^tA_iA_2^2X - 2X^tA_2^2XX^tA_iA_3X)e^{-2z}\\ &+16X^tA_iA_3^2X.\end{aligned}$$

It follows from the ∞-harmonic map equation (3) that φ is ∞-harmonic if and only if

$$\begin{aligned}&16(X^tA_iA_1^2X + X^tA_1^2XX^tA_iA_3X)e^{2z}\\ &+16(X^tA_iA_2^2X - X^tA_2^2XX^tA_iA_3X)e^{-2z}\\ &+16X^tA_iA_3^2X = 0, \quad \text{for all}\, X\in\mathbb{R}^m,\ \text{and}\ i=1,2,3.\end{aligned} \tag{60}$$

Note that Equation (60) is an identity of functions which are analytic. We can substitute the Taylor expansions for $e^{2X^tA_3X}$ and $e^{-2X^tA_3X}$ into (60) and compare the coefficients of the second degree terms to get

$$16(X^tA_i\sum_{j=1}^{3}A_j^2X) = 0, \qquad i=1,2,3.$$

From this we obtain

$$A_i(\sum_{j=1}^{3}A_j^2) + (\sum_{j=1}^{3}A_j^2)A_i = 0,$$

and Lemma 2.1 applies to complete the proof of the Theorem. □

Theorem 7.2. *A quadratic map $\varphi:\mathbb{R}^m\longrightarrow(\mathbb{R}^3, g_{Nil})$, $\varphi(X) = (X^tA_1X, X^tA_2X, X^tA_3X)$ into Nil space is ∞-harmonic if and only if it is a constant map.*

Proof. Using the components of Nil metric (28) with notations g_{ij} replaced by h_{ij} one can easily check that:

$$\begin{aligned}
&\nabla\varphi^i = 2X^tA_i,\\
&|\mathrm{d}\varphi|^2 = \delta^{\alpha\beta}{\varphi_\alpha}^i{\varphi_\beta}^j h_{ij}\circ\varphi\\
&= \textstyle\sum_{\alpha=1}^m {\varphi_\alpha}^i{\varphi_\alpha}^j h_{ij}\\
&= (4X^tA_2^2X)x^2 + 4\sum_{j=1}^3 X^tA_j^2X - 4(X^tA_2A_3 + A_3A_2X)x,
\end{aligned}$$

where $x = X^tA_1X$, and

$$\begin{aligned}
&\nabla\,|\mathrm{d}\varphi|^2 = \tfrac{\partial|\mathrm{d}\varphi|^2}{\partial x_i}\tfrac{\partial}{\partial x_i}\\
&= (8X^tA_2^2)x^2 + (16X^tA_2^2X)(X^tA_1)x + 8\sum_{j=1}^3 X^tA_j^2\\
&-8(X^tA_2A_3 + X^tA_3A_2)x - 8(X^tA_2A_3 + A_3A_2X)X^tA_1.
\end{aligned}$$

By Corollary 1.3, the ∞-harmonic map equation for φ reads

$$\begin{aligned}
&0 = \langle\nabla\,\varphi^i, \nabla\,|\mathrm{d}\varphi|^2\rangle\\
&= \langle 2X^tA_i, (8X^tA_2^2)x^2 + (16X^tA_2^2X)(X^tA_1)x + 8\sum_{j=1}^3 X^tA_j^2 - 8(X^tA_2A_3 + X^tA_3A_2)x\\
&-8(X^t(A_2A_3 + A_3A_2X)X^tA_1\rangle\\
&= 16X^tA_i\sum_{j=1}^3 A_j^2X + P(X), \qquad i = 1, 2, 3.
\end{aligned}$$

where $P(X)$ denotes a polynomial in X of degree greater than 2. Noting that the equation is an identity of polynomials we conclude that if φ is ∞-harmonic, then

$$X^tA_i\sum_{j=1}^3 A_j^2X = 0, \qquad i = 1, 2, 3, \tag{61}$$

which is the same as Equation (8) for $n = 3$ so the rest of the proof is exactly the same as the part in the proof of Theorem 2.2. □

Example 1. We remark that there are many polynomial ∞-harmonic maps $\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}^m$, for instance,

$$\varphi : (\mathbb{R}^3, g_{Nil}) \longrightarrow \mathbb{R}^2$$

with $\varphi(x, y, z) = (z - xy/2, 2z - xy)$ is an ∞-harmonic map which has nonconstant energy density $|\mathrm{d}\varphi|^2 = 5(1 + (x^2 + y^2)/4)$ (see [OTW] for details).

8. Holomorphic ∞-harmonic maps

In this section, we study ∞-harmonicity of holomorphic maps $\mathbb{C}^m \longrightarrow \mathbb{C}^n$. Let $(z_1, \ldots, z_m) \in \mathbb{C}^m$ and $(w_1, \ldots, w_n) \in \mathbb{C}^n$ with $z_j = x_j - iy_j$ $j = 1, \ldots, m$ and $w_\alpha = u_\alpha - iv_\alpha$ $\alpha = 1, \ldots, n$. Then, a map $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^n$, $\varphi(z_1, \ldots, z_m) = (\varphi^1, \ldots, \varphi^n)$ is associated to a map $\varphi : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{2n}$ with $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_m) = (u_1, \ldots, u_n, v_1, \ldots, v_n)$. We write the map as $\varphi(X + iY) = \phi(X, Y) + i\psi(X, Y)$, where $X = (x_1, \ldots, x_m), Y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and the maps $\phi(X, Y) = (u^1(X, Y), \ldots, u^n(X, Y))$ and $\psi(X, Y) = (v^1(X, Y), \ldots, v^n(X, Y))$ are called the real and imaginary parts of φ. We have

Theorem 8.1. *A holomorphic map $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ with $\varphi(X, +iY) = \phi(X, Y) + i\psi(X, Y)$ is ∞-harmonic if and only if its real and imaginary parts $\phi(X, Y)$ and $\psi(X, Y)$ are ∞-harmonic.*

Proof. It is well known that $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is holomorphic if and only if

$$\frac{\partial u^\alpha}{\partial x_j} = \frac{\partial v^\alpha}{\partial y_j}, \quad \frac{\partial u^\alpha}{\partial y_j} = -\frac{\partial v^\alpha}{\partial x_j}; \qquad j = 1, 2, \ldots, m,\ \alpha = 1, 2, \ldots, n. \tag{62}$$

We can easily check that

$$\begin{aligned} |\nabla u^\alpha|^2 &= |\nabla v^\alpha|^2, \\ |\mathrm{d}\varphi|^2 &= \delta^{ij}\varphi_i^\alpha\varphi_j^\beta\delta_{\alpha\beta} = \sum_{\alpha=1}^n |\nabla u^\alpha|^2 + |\nabla v^\alpha|^2 = 2\sum_{\alpha=1}^n |\nabla u^\alpha|^2 = 2\sum_{\alpha=1}^n |\nabla v^\alpha|^2, \\ \nabla\,|\mathrm{d}\varphi|^2 &= 2\sum_{\alpha=1}^n \nabla|\nabla u^\alpha|^2 = 2\sum_{\alpha=1}^n \nabla|\nabla v^\alpha|^2 \\ &= 2\nabla|\nabla\phi|^2 = 2\nabla|\nabla\psi|^2. \end{aligned}$$

Substitute these into the ∞-harmonic map Equation (3) we obtain that φ is ∞-harmonic if and only if

$$\begin{aligned} &2g(\nabla\,\phi^\alpha, \nabla|\nabla\phi|^2) = 0, \\ &2g(\nabla\,\psi^\alpha, \nabla|\nabla\psi|^2) = 0, \qquad \alpha = 1, 2, \ldots, n, \end{aligned} \tag{63}$$

which gives the Theorem. ∎

Remark 7. Explicitly, the ∞-harmonic map equation can be written as:

$$
\begin{cases}
2\Delta_\infty u_1 + \langle \nabla u_1, \nabla |\nabla u_2|^2\rangle + \ldots + \langle \nabla u_1, \nabla |\nabla u_n|^2\rangle = 0\\
\langle \nabla u_2, \nabla |\nabla u_1|^2\rangle + 2\Delta_\infty u_2 + \ldots + \langle \nabla u_2, \nabla |\nabla u_n|^2\rangle = 0\\
\cdots\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots\cdots\\
\langle \nabla u_n, \nabla |\nabla u_1|^2\rangle + \langle \nabla u_n, \nabla |\nabla u_2|^2\rangle + \ldots + +2\Delta_\infty u_n = 0\\
2\Delta_\infty v_1 + \langle \nabla v_1, \nabla |\nabla v_2|^2\rangle + \ldots + \langle \nabla v_1, \nabla |\nabla v_n|^2\rangle = 0\\
\langle \nabla v_2, \nabla |\nabla v_1|^2\rangle + 2\Delta_\infty v_2 + \ldots + \langle \nabla v_2, \nabla |\nabla v_n|^2\rangle = 0\\
\cdots\cdots\cdots \quad \cdots\cdots\cdots \quad \cdots\cdots\cdots\\
\langle \nabla v_n, \nabla |\nabla v_1|^2\rangle + \langle \nabla v_n, \nabla |\nabla v_2|^2\rangle + \ldots + 2\Delta_\infty v_n = 0.
\end{cases} \tag{64}
$$

Theorem 8.2. *Let $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}$ be a nonconstant holomorphic map. Then, φ is an ∞-harmonic map if and only if φ is a composition of an orthogonal projection $\mathbb{C}^m \longrightarrow \mathbb{C}$ followed by a homothety $\mathbb{C} \longrightarrow \mathbb{C}$, i.e., $\varphi(z_1, \ldots, z_m) = \lambda z_i + z_0$, where $\lambda \in \mathbb{R}, z_0 \in \mathbb{C}$ are constants.*

Proof. Notice that a holomorphic map $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}$ is automatically a horizontally weakly conformal harmonic map (see e.g., [BW]). It follows from the relationship among tension, p-tension, and ∞-tension fields of a map

$$
\tau_p(\varphi) = |\mathrm{d}\varphi|^{p-2}\tau_2(\varphi) + (p-2)|\mathrm{d}\varphi|^{p-4}\tau_\infty(\varphi) \tag{65}
$$

that if φ is also an ∞-harmonic map, then it must be p-harmonic for any p. In this case φ is a p-harmonic morphism (being horizontally weakly conformal and p-harmonic map) for any p. By a theorem in [BL], φ must be a horizontally homothetic. Now the classification of horizontally homothetic maps between Euclidean spaces [Ou1] applies to give the required results. □

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