Frobenius Manifolds as a Special Class of Submanifolds in Pseudo-Euclidean Spaces[1]

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Abstract

We introduce a very natural class of *potential* submanifolds in pseudo-Euclidean spaces (each N-dimensional potential submanifold is a special flat torsionless submanifold in a $2N$-dimensional pseudo-Euclidean space) and prove that each N-dimensional Frobenius manifold can be locally represented as an N-dimensional potential submanifold. We show that all potential submanifolds bear natural special structures of Frobenius algebras on their tangent spaces. These special Frobenius structures are generated by the corresponding flat first fundamental form and the set of the second fundamental forms of the submanifolds (in fact, the structural constants are given by the set of the Weingarten operators of the submanifolds). We prove that the associativity equations of two-dimensional topological quantum field theories are very natural reductions of the fundamental nonlinear equations of the theory of submanifolds in pseudo-Euclidean spaces and define locally the class of potential submanifolds. The problem of explicit realization of an arbitrary concrete Frobenius manifold as a potential submanifold in a pseudo-Euclidean space is reduced to solving a linear system of second-order partial differential equations. For concrete Frobenius manifolds, this realization problem can be solved explicitly in elementary and special functions. Moreover, we consider a nonlinear system, which is a natural generalization of the associativity equations, namely, the system describing all flat torsionless submanifolds in pseudo-Euclidean spaces, and prove that this system is integrable by the inverse scattering method. We prove that each flat torsionless submanifold in a pseudo-Euclidean space gives a nonlocal Hamiltonian operator of hydrodynamic type with flat metric, a special pencil of compatible Poisson structures, a recursion operator, infinite sets of integrals of hydrodynamic type in involution and a natural class of integrable hierarchies, which are all directly associated with this flat torsionless submanifold. In particular, using our construction of the reduction to the associativity equations, we obtain that each Frobenius manifold (in point of fact, each solution of the associativity equations) gives a natural nonlocal Hamiltonian operator of hydrodynamic type with flat metric, a natural pencil of compatible Poisson structures (local and nonlocal), a natural recursion operator, natural infinite sets of integrals of hydrodynamic type in involution and a natural class of integrable hierarchies, which are all directly associated with this Frobenius manifold.

Key words and phrases: Frobenius manifold, Frobenius algebra, symmetric algebra, N-parameter deformation of algebra, submanifold in pseudo-Euclidean space, flat submanifold, submanifold with flat normal bundle, flat submanifold with zero torsion, indefinite metric, associativity equations in two-dimensional topological quantum field theories, WDVV equations, topological quantum field theory, integrable nonlinear system, integrable hierarchy, bi-Hamiltonian system, nonlocal Hamiltonian operator of hydrodynamic type, system of hydrodynamic type, compatible Poisson brackets, Poisson pencil, system of integrals in involution, conservation law, recursion operator, pseudo-Riemannian geometry, potential submanifold.

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1 Introduction

We prove that the associativity equations of two-dimensional topological quantum field theories (the Witten–Dijkgraaf–Verlinde–Verlinde equations, see [1]–[4]) for a function (a *potential*) $\Phi = \Phi(u^1, \dots, u^N)$,

$$\sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k} \eta^{kl} \frac{\partial^3 \Phi}{\partial u^l \partial u^m \partial u^n} = \sum_{k=1}^{N}\sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^m \partial u^k} \eta^{kl} \frac{\partial^3 \Phi}{\partial u^l \partial u^j \partial u^n}, \tag{1.1}$$

where η^{ij} is an arbitrary constant nondegenerate symmetric matrix, $\eta^{ij} = \eta^{ji}$, $\eta^{ij} = \text{const}$, $\det(\eta^{ij}) \neq 0$, are very natural reductions of the fundamental nonlinear equations of the theory of submanifolds in pseudo-Euclidean spaces (namely, the Gauss equations, the Codazzi equations and the Ricci equations) and give locally a very natural class of *potential* submanifolds in pseudo-Euclidean spaces. Each N-dimensional potential submanifold is a special flat torsionless submanifold in a $2N$-dimensional pseudo-Euclidean space. All potential submanifolds in pseudo-Euclidean spaces bear natural special structures of Frobenius algebras on their tangent spaces. These special Frobenius structures are generated by the corresponding flat first fundamental form and the set of the second fundamental forms of the submanifolds (in fact, the structural constants are given by the set of the Weingarten operators of the submanifolds).

We recall that the associativity equations (1.1) are consistent and integrable by the inverse scattering method, they possess a rich set of nontrivial solutions, and each solution $\Phi(u^1, \dots, u^N)$ of the associativity equations (1.1) gives N-parameter deformations of special Frobenius algebras (some special commutative associative algebras equipped with nondegenerate invariant symmetric bilinear forms) (see [1]). Indeed, consider algebras $A(u)$ in an N-dimensional vector space with the basis $e_1, \dots, e_N$ and the multiplication (see [1])

$$e_i \circ e_j = c^k_{ij}(u) e_k, \qquad c^k_{ij}(u) = \eta^{ks} \frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j}. \tag{1.2}$$

For all values of the parameters $u = (u^1, \dots, u^N)$ the algebras $A(u)$ are commutative, $e_i \circ e_j = e_j \circ e_i$, and the associativity condition

$$(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k) \tag{1.3}$$

in the algebras $A(u)$ is equivalent to equations (1.1). The matrix η_{ij} inverse to the matrix η^{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$, defines a nondegenerate invariant symmetric bilinear form on the algebras $A(u)$,

$$\langle e_i, e_j \rangle = \eta_{ij}, \qquad \langle e_i \circ e_j, e_k \rangle = \langle e_i, e_j \circ e_k \rangle. \tag{1.4}$$

Recall that locally the tangent space at every point of any Frobenius manifold (see [1]) possesses the structure of Frobenius algebra (1.2)–(1.4), which is determined by a solution of the associativity equations (1.1) and smoothly depends on the point. We prove that each N-dimensional Frobenius manifold can be locally represented as an N-dimensional potential flat torsionless submanifold in a $2N$-dimensional pseudo-Euclidean space. The problem of explicit realization of an arbitrary concrete N-dimensional Frobenius manifold as an N-dimensional potential flat torsionless submanifold in a $2N$-dimensional pseudo-Euclidean space is reduced to solving a linear system of second-order partial differential equations. For concrete Frobenius manifolds, this realization problem can be solved explicitly in elementary and special functions. We shall give many explicit important examples of these realizations in a separate paper.

Moreover, we consider a nonlinear system, which is a natural generalization of the associativity equations (1.1), namely, the system describing all flat torsionless submanifolds in pseudo-Euclidean spaces, and prove that this system is integrable by the inverse scattering method. We also prove that each flat torsionless submanifold in a pseudo-Euclidean space gives a natural nonlocal Hamiltonian operator of hydrodynamic type with flat metric, a natural special pencil of compatible Poisson structures, a natural recursion operator, natural infinite sets of integrals of hydrodynamic type in involution and a natural class of integrable hierarchies, which are all directly associated with this flat torsionless submanifold. In particular, using our construction of the reduction to the associativity equations, we obtain that each Frobenius manifold (in point of fact, each solution of the associativity equations (1.1)) gives a natural nonlocal Hamiltonian operator of hydrodynamic type with flat metric, a natural special pencil of compatible Poisson structures (local and nonlocal), a natural recursion operator, natural infinite sets of integrals of hydrodynamic type in involution and a natural class of integrable hierarchies, which are all directly associated with this Frobenius manifold.

2 Frobenius algebras, Frobenius manifolds and associativity equations

2.1 Frobenius and symmetric algebras

Recall the notion of Frobenius algebra over a field $\mathbb{K}$ (in this paper we consider Frobenius algebras only over $\mathbb{R}$ or $\mathbb{C}$). First of all, we must note that there are various conventional definitions of Frobenius algebras. In particular, sometimes in mathematical literature a finite dimensional algebra $\mathcal{A}$ (with multiplication $\circ$) over a field $\mathbb{K}$ is called *Frobenius* if it is equipped with a linear functional

$$\theta : \mathcal{A} \to \mathbb{K} \tag{2.1}$$

such that if $\theta(a \circ b) = 0$ for all $a \in \mathcal{A}$, then $b = 0$. In this case, $\operatorname{Ker} \theta$ contains no nontrivial ideals. It is also obvious that the bilinear form $f(a, b) = \theta(a \circ b)$ is nondegenerate for every such linear functional in any finite dimensional algebra. If algebra is associative, then we have

$$f(a \circ b, c) = \theta((a \circ b) \circ c) = \theta(a \circ (b \circ c)) = f(a, b \circ c) \tag{2.2}$$

for all $a, b, c \in \mathcal{A}$ (*invariance* or *associativity* of bilinear form).

Definition 2.1 A bilinear form $f : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ in an algebra $\mathcal{A}$ is called *invariant* (or *associative*) if

$$f(a \circ b, c) = f(a, b \circ c) \tag{2.3}$$

for all $a, b, c \in \mathcal{A}$.

Consider the following conventional general definition of Frobenius algebra.

Definition 2.2 A finite dimensional algebra $\mathcal{A}$ over a field $\mathbb{K}$ is called *Frobenius* if it is equipped with a nondegenerate invariant bilinear form.

Generally speaking, even associativity of algebra is not assumed here (we note that often some of the following additional conditions are included in definition of Frobenius algebras: symmetry of invariant bilinear form, presence of a unit in algebra, associativity of algebra, and commutativity of algebra).

Consider an arbitrary Frobenius algebra $(\mathcal{A}, f)$, an arbitrary element $w \in \mathcal{A}$ and the corresponding linear functional $\theta_w(a) = f(a, w)$ in $\mathcal{A}$. Then we have $\theta_w(a \circ b) = f(a \circ b, w) = f(a, b \circ w)$. Therefore, if $\theta_w(a \circ b) = f(a, b \circ w) = 0$ for all $a \in \mathcal{A}$, then $b \circ w = 0$. If w is an element of algebra $\mathcal{A}$ such that $b \circ w = 0$ implies $b = 0$, then $\theta_w(a)$ is a linear functional of type (2.1) and $\operatorname{Ker} \theta_w$ contains no ideals. For example, if algebra contains a unit e, then the unit e gives a linear functional of type (2.1), $\theta_e(a) = f(a, e)$, and $\operatorname{Ker} \theta_e$ contains no ideals. Moreover, for any algebra with a unit e, any invariant bilinear form f is completely generated by the linear functional $\theta_e(a) = f(a, e)$, since $f(a, b) = f(a, b \circ e) = f(a \circ b, e) = \theta_e(a \circ b)$.

Example 2.1 Matrix algebra $M_n(\mathbb{K})$.

Consider the algebra $M_n(\mathbb{K})$ of $n \times n$ matrices over a field $\mathbb{K}$, the linear functional (trace of matrices)

$$\theta(a) = \mathrm{Tr}\,(a),\ \ a \in M_n(\mathbb{K}),$$

and the bilinear form $f(a,b) = \theta(ab)$. The bilinear form is invariant, since the matrix algebra is associative. It is easy to prove that the bilinear form is nondegenerate, and $(M_n(\mathbb{K}), f)$ is a noncommutative associative Frobenius algebra with a unit over $\mathbb{K}$. Note that the bilinear form $f(a,b) = \theta(ab)$ is symmetric, $\theta(ab) = \theta(ba)$. Recall that a finite dimensional associative algebra with a unit over a field $\mathbb{K}$ is called *symmetric* if it is equipped with a symmetric nondegenerate associative bilinear form (see [22]). Therefore, $(M_n(\mathbb{K}), f)$ is a symmetric algebra.

Example 2.2 Group algebra $\mathbb{K}G$.

Let G be a finite group. Consider the group algebra $\mathbb{K}G$ over a field $\mathbb{K}$,

$$\mathbb{K}G = \{a \mid a = \sum_{g\in G} \alpha_g g,\ \ \alpha_g \in \mathbb{K}\}.$$

$\mathbb{K}G$ is an associative algebra with a unit over $\mathbb{K}$. Let e be the unit of the group G. Consider the linear functional

$$\theta(a) = \alpha_e(a), \quad a = \sum_{g\in G} \alpha_g(a) g \in \mathbb{K}G, \quad \alpha_g(a) \in \mathbb{K},$$

and the bilinear form $f(a,b) = \theta(ab)$. The bilinear form is invariant, since the group algebra is associative. It is easy to prove that the bilinear form is nondegenerate. Indeed, we have

$$f(g^{-1}, a) = \theta(g^{-1}a) = \alpha_g(a)$$

for all $g \in G$. Therefore, if $f(g,a) = \theta(ga) = 0$ for all $g \in G$, then $\alpha_g(a) = 0$ for all $g \in G$, i.e., $a = 0$. Hence the bilinear form f is nondegenerate, and $(\mathbb{K}G, f)$ is a noncommutative associative Frobenius algebra with a unit over $\mathbb{K}$ (it is commutative only for Abelian groups). Note that the bilinear form $f(a,b) = \theta(ab)$ is symmetric for any group G, $\theta(ab) = \theta(ba)$. Therefore, $(\mathbb{K}G, f)$ is a symmetric algebra.

2.2 Frobenius manifolds

Consider an N-dimensional pseudo-Riemannian manifold M with a metric g and a structure of Frobenius algebra $(T_uM, \circ, g)$, $T_uM \times T_uM \overset{\circ}{\to} T_uM$, on each tangent space T_uM at any point $u \in M$ smoothly depending on the point such that the metric g is the corresponding nondegenerate invariant symmetric bilinear form on each tangent space T_uM, $g(X \circ Y, Z) = g(X, Y \circ Z)$, where X, Y and Z are arbitrary vector fields on M.

This class of pseudo-Riemannian manifolds equipped with Frobenius structures could be naturally called Frobenius, but in this paper we shall consider well-known and generally accepted Dubrovin's definition of Frobenius manifolds [1], which is motivated by two-dimensional topological quantum field theories and quantum cohomology and imposes very severe additional constraints on Frobenius structures of Frobenius manifolds.

Definition 2.3 (Dubrovin [1])

An N-dimensional pseudo-Riemannian manifold M with a metric g and a structure of Frobenius algebra $(T_uM, \circ, g)$, $T_uM \times T_uM \overset{\circ}{\to} T_uM$, on each tangent space T_uM at any point $u \in M$ smoothly depending on the point is called *Frobenius* if

(1) the metric g is a nondegenerate invariant symmetric bilinear form on each tangent space T_uM,

$$g(X \circ Y, Z) = g(X, Y \circ Z), \tag{2.4}$$

(2) the Frobenius algebra is commutative,

$$X \circ Y = Y \circ X \tag{2.5}$$

for all vector fields X and Y on M,

(3) the Frobenius algebra is associative,

$$(X \circ Y) \circ Z = X \circ (Y \circ Z) \tag{2.6}$$

for all vector fields X, Y and Z on M,

(4) the metric g is flat,

(5) $A(X,Y,Z) = g(X \circ Y, Z)$ is a symmetric tensor on M (it is obvious that, by virtue of (1) and (2), we have $g(X \circ Y, Z) = g(X, Y \circ Z) = g(Y \circ Z, X) = g(Y, Z \circ X) = g(Z \circ X, Y) = g(Z, X \circ Y) = g(Z, Y \circ X) = g(Z \circ Y, X) = g(X, Z \circ Y) = g(X \circ Z, Y) = g(Y, X \circ Z) = g(Y \circ X, Z)$) such that the tensor $(\nabla_W A)(X,Y,Z)$ is symmetric with respect to all vector fields X, Y, Z and W on M (∇ is the covariant differentiation generated by the Levi-Civita connection of the metric g),

(6) the Frobenius algebra possesses a unit, and the unit vector field U, for which $X \circ U = U \circ X = X$ for each vector field X on M, is covariantly constant, i.e.,

$$\nabla U = 0, \tag{2.7}$$

where ∇ is the covariant differentiation generated by the Levi-Civita connection of the metric g,

(7) the manifold M is equipped with a vector field E (*Euler vector field*) such that

$$\nabla \nabla E = 0, \tag{2.8}$$

$$\mathcal{L}_E (X \circ Y) - (\mathcal{L}_E X) \circ Y - X \circ (\mathcal{L}_E Y) = X \circ Y, \tag{2.9}$$

$$\mathcal{L}_E \, g(X,Y) - g(\mathcal{L}_E X, Y) - g(X, \mathcal{L}_E Y) = K \, g(X,Y), \tag{2.10}$$

$$\mathcal{L}_E U = -U, \tag{2.11}$$

where K is an arbitrary fixed constant, $\mathcal{L}_E$ is the Lie derivative along the Euler vector field, and ∇ is the covariant differentiation generated by the Levi-Civita connection of the metric g.

A beautiful theory of these very special Frobenius structures and Frobenius manifolds and many important examples were constructed by Dubrovin in connection with two-dimensional topological quantum field theories and quantum cohomology [1]. No doubt that these very special Frobenius structures and Frobenius manifolds should be called Dubrovin's. A lot of very important examples of Frobenius manifolds arises in the theory of Gromov–Witten invariants, the quantum cohomology, the singularity theory, the enumerative geometry, the topological field theories and the modern differential geometry, mathematical and theoretical physics.

In this paper we describe a very natural special class of submanifolds in pseudo-Euclidean spaces bearing natural Frobenius structures satisfying the conditions (1)–(5), namely, the class of potential submanifolds. Moreover, we show that each manifold satisfying the conditions (1)–(5) can be locally realized as a potential submanifold in a pseudo-Euclidean space [5]–[7]. For any concrete Frobenius structure satisfying the conditions (1)–(5) and for any given Frobenius manifold, the corresponding realization problem is reduced to solving a system of linear second-order partial differential equations.

2.3 Associativity equations

Consider an arbitrary manifold satisfying the conditions (1)–(5). Let $u = (u^1, \dots, u^N)$ be arbitrary flat coordinates of the flat metric g. In flat local coordinates, the metric $g(u)$ is a constant nondegenerate symmetric matrix η_{ij}, $\eta_{ij} = \eta_{ji}$, $\det(\eta_{ij}) \neq 0$, $\eta_{ij} =$ const, $g(X, Y) = \eta_{ij} X^i(u) Y^j(u)$.

In these flat local coordinates, for structural functions $c^i_{jk}(u)$ of the Frobenius structure on the manifold,

$$X \circ Y = W, \quad W^i(u) = c^i_{jk}(u) X^j(u) Y^k(u),$$

and for the symmetric tensor $A_{ijk}(u)$, we have

$$A(X, Y, Z) = A_{ijk}(u) X^i(u) Y^j(u) Z^k(u) = g(X \circ Y, Z) =$$
$$= g(W, Z) = \eta_{ij} W^i(u) Z^j(u) = \eta_{ij} c^i_{kl}(u) X^k(u) Y^l(u) Z^j(u).$$

Therefore,

$$A_{ijk}(u) = \eta_{sk} c^s_{ij}(u). \tag{2.12}$$

According to (5) $(\nabla_l A_{ijk})(u)$ is a symmetric tensor, i.e., in the flat local coordinates we also have

$$\frac{\partial A_{ijk}}{\partial u^l} = \frac{\partial A_{ijl}}{\partial u^k}.$$

Hence there locally exist functions $B_{ij}(u)$ such that

$$A_{ijk}(u) = \frac{\partial B_{ij}}{\partial u^k}.$$

We can consider that the matrix $B_{ij}(u)$ is symmetric, $B_{ij}(u) = B_{ji}(u)$. Indeed, if

$$A_{ijk}(u) = \frac{\partial \widetilde{B}_{ij}}{\partial u^k},$$

then

$$\frac{\partial \widetilde{B}_{ij}}{\partial u^k} = \frac{\partial \widetilde{B}_{ji}}{\partial u^k}$$

for any k, since the tensor $A_{ijk}(u)$ is symmetric. Hence, $\widetilde{B}_{ij}(u) = \widetilde{B}_{ji}(u) + C_{ij}$, where $C_{ij} =$ const, $C_{ij} = -C_{ji}$. Thus, if we take $B_{ij}(u) = \widetilde{B}_{ij}(u) - (1/2) C_{ij}$, then $B_{ij}(u) = B_{ji}(u)$ and

$$A_{ijk}(u) = \frac{\partial B_{ij}}{\partial u^k}.$$

Since the tensor $A_{ijk}(u)$ is symmetric, we have also

$$\frac{\partial B_{ij}}{\partial u^k} = \frac{\partial B_{ik}}{\partial u^j}.$$

Hence there locally exist functions $F_i(u)$ such that

$$B_{ij}(u) = \frac{\partial F_i}{\partial u^j}.$$

Since the matrix $B_{ij}(u)$ is symmetric, we have

$$\frac{\partial F_i}{\partial u^j} = \frac{\partial F_j}{\partial u^i}.$$

Hence there locally exist a function (a *potential*) $\Phi(u)$ such that

$$F_i(u) = \frac{\partial \Phi}{\partial u^i}.$$

Thus,

$$A_{ijk}(u) = \frac{\partial B_{ij}}{\partial u^k} = \frac{\partial^2 F_i}{\partial u^j \partial u^k} = \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k}.$$

From (2.12) for the structural functions $c^i_{jk}(u)$ we have

$$c^i_{jk}(u) = \eta^{is} A_{sjk}(u) = \eta^{is} \frac{\partial^3 \Phi}{\partial u^s \partial u^j \partial u^k}, \tag{2.13}$$

where the matrix η^{ij} is inverse to the matrix η_{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$.

For any values of the parameters $u = (u^1, \ldots, u^N)$, the structural functions (2.13) give a commutative Frobenius algebra

$$\partial_i \circ \partial_j = c^k_{ij}(u)\partial_k = \eta^{ks} \frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j} \partial_k \tag{2.14}$$

equipped with a symmetric invariant nondegenerate bilinear form

$$\langle \partial_i, \partial_j \rangle = \eta_{ij} \tag{2.15}$$

for any constant nondegenerate symmetric matrix η_{ij} and for any function $\Phi(u)$, but, generally speaking, this algebra is not associative. All the conditions (1)–(5) except the associativity condition (3) are obviously satisfied for all these N-parameter deformations of nonassociative Frobenius algebras.

The associativity condition (3) is equivalent to a nontrivial overdetermined system of nonlinear partial differential equations for the potential $\Phi(u)$,

$$\sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k} \eta^{kl} \frac{\partial^3 \Phi}{\partial u^l \partial u^m \partial u^n} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\partial^3 \Phi}{\partial u^i \partial u^m \partial u^k} \eta^{kl} \frac{\partial^3 \Phi}{\partial u^l \partial u^j \partial u^n}, \tag{2.16}$$

which is well known as the associativity equations of two-dimensional topological quantum field theories (the Witten–Dijkgraaf–Verlinde–Verlinde or the WDVV equations, see [1]–[4]); it is consistent, integrable by the inverse scattering method and possesses a rich set of nontrivial solutions (see [1]).

It is obvious that each solution $\Phi(u^1, \ldots, u^N)$ of the associativity equations (2.16) gives N-parameter deformations of commutative associative Frobenius algebras (2.14) equipped with nondegenerate invariant symmetric bilinear forms (2.15). These Frobenius structures satisfy to all the conditions (1)–(5).

Further in this paper we show that the associativity equations (2.16) are very natural reductions of the fundamental nonlinear equations of the theory of submanifolds in pseudo-Euclidean spaces and give a natural class of *potential* flat torsionless submanifolds [5]–[7]. All potential flat torsionless submanifolds in pseudo-Euclidean spaces bear natural structures of Frobenius algebras (2.14), (2.15) on their tangent spaces. These Frobenius structures are generated by the corresponding flat first fundamental form and the set of the second fundamental forms of the submanifolds.

3 Gauss, Codazzi, and Ricci equations and Bonnet theorem in the theory of submanifolds in Euclidean spaces

3.1 Submanifolds in Euclidean spaces

Let us consider an arbitrary smooth N-dimensional submanifold M^N in an $(N+L)$-dimensional Euclidean space E^{N+L}, $M^N \subset E^{N+L}$, and introduce the standard classical notation. Let the submanifold M^N be given locally by a smooth vector function $r(u^1, \ldots, u^N)$ of N independent variables $(u^1, \ldots, u^N)$ (some independent parameters on the submanifold), $r(u^1, \ldots, u^N) = (z^1(u^1, \ldots, u^N), \ldots, z^{N+L}(u^1, \ldots, u^N))$, where $(z^1, \ldots, z^{N+L})$ are Cartesian coordinates in the Euclidean space E^{N+L}, $(z^1, \ldots, z^{N+L}) \in E^{N+L}$, $(u^1, \ldots, u^N)$ are local coordinates (parameters) on M^N, $\operatorname{rank}(\partial z^i/\partial u^j) = N$ (here $1 \leq i \leq N+L$, $1 \leq j \leq N$). Then $\partial r/\partial u^i = r_{u^i}$, $1 \leq i \leq N$, are tangent vectors at any point $u = (u^1, \ldots, u^N)$ on M^N. Let $\mathbf{N}_u$ be the normal space of the submanifold M^N at an arbitrary point $u = (u^1, \ldots, u^N)$ on M^N, $\mathbf{N}_u = \langle n_1, \ldots, n_L\rangle$, where n_α, $1 \leq \alpha \leq L$, is an orthonormalized basis of the normal space (orthonormalized normals), $(n_\alpha, r_{u^i}) = 0$, $1 \leq \alpha \leq L$, $1 \leq i \leq N$, $(n_\alpha, n_\beta) = \delta_{\alpha\beta}$, $1 \leq \alpha, \beta \leq L$. Then $\mathbf{I} = ds^2 = g_{ij}(u) du^i du^j$, $g_{ij}(u) = (r_{u^i}, r_{u^j})$, is the first fundamental form, and $\mathbf{II}_\alpha = \omega_{\alpha,ij}(u) du^i du^j$, $\omega_{\alpha,ij}(u) = (n_\alpha, r_{u^i u^j})$, $1 \leq \alpha \leq L$, are the second fundamental forms of the submanifold M^N.

3.2 Torsion forms of submanifolds in Euclidean spaces

Since the set of vectors $(r_{u^1}(u), \ldots, r_{u^N}(u), n_1(u), \ldots, n_L(u))$ forms a basis in E^{N+L} at each point of the submanifold M^N, we can decompose each of the vectors $n_{\alpha,u^i}(u)$, $1 \leq \alpha \leq L$, $1 \leq i \leq N$, with respect to this basis, namely,

$$n_{\alpha,u^i}(u) = \sum_{k=1}^{N} A^k_{\alpha,i}(u) r_{u^k}(u) + \sum_{\beta=1}^{L} \varkappa_{\alpha\beta,i}(u) n_\beta(u),$$

where $A^k_{\alpha,i}(u)$ and $\varkappa_{\alpha\beta,i}(u)$ are some coefficients depending on u (the *Weingarten decomposition*). It is easy to prove that $A^k_{\alpha,i}(u) = -\omega_{\alpha,ij}(u) g^{jk}(u)$, where $g^{jk}(u)$ is the contravariant metric inverse to the first fundamental form $g_{ij}(u)$, $g^{is}(u) g_{sj}(u) = \delta^i_j$. The coefficients $\varkappa_{\alpha\beta,i}(u)$ are called the *torsion coefficients of the submanifold* M^N, $\varkappa_{\alpha\beta,i}(u) = (n_{\alpha,u^i}(u), n_\beta(u))$. It is also easy to prove that the coefficients $\varkappa_{\alpha\beta,i}(u)$ are skew-symmetric with respect to the indices α and β, $\varkappa_{\alpha\beta,i}(u) = -\varkappa_{\beta\alpha,i}(u)$, and form covariant tensors (1-forms) with respect to the index i on the submanifold M^N. The 1-forms $\varkappa_{\alpha\beta,i}(u) du^i$ are called the *torsion forms of the submanifold* M^N.

3.3 Fundamental nonlinear equations in the theory of submanifolds in Euclidean spaces

It is well known that for each submanifold M^N the forms $g_{ij}(u)$, $\omega_{\alpha,ij}(u)$ and $\varkappa_{\alpha\beta,i}(u)$ satisfy the Gauss equations, the Codazzi equations and the Ricci equations, which are the fundamental equations of the theory of submanifolds. In our case, the Gauss equations have the form

$$R_{ijkl}(u) = \sum_{\alpha=1}^{L} \left(\omega_{\alpha,jl}(u)\omega_{\alpha,ik}(u) - \omega_{\alpha,jk}(u)\omega_{\alpha,il}(u)\right), \tag{3.1}$$

where $R_{ijkl}(u)$ is the Riemannian curvature tensor of the first fundamental form $g_{ij}(u)$, the Codazzi equations have the form

$$\nabla_k \omega_{\alpha,ij}(u) - \nabla_j \omega_{\alpha,ik}(u) = \sum_{\beta=1}^{L} (\varkappa_{\alpha\beta,k}(u)\omega_{\beta,ij}(u) - \varkappa_{\alpha\beta,j}(u)\omega_{\beta,ik}(u)), \tag{3.2}$$

where ∇_k is the covariant differentiation generated by the Levi-Civita connection of the first fundamental form $g_{ij}(u)$, and the Ricci equations have the form

$$\nabla_k \varkappa_{\alpha\beta,i}(u) - \nabla_i \varkappa_{\alpha\beta,k}(u) + \sum_{\gamma=1}^{L} \left(\varkappa_{\alpha\gamma,i}(u)\varkappa_{\gamma\beta,k}(u) - \varkappa_{\alpha\gamma,k}(u)\varkappa_{\gamma\beta,i}(u)\right) +$$

$$+\sum_{l=1}^{N}\sum_{j=1}^{N} g^{lj}(u)\ (\omega_{\alpha,kl}(u)\omega_{\beta,ji}(u) - \omega_{\alpha,il}(u)\omega_{\beta,jk}(u)) = 0. \tag{3.3}$$

3.4 Bonnet theorem for submanifolds in Euclidean spaces

Theorem (Bonnet). *Let K^N be an arbitrary smooth N-dimensional Riemannian manifold with a metric $g_{ij}(u)du^i du^j$. Let some 2-forms $\omega_{\alpha,ij}(u)du^i du^j$, $1 \le \alpha \le L$, and some 1-forms $\varkappa_{\alpha\beta,i}(u)du^i$, $1 \le \alpha, \beta \le L$, be given in a simply connected domain of the manifold K^N. If $\omega_{\alpha,ij}(u) = \omega_{\alpha,ji}(u)$, $\varkappa_{\alpha\beta,i}(u) = -\varkappa_{\beta\alpha,i}(u)$, and the Gauss equations* (3.1)*, the Codazzi equations* (3.2) *and the Ricci equations* (3.3) *are satisfied for the forms $g_{ij}(u)$, $\omega_{\alpha,ij}(u)$ and $\varkappa_{\alpha\beta,i}(u)$, then there exists a unique* (*up to motions*) *smooth N-dimensional submanifold M^N in an $(N+L)$-dimensional Euclidean space E^{N+L} with the first fundamental form $ds^2 = g_{ij}(u)du^i du^j$, the second fundamental forms $\omega_{\alpha,ij}(u)du^i du^j$ and the torsion forms $\varkappa_{\alpha\beta,i}(u)du^i$.*

Similar fundamental equations and the Bonnet theorem hold for all *totally nonisotropic submanifolds in pseudo-Euclidean spaces* (we recall that if we have a submanifold in an arbitrary pseudo-Euclidean space E^m_n, then the metric induced on the submanifold from the ambient pseudo-Euclidean space E^m_n is nondegenerate if and only if this submanifold is totally nonisotropic, i.e., it is not tangent to isotropic cones of the ambient pseudo-Euclidean space E^m_n at its points).

4 Description of flat submanifolds with zero torsion in pseudo-Euclidean spaces

4.1 Submanifolds with zero torsion in pseudo-Euclidean spaces

Let us consider totally nonisotropic smooth N-dimensional submanifolds with *zero torsion* in an arbitrary $(N+L)$-dimensional pseudo-Euclidean space, i.e., all torsion forms of submanifolds of this class vanish, $\varkappa_{\alpha\beta,i}(u) = 0$. In the normal spaces $\mathbf{N}_u$, we also use the bases n_α, $1 \le \alpha \le L$, with arbitrary admissible constant Gram matrices $\mu_{\alpha\beta}$, $(n_\alpha, n_\beta) = \mu_{\alpha\beta}$, $\mu_{\alpha\beta} = \text{const}$, $\mu_{\alpha\beta} = \mu_{\beta\alpha}$, $\det(\mu_{\alpha\beta}) \ne 0$ (the signature of the metric $\mu_{\alpha\beta}$ is completely determined by the signature of the first fundamental form of the corresponding submanifold and the signature of the corresponding ambient pseudo-Euclidean space).

For torsionless N-dimensional submanifolds in an arbitrary $(N+L)$-dimensional pseudo-Euclidean space, we obtain the following system of fundamental equations: the Gauss equations

$$R_{ijkl}(u) = \sum_{\alpha=1}^{L}\sum_{\beta=1}^{L} \mu^{\alpha\beta}(\omega_{\alpha,ik}(u)\omega_{\beta,jl}(u) - \omega_{\alpha,il}(u)\omega_{\beta,jk}(u)), \tag{4.1}$$

where $\mu^{\alpha\beta}$ is inverse to the matrix $\mu_{\alpha\beta}$, $\mu^{\alpha\gamma}\mu_{\gamma\beta} = \delta^\alpha_\beta$, the Codazzi equations

$$\nabla_k \omega_{\alpha,ij}(u) = \nabla_j \omega_{\alpha,ik}(u), \tag{4.2}$$

and the Ricci equations

$$\sum_{i=1}^{N}\sum_{j=1}^{N} g^{ij}(u)\,(\omega_{\alpha,ik}(u)\omega_{\beta,jl}(u) - \omega_{\alpha,il}(u)\omega_{\beta,jk}(u)) = 0. \tag{4.3}$$

4.2 Second fundamental forms of flat torsionless submanifolds in pseudo-Euclidean spaces and Hessians

Now let $g_{ij}(u)$ be a flat metric, i.e., we consider flat torsionless N-dimensional submanifolds M^N in an $(N+L)$-dimensional pseudo-Euclidean space. Then we can consider that $u = (u^1, \ldots, u^N)$ are certain flat coordinates of the metric $g_{ij}(u)$ on M^N. In flat coordinates, the metric is a constant nondegenerate symmetric matrix η_{ij}, $\eta_{ij} = \eta_{ji}$, $\eta_{ij} = \text{const}$, $\det(\eta_{ij}) \neq 0$, and the Codazzi equations (4.2) have the form

$$\frac{\partial \omega_{\alpha,ij}}{\partial u^k} = \frac{\partial \omega_{\alpha,ik}}{\partial u^j}. \tag{4.4}$$

Therefore, there locally exist some functions $\chi_{\alpha,i}(u)$, $1 \leq \alpha \leq L$, $1 \leq i \leq N$, such that

$$\omega_{\alpha,ij}(u) = \frac{\partial \chi_{\alpha,i}}{\partial u^j}. \tag{4.5}$$

From symmetry of the second fundamental forms $\omega_{\alpha,ij}(u) = \omega_{\alpha,ji}(u)$, we have

$$\frac{\partial \chi_{\alpha,i}}{\partial u^j} = \frac{\partial \chi_{\alpha,j}}{\partial u^i}. \tag{4.6}$$

Therefore, there locally exist some functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$, such that

$$\chi_{\alpha,i}(u) = \frac{\partial \psi_\alpha}{\partial u^i}, \qquad \omega_{\alpha,ij}(u) = \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^j}. \tag{4.7}$$

We have thus proved the following important lemma.

Lemma 4.1 [6], [7] *All the second fundamental forms of each flat torsionless submanifold in a pseudo-Euclidean space are Hessians in any flat coordinates in any simply connected domain on the submanifold.*

4.3 Fundamental nonlinear equations for flat torsionless submanifolds in pseudo-Euclidean spaces

It follows from Lemma 4.1 that in any flat coordinates, the Gauss equations (4.1) have the form

$$\sum_{\alpha=1}^{L} \sum_{\beta=1}^{L} \mu^{\alpha\beta} \left(\frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^k} \frac{\partial^2 \psi_\beta}{\partial u^j \partial u^l} - \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^l} \frac{\partial^2 \psi_\beta}{\partial u^j \partial u^k} \right) = 0, \tag{4.8}$$

and the Ricci equations (4.3) have the form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \eta^{ij} \left(\frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^k} \frac{\partial^2 \psi_\beta}{\partial u^j \partial u^l} - \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^l} \frac{\partial^2 \psi_\beta}{\partial u^j \partial u^k} \right) = 0, \tag{4.9}$$

where η^{ij} is inverse to the matrix η_{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$.

Theorem 4.1 [5]–[7] *The class of N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces is described (in flat coordinates) by the system of nonlinear equations (4.8), (4.9) for functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$. Here, η^{ij} and $\mu^{\alpha\beta}$ are arbitrary constant nondegenerate symmetric matrices, $\eta^{ij} = \eta^{ji}$, $\eta^{ij} = \text{const}$, $\det(\eta^{ij}) \neq 0$, $\mu^{\alpha\beta} = \text{const}$, $\mu^{\alpha\beta} = \mu^{\beta\alpha}$, $\det(\mu^{\alpha\beta}) \neq 0$; the signature of the ambient $(N+L)$-dimensional pseudo-Euclidean space is the sum of the signatures of the metrics η^{ij} and $\mu^{\alpha\beta}$; $\mathbf{I} = ds^2 = \eta_{ij} du^i du^j$ is the first fundamental form, where η_{ij} is inverse to the matrix η^{ij}, $\eta^{is}\eta_{sj} = \delta^i_j$, and $\mathbf{II}_\alpha = (\partial^2 \psi_\alpha / (\partial u^i \partial u^j)) du^i du^j$, $1 \leq \alpha \leq L$, are the second fundamental forms given by the Hessians of the functions $\psi_\alpha(u)$, $1 \leq \alpha \leq L$, for the corresponding flat torsionless submanifold.*

According to the Bonnet theorem, any solution $\psi_\alpha(u)$, $1 \le \alpha \le L$, of the nonlinear system (4.8), (4.9) determines a unique (up to motions) N-dimensional flat torsionless submanifold of the corresponding $(N+L)$-dimensional pseudo-Euclidean space with the first fundamental form $\eta_{ij} du^i du^j$ and the second fundamental forms $\omega_\alpha(u) = (\partial^2 \psi_\alpha / (\partial u^i \partial u^j)) du^i du^j$, $1 \le \alpha \le L$, given by the Hessians of the functions $\psi_\alpha(u)$, $1 \le \alpha \le L$. It is obvious that we can always add arbitrary terms linear in the coordinates $(u^1, \dots, u^N)$ to any solution of the system (4.8), (4.9), but the set of the second fundamental forms and the corresponding submanifold will be the same. Moreover, any two sets of the second fundamental forms of the form $\omega_{\alpha,ij}(u) = \partial^2 \psi_\alpha / (\partial u^i \partial u^j)$, $1 \le \alpha \le L$, coincide if and only if the corresponding functions $\psi_\alpha(u)$, $1 \le \alpha \le L$, coincide up to terms linear in the coordinates; hence we must not distinguish solutions of the nonlinear system (4.8), (4.9) up to terms linear in the coordinates $(u^1, \dots, u^N)$.

5 Integrability of the nonlinear equations for flat torsionless submanifolds in pseudo-Euclidean spaces

5.1 Linear problem with parameters for the nonlinear equations describing all flat torsionless submanifolds in pseudo-Euclidean spaces

Consider the following linear problem with parameters for vector functions $\partial a(u)/\partial u^i$, $1 \le i \le N$, and $b_\alpha(u)$, $1 \le \alpha \le L$:

$$\frac{\partial^2 a}{\partial u^i \partial u^j} = \lambda\, \mu^{\alpha\beta} \omega_{\alpha,ij}(u) b_\beta(u), \qquad \frac{\partial b_\alpha}{\partial u^i} = \rho\, \eta^{kj} \omega_{\alpha,ij}(u) \frac{\partial a}{\partial u^k}, \tag{5.1}$$

where η^{ij}, $1 \le i, j \le N$, and $\mu^{\alpha\beta}$, $1 \le \alpha, \beta \le L$, are arbitrary constant nondegenerate symmetric matrices, $\eta^{ij} = \eta^{ji}$, $\eta^{ij} = \mathrm{const}$, $\det(\eta^{ij}) \ne 0$, $\mu^{\alpha\beta} = \mathrm{const}$, $\mu^{\alpha\beta} = \mu^{\beta\alpha}$, $\det(\mu^{\alpha\beta}) \ne 0$; λ and ρ are arbitrary constants (parameters) [7]. Of course, only one of the parameters is essential (but it is really essential). It is obvious that the coefficients $\omega_{\alpha,ij}(u)$, $1 \le \alpha \le L$, here must be symmetric matrix functions, $\omega_{\alpha,ij}(u) = \omega_{\alpha,ji}(u)$.

The consistency conditions for the linear system (5.1) are equivalent to the nonlinear system (4.8), (4.9) describing the class of N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces. Indeed, we have

$$\begin{aligned}\frac{\partial^3 a}{\partial u^i \partial u^j \partial u^k} &= \lambda\, \mu^{\alpha\beta} \frac{\partial \omega_{\alpha,ij}}{\partial u^k} b_\beta(u) + \lambda\, \mu^{\alpha\beta} \omega_{\alpha,ij}(u) \frac{\partial b_\beta}{\partial u^k} = \\ &= \lambda\, \mu^{\alpha\beta} \frac{\partial \omega_{\alpha,ij}}{\partial u^k} b_\beta(u) + \lambda\, \mu^{\alpha\beta} \omega_{\alpha,ij}(u) \rho\, \eta^{ls} \omega_{\beta,ks}(u) \frac{\partial a}{\partial u^l} = \\ &= \lambda\, \mu^{\alpha\beta} \frac{\partial \omega_{\alpha,ik}}{\partial u^j} b_\beta(u) + \lambda\, \mu^{\alpha\beta} \omega_{\alpha,ik}(u) \rho\, \eta^{ls} \omega_{\beta,js}(u) \frac{\partial a}{\partial u^l},\end{aligned} \tag{5.2}$$

whence we obtain

$$\frac{\partial \omega_{\alpha,ij}(u)}{\partial u^k} = \frac{\partial \omega_{\alpha,ik}(u)}{\partial u^j} \tag{5.3}$$

and

$$\mu^{\alpha\beta} \omega_{\alpha,ij}(u) \omega_{\beta,ks}(u) = \mu^{\alpha\beta} \omega_{\alpha,ik}(u) \omega_{\beta,js}(u). \tag{5.4}$$

Moreover,

$$\begin{aligned}\frac{\partial^2 b_\alpha}{\partial u^i \partial u^l} &= \rho\, \eta^{kj} \frac{\partial \omega_{\alpha,ij}}{\partial u^l} \frac{\partial a}{\partial u^k} + \rho\, \eta^{kj} \omega_{\alpha,ij}(u) \frac{\partial^2 a}{\partial u^k \partial u^l} = \\ &= \rho\, \eta^{kj} \frac{\partial \omega_{\alpha,ij}}{\partial u^l} \frac{\partial a}{\partial u^k} + \rho\, \eta^{kj} \omega_{\alpha,ij}(u) \lambda\, \mu^{\gamma\beta} \omega_{\gamma,kl}(u) b_\beta(u) = \\ &= \rho\, \eta^{kj} \frac{\partial \omega_{\alpha,lj}}{\partial u^i} \frac{\partial a}{\partial u^k} + \rho\, \eta^{kj} \omega_{\alpha,lj}(u) \lambda\, \mu^{\gamma\beta} \omega_{\gamma,ki}(u) b_\beta(u),\end{aligned} \tag{5.5}$$

whence we have

$$\frac{\partial \omega_{\alpha,ij}}{\partial u^l} = \frac{\partial \omega_{\alpha,lj}}{\partial u^i} \tag{5.6}$$

and

$$\eta^{kj}\omega_{\alpha,ij}(u)\omega_{\gamma,kl}(u) = \eta^{kj}\omega_{\alpha,lj}(u)\omega_{\gamma,ki}(u). \tag{5.7}$$

It follows from (5.3) and (5.6) that there locally exist some functions $\psi_\alpha(u)$, $1 \le \alpha \le L$, such that

$$\omega_{\alpha,ij}(u) = \frac{\partial^2 \psi_\alpha}{\partial u^i \partial u^j}. \tag{5.8}$$

Then the relations (5.4) and (5.7) are equivalent to the nonlinear system (4.8), (4.9) for the functions $\psi_\alpha(u)$, $1 \le \alpha \le L$.

Theorem 5.1 [7] *The nonlinear system* (4.8), (4.9) *is integrable by the inverse scattering method.*

5.2 Integrable invariant description of flat torsionless submanifolds in pseudo-Euclidean spaces

In arbitrary local coordinates, we obtain the following integrable description of all N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces.

Theorem 5.2 [6], [7] *For each N-dimensional flat torsionless submanifold in an $(N+L)$-dimensional pseudo-Euclidean space with a flat first fundamental form $g_{ij}(u)$, there locally exist functions $\psi_\alpha(u)$, $1 \le \alpha \le L$, such that the second fundamental forms have the form*

$$(\omega_\alpha)_{ij}(u) = \nabla_i \nabla_j \psi_\alpha, \tag{5.9}$$

where ∇_i is the covariant differentiation defined by the Levi-Civita connection generated by the metric $g_{ij}(u)$. The class of N-dimensional flat torsionless submanifolds in $(N+L)$-dimensional pseudo-Euclidean spaces is described by the following integrable system of nonlinear equations for the functions $\psi_\alpha(u)$, $1 \le \alpha \le L$:

$$\sum_{n=1}^{N} \nabla^n \nabla_i \psi_\alpha \nabla_n \nabla_l \psi_\beta = \sum_{n=1}^{N} \nabla^n \nabla_i \psi_\beta \nabla_n \nabla_l \psi_\alpha, \tag{5.10}$$

$$\sum_{\alpha=1}^{L}\sum_{\beta=1}^{L} \mu^{\alpha\beta} \nabla_i \nabla_j \psi_\alpha \nabla_k \nabla_l \psi_\beta = \sum_{\alpha=1}^{L}\sum_{\beta=1}^{L} \mu^{\alpha\beta} \nabla_i \nabla_k \psi_\alpha \nabla_j \nabla_l \psi_\beta, \tag{5.11}$$

where ∇_i is the covariant differentiation defined by the Levi-Civita connection generated by a flat metric $g_{ij}(u)$, $\nabla^i = g^{is}(u)\nabla_s$, $g^{is}(u)g_{sj}(u) = \delta^i_j$. Moreover, in this case, the systems of hydrodynamic type

$$u^i_{t_\alpha} = \left(\nabla^i \nabla_j \psi_\alpha\right) u^j_x, \qquad 1 \le \alpha \le L, \tag{5.12}$$

are commuting integrable bi-Hamiltonian systems of hydrodynamic type.

Any solution $\psi_\alpha(u)$, $1 \le \alpha \le L$, of the integrable nonlinear system (5.10), (5.11) *determines a unique* (*up to motions*) *N-dimensional flat torsionless submanifold of the corresponding $(N+L)$-dimensional pseudo-Euclidean space with the first fundamental form $g_{ij}(u)du^i du^j$ and the second fundamental forms* (5.9).

6 Reduction to the associativity equations of two-dimensional topological quantum field theories and potential flat torsionless submanifolds in pseudo-Euclidean spaces

6.1 Special case of flat torsionless submanifolds, when the Gauss and the Ricci equations coincide, and the associativity equations

We now also find some natural and very important integrable reductions of the nonlinear system (4.8), (4.9). We show that the class of flat torsionless submanifolds in pseudo-Euclidean spaces is quite rich, and we describe a nontrivial and very important family of submanifolds of this class. This family is generated by the associativity equations of two-dimensional topological quantum field theories (the WDVV equations). First of all, we note that although the Gauss equations (4.8) and the Ricci equations (4.9) for flat torsionless submanifolds in pseudo-Euclidean spaces are essentially different, they are fantastically similar. The case of a natural reduction under which the Gauss equations (4.8) and the Ricci equations (4.9) merely coincide is of particular interest. Such a reduction readily leads to the associativity equations of two-dimensional topological quantum field theories.

Theorem 6.1 [5]–[7] *If $L = N$, $\mu^{ij} = c\eta^{ij}$, $1 \leq i, j \leq N$, c is an arbitrary nonzero constant, and $\psi_\alpha(u) = \partial\Phi/\partial u^\alpha$, $1 \leq \alpha \leq N$, where $\Phi = \Phi(u^1, \ldots, u^N)$, then the Gauss equations* (4.8) *coincide with the Ricci equations* (4.9)*, and each of them coincides with the associativity equations* (2.16) *of two-dimensional topological quantum field theories* (*the WDVV equations*) *for the potential $\Phi(u)$.*

6.2 Potential flat torsionless submanifolds in pseudo-Euclidean spaces and the associativity equations

Definition 6.1 [7] A flat torsionless N-dimensional submanifold in a $2N$-dimensional pseudo-Euclidean space with a flat first fundamental form $g_{ij}(u)du^i du^j$ is called *potential* if there always locally exist a certain function $\Phi(u)$ in a neighborhood on the submanifold such that the second fundamental forms of this submanifold locally in this neighborhood have the form

$$(\omega_i)_{jk}(u)du^j du^k = (\nabla_i \nabla_j \nabla_k \Phi(u))\, du^j du^k, \quad 1 \leq i \leq N, \tag{6.1}$$

where ∇_i is the covariant differentiation defined by the Levi-Civita connection generated by the flat metric $g_{ij}(u)$.

Theorem 6.2 [5]–[7] *The associativity equations of two-dimensional topological quantum field theories describe a special class of N-dimensional flat submanifolds without torsion in $2N$-dimensional pseudo-Euclidean spaces, namely, exactly the class of* potential flat torsionless submanifolds*.*

According to the Bonnet theorem, any solution $\Phi(u)$ of the associativity equations (2.16) (with an arbitrary fixed constant metric η_{ij}) determines a unique (up to motions) N-dimensional potential flat torsionless submanifold of the corresponding $2N$-dimensional pseudo-Euclidean space with the first fundamental form $\eta_{ij}du^i du^j$ and the second fundamental forms $\omega_n(u) = (\partial^3\Phi/(\partial u^n \partial u^i \partial u^j))du^i du^j$ given by the third derivatives of the potential $\Phi(u)$. Here, we do not distinguish solutions of the associativity equations (2.16) up to terms quadratic in the coordinates u.

Theorem 6.3 [6], [7] *On each potential flat torsionless submanifold in a pseudo-Euclidean space, there is a structure of a Frobenius algebra given* (*in flat coordinates*) *by the flat first fundamental form η_{ij} and the*

Weingarten operators $(A_s)^i_j(u) = -\eta^{ik}(\omega_s)_{kj}(u)$:

$$\langle e_i, e_j\rangle = \eta_{ij}, \qquad e_i \circ e_j = c^k_{ij}(u) e_k, \qquad e_i = \frac{\partial}{\partial u^i},$$
$$c^k_{ij}(u^1, \dots, u^N) = -(A_i)^k_j(u) = \eta^{ks}(\omega_i)_{sj}(u^1, \dots, u^N). \tag{6.2}$$

In arbitrary local coordinates, this Frobenius structure has the form

$$\langle e_i, e_j\rangle = g_{ij}, \qquad e_i \circ e_j = c^k_{ij}(u) e_k, \qquad e_i = \frac{\partial}{\partial u^i},$$
$$c^k_{ij}(u^1, \dots, u^N) = -(A_i)^k_j(u) = g^{ks}(u^1, \dots, u^N)(\omega_i)_{sj}(u^1, \dots, u^N), \tag{6.3}$$

where $g^{ij}(u)$ *is the contravariant metric inverse to the first fundamental form* $g_{ij}(u)$, $g^{is}(u)g_{sj}(u) = \delta^i_j$, *and* $(\omega_k)_{ij}(u)du^i du^j$, $1 \le k \le N$, *are the second fundamental forms.*

Theorem 6.4 [6], [7] *Each N-dimensional Frobenius manifold can be locally represented as a potential flat torsionless N-dimensional submanifold in a $2N$-dimensional pseudo-Euclidean space.*

7 Realization of Frobenius manifolds as submanifolds in pseudo-Euclidean spaces

It is important to note that we have at least two essentially different possibilities for signature of the corresponding ambient $2N$-dimensional pseudo-Euclidean space, namely, we can always consider the ambient $2N$-dimensional pseudo-Euclidean space of zero signature, and we can also consider the ambient $2N$-dimensional pseudo-Euclidean space whose signature is equal to doubled signature of the metric η_{ij}. Thus, if the metric η_{ij} of a Frobenius manifold has a nonzero signature, then according to our construction we have two essentially different possibilities for realization of the Frobenius manifold as a potential flat torsionless submanifold.

Theorem 7.1 [8] *For an arbitrary Frobenius manifold, which is locally given by a solution $\Phi(u^1, \dots, u^N)$ of the associativity equations* (2.16)*, the corresponding potential flat torsionless submanifold in a $2N$-dimensional pseudo-Euclidean space that realizes this Frobenius manifold is given by the $2N$-component vector function $r(u^1, \dots, u^N)$ satisfying the following compatible linear system of second-order partial differential equations*:

$$\frac{\partial^2 r}{\partial u^i \partial u^j} = c\eta^{kl} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k} \frac{\partial n}{\partial u^l}, \tag{7.1}$$
$$\frac{\partial^2 n}{\partial u^i \partial u^j} = -\eta^{kl} \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k} \frac{\partial r}{\partial u^l}, \tag{7.2}$$

where $n(u^1, \dots, u^N)$ is a $2N$-component vector function, c is an arbitrary nonzero constant (*a deformation parameter preserving the corresponding Frobenius structure*)*. In particular, two essentially different cases $c = 1$ and $c = -1$ correspond to ambient $2N$-dimensional pseudo-Euclidean spaces of different signatures* (*if the metric η_{ij} has a nonzero signature*)*. The consistency of the linear system* (7.1), (7.2) *is equivalent to the associativity equations* (2.16)*.*

8 General nonlocal Hamiltonian operators of hydrodynamic type

Now we consider applications of our construction to the theory of integrable systems, the theory of nonlocal Hamiltonian operators of hydrodynamic type, the theory of compatible Poisson structures and the theory of bi-Hamiltonian integrable hierarchies of hydrodynamic type. Recall that the theory of nonlocal Hamiltonian operators of hydrodynamic type was invented by the author and Ferapontov in 1990–1991 [11], [12] in connection with vital necessities of the Hamiltonian theory of systems of hydrodynamic type proposed by Dubrovin and Novikov [9] and developed by Tsarev [10]. In this paper we give an integrable description of all nonlocal Hamiltonian operators of hydrodynamic type with flat metrics. This nontrivial special class of Hamiltonian operators is generated by flat torsionless submanifolds in pseudo-Euclidean spaces and closely connected with the associativity equations of two-dimensional topological quantum field theories and the theory of Frobenius manifolds. The Hamiltonian operators of this class are of special interest for many other reasons too. In particular, any such Hamiltonian operator always determines integrable structural flows (some systems of hydrodynamic type), always gives a nontrivial pencil of compatible Hamiltonian operators and generates bi-Hamiltonian integrable hierarchies of hydrodynamic type. The affinors of any such Hamiltonian operator generate some special integrals in involution. The nonlinear systems describing integrals in involution are of independent great interest. The equations of associativity of two-dimensional topological quantum field theories (the WDVV equations) describe an important special class of integrals in involution, a special class of nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, a special class of compatible local and nonlocal Poisson structures and important special classes of bi-Hamiltonian integrable hierarchies of systems of hydrodynamic type. Moreover, we show that each flat torsionless submanifold in a pseudo-Euclidean space (recall that this class of submanifolds is described in our paper by an integrable system [7]) gives a set of integrals in involution, nontrivial local and nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, a pencil of compatible Poisson structures and generates bi-Hamiltonian integrable hierarchies of systems of hydrodynamic type.

Recall that general nonlocal Hamiltonian operators of hydrodynamic type, namely, Hamiltonian operators of the form

$$P^{ij} = g^{ij}(u(x))\frac{d}{dx} + b^{ij}_k(u(x))\,u^k_x + \sum_{n=1}^{L} \varepsilon^n (w_n)^i_k(u(x))u^k_x \left(\frac{d}{dx}\right)^{-1} \circ (w_n)^j_s(u(x))u^s_x, \tag{8.1}$$

where $\det(g^{ij}(u)) \neq 0$, $\varepsilon^n = \pm 1$, $1 \le n \le L$, $u^1, \ldots, u^N$ are local coordinates, $u = (u^1, \ldots, u^N)$, $u^i(x)$, $1 \le i \le N$, are functions (fields) of one independent variable x, and the coefficients $g^{ij}(u)$, $b^{ij}_k(u)$, $(w_n)^i_j(u)$, $1 \le i, j, k \le N$, $1 \le n \le L$, are smooth functions of local coordinates, were studied by Ferapontov in [12] (see also [9], [11]).

Hamiltonian operators of the general form (8.1) (local and nonlocal) play a key role in the Hamiltonian theory of systems of hydrodynamic type [9]–[12]. Recall that an operator M^{ij} is said to be *Hamiltonian* if the operator defines a Poisson bracket

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} M^{ij} \frac{\delta J}{\delta u^j(x)} dx \tag{8.2}$$

on arbitrary functionals I and J on the space of the fields $u^i(x)$, i.e., the bracket (8.2) is skew-symmetric and satisfies the Jacobi identity.

It was proved in [12] that the operator (8.1) is Hamiltonian if and only if $g^{ij}(u)$ is a symmetric (pseudo-Riemannian) contravariant metric and the following relations are satisfied for the coefficients of the operator:

1) $b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u)$, where $\Gamma^j_{sk}(u)$ is the Levi-Civita connection generated by the contravariant metric $g^{ij}(u)$,

2) $g^{ik}(u)(w_n)^j_k(u) = g^{jk}(u)(w_n)^i_k(u)$,

3) $\nabla_k (w_n)^i_j(u) = \nabla_j (w_n)^i_k(u)$, where ∇_k is the covariant differentiation generated by the Levi-Civita connection $\Gamma^j_{sk}(u)$ of the metric $g^{ij}(u)$,

4) $R^{ij}_{kl}(u) = \sum_{n=1}^{L} \varepsilon^n \left((w_n)^i_l(u)(w_n)^j_k(u) - (w_n)^j_l(u)(w_n)^i_k(u) \right)$, where

$$R^{ij}_{kl}(u) = g^{is}(u) R^j_{skl}(u)$$

is the Riemannian curvature tensor of the metric $g^{ij}(u)$,

5) $[w_n(u), w_m(u)] = 0$, i.e., the family $(w_n)^i_j(u)$, $1 \le n \le L$, of (1, 1)-tensors (*affinors*) is commutative.

Each Hamiltonian operator of the form (8.1) exactly corresponds to an N-dimensional submanifold with flat normal bundle embedded in a pseudo-Euclidean space E^{N+L}. Here, the covariant metric $g_{ij}(u)$ (for which $g_{is}(u)g^{sj}(u) = \delta^j_i$) is the first fundamental form, and the affinors $w_n(u)$, $1 \le n \le L$, are the Weingarten operators of this embedded submanifold ($g_{is}(u)(w_n)^s_j(u)$ are the corresponding second fundamental forms). Correspondingly, the relations 2)–4) are the Gauss–Codazzi equations for an N-dimensional submanifold with zero torsion embedded in a pseudo-Euclidean space E^{N+L} [12]. The relations 5) are equivalent to the Ricci equations for this embedded submanifold.

Having in mind further applications to arbitrary Frobenius manifolds, we prefer to consider general nonlocal Hamiltonian operators of hydrodynamic type in the form

$$P^{ij} = g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x))\, u^k_x + \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} (w_m)^i_k(u(x)) u^k_x \left(\frac{d}{dx} \right)^{-1} \circ (w_n)^j_s(u(x)) u^s_x, \tag{8.3}$$

where $\det(g^{ij}(u)) \ne 0$, μ^{mn} is an arbitrary nondegenerate symmetric constant matrix. Each operator of the form (8.3) can be reduced to the form (8.1) (and conversely, each operator of the form (8.3) can be obtained from some operator of the form (8.1)) by a linear transformation $w_n(u) = c^l_n \widetilde{w}_l(u)$ in the vector space of affinors $w_n(u)$, $1 \le n \le L$; here c^l_n is an arbitrary nondegenerate constant matrix. Among all the conditions 1)–5) for the Hamiltonian property of the operator (8.1), these transformations change only the condition 4) for the Riemannian curvature tensor of the metric. The condition 4) for the operator (8.3) takes the form

$$R^{ij}_{kl}(u) = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \left((w_m)^i_l(u)(w_n)^j_k(u) - (w_m)^j_l(u)(w_n)^i_k(u) \right),$$

and all the other conditions 1)–3) and 5) for the Hamiltonian property remain unchanged.

Consider all the relations for the coefficients of the nonlocal Hamiltonian operator (8.3) in a form convenient for further use.

Lemma 8.1 [6] *The operator* (8.3), *where* $\det(g^{ij}(u)) \ne 0$, *is Hamiltonian if and only if its coefficients satisfy the relations*

$$g^{ij} = g^{ji}, \tag{8.4}$$

$$\frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k, \tag{8.5}$$

$$g^{is} b^{jk}_s = g^{js} b^{ik}_s, \tag{8.6}$$

$$g^{is} (w_n)^j_s = g^{js} (w_n)^i_s, \tag{8.7}$$

$$(w_n)^i_s (w_m)^s_j = (w_m)^i_s (w_n)^s_j, \tag{8.8}$$

$$g^{is} g^{jr} \frac{\partial (w_n)^k_r}{\partial u^s} - g^{jr} b^{ik}_s (w_n)^s_r = g^{js} g^{ir} \frac{\partial (w_n)^k_r}{\partial u^s} - g^{ir} b^{jk}_s (w_n)^s_r, \tag{8.9}$$

$$g^{is} \left(\frac{\partial b^{jk}_s}{\partial u^r} - \frac{\partial b^{jk}_r}{\partial u^s} \right) + b^{ij}_s b^{sk}_r - b^{ik}_s b^{sj}_r = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} g^{is} \left((w_m)^j_r (w_n)^k_s - (w_m)^j_s (w_n)^k_r \right). \tag{8.10}$$

9 Nonlocal Hamiltonian operators of hydrodynamic type with flat metrics and special pencils of Hamiltonian operators

Let us consider the important special case of the nonlocal Hamiltonian operators of the form (8.3) when the metric $g^{ij}(u)$ is flat. Recall that each flat metric uniquely determines a local Hamiltonian operator of hydrodynamic type (i.e., a Hamiltonian operator of the form (8.3) with zero affinors) known as a Dubrovin–Novikov Hamiltonian operator [9]. We prove that for each flat metric there also exist a remarkable class of nonlocal Hamiltonian operators of hydrodynamic type with this flat metric and nontrivial affinors; moreover, these Hamiltonian operators have important applications in the theory of Frobenius manifolds and integrable hierarchies. First of all, note the following important property of nonlocal Hamiltonian operators of hydrodynamic type with flat metrics. Recall that two Hamiltonian operators are said to be *compatible* if any linear combination of these Hamiltonian operators is also a Hamiltonian operator [13], i.e., they form a *pencil of Hamiltonian operators* and, correspondingly, they form a *pencil of Poisson brackets.*

Lemma 9.1 [6] *The metric $g^{ij}(u)$ of a Hamiltonian operator of the form* (8.3) *is flat if and only if this operator defines the pencil*

$$\begin{aligned} P^{ij}_{\lambda_1,\lambda_2} &= \lambda_1 \left(g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x))\, u^k_x \right) + \\ &+ \lambda_2 \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} (w_m)^i_k(u(x)) u^k_x \left(\frac{d}{dx} \right)^{-1} \circ (w_n)^j_s(u(x)) u^s_x, \end{aligned} \tag{9.1}$$

of compatible Hamiltonian operators, where λ_1 and λ_2 are arbitrary constants.

Indeed, if the operator (8.3) is Hamiltonian, then its coefficients satisfy the relations (8.4)–(8.10). It is obvious that in this case the relations (8.4)–(8.9) for the operator (9.1) are always satisfied for any constants λ_1 and λ_2, and the relation (8.10) is satisfied for any constants λ_1 and λ_2 if and only if the left- and right-hand sides of this relation are zero identically.

It follows from the relations (8.4)–(8.6) for the Hamiltonian operator (8.3) that the Riemannian curvature tensor of the metric $g^{ij}(u)$ has the form

$$R^{ijk}_r(u) = g^{is}(u) R^{jk}_{sr}(u) = g^{is}(u) \left(\frac{\partial b^{jk}_s}{\partial u^r} - \frac{\partial b^{jk}_r}{\partial u^s} \right) + b^{ij}_s(u) b^{sk}_r(u) - b^{ik}_s(u) b^{sj}_r(u). \tag{9.2}$$

Consequently, if the metric $g^{ij}(u)$ of a Hamiltonian operator of the form (8.3) is flat, i.e., $R^{ijk}_r(u) = 0$, then the relation (8.10) becomes

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} g^{is} \left((w_m)^j_r(u) (w_n)^k_s(u) - (w_m)^j_s(u) (w_n)^k_r(u) \right) = 0.$$

Thus the metric $g^{ij}(u)$ of a Hamiltonian operator of the form (8.3) is flat if and only if the left- and right-hand sides of the relation (8.10) for the Hamiltonian operator (8.3) are zero identically. In this case, the left- and right-hand sides of the relation (8.10) for the operator (9.1) are also zero identically for any constants λ_1 and λ_2, i.e., we obtain a pencil of compatible Hamiltonian operators (9.1). Note also that for the pencil of Hamiltonian operators $P^{ij}_{\lambda_1,\lambda_2}$ given by the formula (9.1) it readily follows from the Dubrovin–Novikov theorem [9] applied to the local operator $P^{ij}_{1,0}$ that the metric $g^{ij}(u)$ is flat. Lemma 9.1 is proved.

Thus if the metric $g^{ij}(u)$ of a Hamiltonian operator of the form (8.3) is flat, then the operator

$$P^{ij}_{0,1} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} (w_m)^i_k(u(x)) u^k_x \left(\frac{d}{dx}\right)^{-1} \circ (w_n)^j_s(u(x)) u^s_x \tag{9.3}$$

is also a Hamiltonian operator obtained by the degeneration as $\lambda_1 \to 0$. Moreover, in this case, this Hamiltonian operator is always compatible with the local Hamiltonian operator of hydrodynamic type (the Dubrovin–Novikov operator)

$$P^{ij}_{1,0} = g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x))\, u^k_x. \tag{9.4}$$

The compatible Hamiltonian operators (9.3) and (9.4) always generate the corresponding integrable bi-Hamiltonian hierarchies. We construct these integrable hierarchies further in Section 13.

10 Integrability of structural flows

We recall that systems of hydrodynamic type

$$u^i_{t_n} = (w_n)^i_j(u) u^j_x, \qquad 1 \le n \le L, \tag{10.1}$$

are called *structural flows* of the nonlocal Hamiltonian operator of hydrodynamic type (8.3) (see [12], [14]).

Lemma 10.1 [6] *All the structural flows* (10.1) *of any nonlocal Hamiltonian operator of hydrodynamic type with flat metric are commuting integrable bi-Hamiltonian systems of hydrodynamic type.*

Maltsev and Novikov proved in [14] (see also [12]) that the structural flows of any nonlocal Hamiltonian operator of hydrodynamic type (8.3) are always Hamiltonian with respect to this Hamiltonian operator. Let us consider an arbitrary nonlocal Hamiltonian operator of hydrodynamic type (8.3) with a flat metric $g^{ij}(u)$ and the pencil of compatible Hamiltonian operators (9.1) corresponding to this Hamiltonian operator. The corresponding structural flows are necessarily Hamiltonian with respect to each of the operators in the Hamiltonian pencil (9.1) and, consequently, they are integrable bi-Hamiltonian systems.

11 Integrable description of nonlocal Hamiltonian operators of hydrodynamic type with flat metrics

Let us describe all the nonlocal Hamiltonian operators of hydrodynamic type with flat metrics. The form of the Hamiltonian operator (8.3) is invariant with respect to local changes of coordinates, and also all the coefficients of the operator are transformed as the corresponding differential-geometric objects. Since the metric is flat, there exist local coordinates in which the metric is reduced to a constant matrix η^{ij}, $\eta^{ij} = \text{const}$, $\det(\eta^{ij}) \neq 0$, $\eta^{ij} = \eta^{ji}$. In these local coordinates, all the coefficients of the Levi-Civita connection are zero, and the Hamiltonian operator has the form

$$\widetilde{P}^{ij} = \eta^{ij} \frac{d}{dx} + \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} (\widetilde{w}_m)^i_k(u(x)) u^k_x \left(\frac{d}{dx}\right)^{-1} \circ (\widetilde{w}_n)^j_s(u(x)) u^s_x. \tag{11.1}$$

Description of nonlocal Hamiltonian operators of hydrodynamic type with flat metrics coincides with description of flat torsionless submanifolds in pseudo-Euclidean spaces.

Theorem 11.1 [5], [6] *The operator* (11.1)*, where* η^{ij} *and* μ^{mn} *are arbitrary nondegenerate symmetric constant matrices, is Hamiltonian if and only if there exist functions* $\psi_n(u)$, $1 \le n \le L$, *such that*

$$(\widetilde{w}_n)^i_j(u) = \eta^{is} \frac{\partial^2 \psi_n}{\partial u^s \partial u^j}, \tag{11.2}$$

and the integrable system (4.8), (4.9) *of nonlinear equations describing all flat torsionless submanifolds in pseudo-Euclidean spaces is satisfied.*

The relations (8.4)–(8.6) for any operator of the form (11.1) are automatically fulfilled, and the relation (8.9) for any operator of the form (11.1) has the form

$$\frac{\partial (\widetilde{w}_n)^k_r}{\partial u^s} = \frac{\partial (\widetilde{w}_n)^k_s}{\partial u^r}, \tag{11.3}$$

and, consequently, there locally exist functions $\varphi^i_n(u)$, $1 \le i \le N$, $1 \le n \le L$, such that

$$(\widetilde{w}_n)^i_j(u) = \frac{\partial \varphi^i_n}{\partial u^j}. \tag{11.4}$$

Then relation (8.7) becomes

$$\eta^{is} \frac{\partial \varphi^j_n}{\partial u^s} = \eta^{js} \frac{\partial \varphi^i_n}{\partial u^s} \tag{11.5}$$

or, equivalently,

$$\frac{\partial \left(\eta_{is} \varphi^s_n\right)}{\partial u^j} = \frac{\partial \left(\eta_{js} \varphi^s_n\right)}{\partial u^i}, \tag{11.6}$$

where the matrix η_{ij} is inverse to the matrix η^{ij}, $\eta_{is}\eta^{sj} = \delta^j_i$. It follows from the relation (11.6) that there locally exist functions $\psi_n(u)$, $1 \le n \le L$, such that

$$\eta_{is} \varphi^s_n = \frac{\partial \psi_n}{\partial u^i}. \tag{11.7}$$

Thus

$$\varphi^i_n = \eta^{is} \frac{\partial \psi_n}{\partial u^s}, \quad (\widetilde{w}_n)^i_j(u) = \eta^{is} \frac{\partial^2 \psi_n}{\partial u^s \partial u^j}. \tag{11.8}$$

In this case, the relations (8.8) and (8.10) become (4.9) and (4.8) respectively.

The nonlinear equations (4.8) and (4.9) describing all nonlocal Hamiltonian operators of hydrodynamic type with flat metrics are exactly equivalent to the conditions that a flat N-dimensional submanifold with flat normal bundle, with the first fundamental form $\eta_{ij} du^i du^j$ and the second fundamental forms $\omega_n(u)$ given by Hessians of L functions $\psi_n(u)$, $1 \le n \le L$,

$$\omega_n(u) = \frac{\partial^2 \psi_n}{\partial u^i \partial u^j} du^i du^j,$$

is embedded in an $(N + L)$-dimensional pseudo-Euclidean space.

12 Integrable description of a special class of pencils of Hamiltonian operators

Now we give an integrable description of a special class of pencils of Hamiltonian operators and a special class of integrable bi-Hamiltonian hierarchies of hydrodynamic type.

Theorem 12.1 [6] *If functions $\psi_n(u)$, $1 \le n \le L$, are a solution of the integrable nonlinear system* (4.8), (4.9), *then the systems of hydrodynamic type* (*the structural flows of the corresponding nonlocal Hamiltonian operator of hydrodynamic type with flat metric*)

$$u^i_{t_n} = \eta^{is} \frac{\partial^2 \psi_n}{\partial u^s \partial u^j} u^j_x, \qquad 1 \le n \le L, \tag{12.1}$$

are commuting integrable bi-Hamiltonian systems of hydrodynamic type. Moreover, in this case the nonlocal operator

$$M_1^{ij} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \eta^{jr} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^2 \psi_n}{\partial u^r \partial u^s} u^s_x \tag{12.2}$$

is also a Hamiltonian operator, and this nonlocal Hamiltonian operator is compatible with the constant Hamiltonian operator

$$M_2^{ij} = \eta^{ij} \frac{d}{dx}. \tag{12.3}$$

In arbitrary local coordinates, we obtain the following integrable description of all nonlocal Hamiltonian operators of hydrodynamic type with flat metrics and the corresponding pencils of Hamiltonian operators.

Theorem 12.2 [6] *The operator* (8.3) *with a flat metric $g^{ij}(u)$ is Hamiltonian if and only if $b^{ij}_k(u) = -g^{is}(u)\Gamma^j_{sk}(u)$, where $\Gamma^j_{sk}(u)$ is the flat connection generated by the flat metric $g^{ij}(u)$, and there locally exist functions $\psi_n(u)$, $1 \le n \le L$, such that*

$$(w_n)^i_j(u) = \nabla^i \nabla_j \psi_n, \tag{12.4}$$

and the integrable system (5.10), (5.11) *of nonlinear equations describing all flat torsionless submanifolds in pseudo-Euclidean spaces is satisfied. In particular, in this case the operator*

$$\begin{aligned} M^{ij}_{\lambda_1,\lambda_2} &= \lambda_1 \left(g^{ij}(u(x)) \frac{d}{dx} - g^{is}(u(x)) \Gamma^j_{sk}(u(x))\, u^k_x \right) + \\ &+ \lambda_2 \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \nabla^i \nabla_k \psi_m u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \nabla^j \nabla_s \psi_n u^s_x \end{aligned} \tag{12.5}$$

is a Hamiltonian operator for any constants λ_1 and λ_2, and the systems of hydrodynamic type

$$u^i_{t_n} = \nabla^i \nabla_j \psi_n u^j_x, \qquad 1 \le n \le L, \tag{12.6}$$

are always commuting integrable bi-Hamiltonian systems of hydrodynamic type.

Hence, each flat torsionless submanifold in a pseudo-Euclidean space generates a nonlocal Hamiltonian operator of hydrodynamic type with flat metric, gives a special class of pencils of Hamiltonian operators and special integrable bi-Hamiltonian hierarchies of hydrodynamic type. Now we construct an infinite integrable bi-Hamiltonian hierarchy of hydrodynamic type generated by an arbitrary flat torsionless submanifold in a pseudo-Euclidean space.

13 Integrable hierarchy generated by an arbitrary flat torsionless submanifold in pseudo-Euclidean space

Consider the recursion operator

$$R^i_j = \left(M_1 (M_2)^{-1} \right)^i_j = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^s_x \left(\frac{d}{dx} \right)^{-1} \tag{13.1}$$

corresponding to the compatible Hamiltonian operators (12.2) and (12.3). Let us apply this recursion operator (13.1) to the system of translations with respect to x,

$$u^i_t = u^i_x. \tag{13.2}$$

Each system in the hierarchy

$$u^i_{t_s} = (R^s)^i_j u^j_x, \qquad s \in \mathbb{Z}, \tag{13.3}$$

is a multi-Hamiltonian integrable system of hydrodynamic type. In particular, each system of the form

$$u^i_{t_1} = R^i_j u^j_x, \tag{13.4}$$

i.e., the system

$$u^i_{t_1} = \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn}\eta^{ip}\frac{\partial^2\psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \circ \frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^j u^s_x, \tag{13.5}$$

is integrable.

Since

$$\frac{\partial}{\partial u^r}\left(\frac{\partial^2\psi_n}{\partial u^j \partial u^s} u^j\right) = \frac{\partial^3 \psi_n}{\partial u^j \partial u^s \partial u^r} u^j + \frac{\partial^2 \psi_n}{\partial u^r \partial u^s} = \frac{\partial}{\partial u^s}\left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^r} u^j\right), \tag{13.6}$$

there locally exist functions $F_n(u)$, $1 \leq n \leq L$, such that

$$\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} u^j = \frac{\partial F_n}{\partial u^s}, \qquad F_n = \frac{\partial \psi_n}{\partial u^j} u^j - \psi_n. \tag{13.7}$$

Thus the system of hydrodynamic type (13.5) has the local form

$$u^i_{t_1} = \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn}\eta^{ip} F_n(u) \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x. \tag{13.8}$$

This system of hydrodynamic type is bi-Hamiltonian with respect to the compatible Hamiltonian operators (12.2) and (12.3):

$$u^i_{t_1} = \sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn}\eta^{ip}\eta^{jr}\frac{\partial^2\psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \left(\frac{\partial^2 \psi_n}{\partial u^r \partial u^s} u^s_x \frac{\delta H_1}{\delta u^j(x)}\right), \tag{13.9}$$

$$H_1 = \int h_1(u(x))dx, \qquad h_1(u(x)) = \frac{1}{2}\eta_{ij} u^i(x) u^j(x), \tag{13.10}$$

$$u^i_{t_1} = \eta^{ij}\frac{d}{dx}\frac{\delta H_2}{\delta u^j(x)}, \qquad H_2 = \int h_2(u(x))dx, \tag{13.11}$$

since in our case there always locally exists a function $h_2(u)$ such that

$$\sum_{m=1}^{L}\sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} F_n(u) = \frac{\partial^2 h_2}{\partial u^j \partial u^k}. \tag{13.12}$$

Indeed, we have

$$\frac{\partial}{\partial u^i}\left(\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}F_n(u)\right)=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^3\psi_m}{\partial u^i\partial u^j\partial u^k}F_n(u)+$$
$$+\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}\frac{\partial F_n}{\partial u^i}=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^3\psi_m}{\partial u^i\partial u^j\partial u^k}F_n(u)+$$
$$+\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}\frac{\partial^2\psi_n}{\partial u^i\partial u^s}u^s=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^3\psi_m}{\partial u^i\partial u^j\partial u^k}F_n(u)+$$
$$+\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^i}\frac{\partial^2\psi_n}{\partial u^k\partial u^s}u^s, \tag{13.13}$$

where we have used the relation (4.8). Consequently, by virtue of symmetry with respect to the indices i and j, we obtain

$$\frac{\partial}{\partial u^i}\left(\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}F_n(u)\right)=\frac{\partial}{\partial u^j}\left(\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^i\partial u^k}F_n(u)\right), \tag{13.14}$$

i.e., there locally exist functions $a_k(u)$, $1\le k\le N$, such that

$$\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}F_n(u)=\frac{\partial a_k}{\partial u^j}. \tag{13.15}$$

By virtue of symmetry with respect to the indices j and k, we obtain

$$\frac{\partial a_k}{\partial u^j}=\frac{\partial a_j}{\partial u^k}, \tag{13.16}$$

i.e., there locally exists a function $h_2(u)$ such that

$$a_k(u)=\frac{\partial h_2}{\partial u^k}. \tag{13.17}$$

Thus

$$\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}F_n(u)=\frac{\partial a_k}{\partial u^j}=\frac{\partial^2 h_2}{\partial u^j\partial u^k}. \tag{13.18}$$

Consider the next equation in the integrable hierarchy (13.3):

$$u^i_{t_2}=(R^2)^i_j u^j_x=$$
$$=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\eta^{ip}\frac{\partial^2\psi_m}{\partial u^p\partial u^k}u^k_x\left(\frac{d}{dx}\right)^{-1}\circ\frac{\partial^2\psi_n}{\partial u^j\partial u^s}u^s_x\left(\frac{d}{dx}\right)^{-1}\circ\eta^{jr}\frac{d}{dx}\frac{\delta H_2}{\delta u^r(x)}=$$
$$=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\eta^{ip}\frac{\partial^2\psi_m}{\partial u^p\partial u^k}u^k_x\left(\frac{d}{dx}\right)^{-1}\circ\frac{\partial^2\psi_n}{\partial u^j\partial u^s}u^s_x\eta^{jr}\frac{\partial h_2}{\partial u^r}. \tag{13.19}$$

Let us prove that in our case there always locally exist functions $G_n(u)$, $1\le n\le L$, such that

$$\frac{\partial^2\psi_n}{\partial u^j\partial u^s}\eta^{jr}\frac{\partial h_2}{\partial u^r}=\frac{\partial G_n}{\partial u^s}. \tag{13.20}$$

Indeed, we have

$$\frac{\partial}{\partial u^p}\left(\frac{\partial^2\psi_n}{\partial u^j\partial u^s}\eta^{jr}\frac{\partial h_2}{\partial u^r}\right)=\frac{\partial^3\psi_n}{\partial u^j\partial u^s\partial u^p}\eta^{jr}\frac{\partial h_2}{\partial u^r}+\frac{\partial^2\psi_n}{\partial u^j\partial u^s}\eta^{jr}\frac{\partial^2 h_2}{\partial u^r\partial u^p}=$$
$$=\frac{\partial^3\psi_n}{\partial u^j\partial u^s\partial u^p}\eta^{jr}\frac{\partial h_2}{\partial u^r}+\frac{\partial^2\psi_n}{\partial u^j\partial u^s}\eta^{jr}\left(\sum_{k=1}^{L}\sum_{l=1}^{L}\mu^{kl}\frac{\partial^2\psi_k}{\partial u^r\partial u^p}F_l(u)\right)=$$
$$=\frac{\partial^3\psi_n}{\partial u^j\partial u^s\partial u^p}\eta^{jr}\frac{\partial h_2}{\partial u^r}+\sum_{k=1}^{L}\sum_{l=1}^{L}\mu^{kl}\eta^{jr}\frac{\partial^2\psi_n}{\partial u^s\partial u^j}\frac{\partial^2\psi_k}{\partial u^r\partial u^p}F_l(u)=$$
$$=\frac{\partial^3\psi_n}{\partial u^j\partial u^s\partial u^p}\eta^{jr}\frac{\partial h_2}{\partial u^r}+\sum_{k=1}^{L}\sum_{l=1}^{L}\mu^{kl}\eta^{jr}\frac{\partial^2\psi_k}{\partial u^s\partial u^j}\frac{\partial^2\psi_n}{\partial u^r\partial u^p}F_l(u)=$$
$$=\frac{\partial^3\psi_n}{\partial u^j\partial u^s\partial u^p}\eta^{jr}\frac{\partial h_2}{\partial u^r}+\sum_{k=1}^{L}\sum_{l=1}^{L}\mu^{kl}\eta^{jr}\frac{\partial^2\psi_k}{\partial u^s\partial u^r}\frac{\partial^2\psi_n}{\partial u^j\partial u^p}F_l(u), \tag{13.21}$$

where we have used the relation (4.9) and the symmetry of the matrix η^{jr}. Thus we have proved that the expression under consideration is symmetric with respect to the indices p and s, i.e.,

$$\frac{\partial}{\partial u^p}\left(\frac{\partial^2\psi_n}{\partial u^j\partial u^s}\eta^{jr}\frac{\partial h_2}{\partial u^r}\right)=\frac{\partial}{\partial u^s}\left(\frac{\partial^2\psi_n}{\partial u^j\partial u^p}\eta^{jr}\frac{\partial h_2}{\partial u^r}\right). \tag{13.22}$$

Consequently, there locally exist functions $G_n(u)$, $1\le n\le L$, such that the relation (13.20) is satisfied, and therefore, we have proved that the second flow in the integrable hierarchy (13.3) has the form of a local system of hydrodynamic type

$$u^i_{t_2}=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\eta^{ip}G_n(u)\frac{\partial^2\psi_m}{\partial u^p\partial u^k}u^k_x. \tag{13.23}$$

Repeating the preceding argument word for word, we prove by induction that if the functions $\psi_n(u)$, $1\le n\le L$, are a solution of the system of equations (4.8), (4.9), then for each $s\ge 1$ and for the corresponding function $h_s(u(x))$ (starting from the function $h_1(u(x))=\frac{1}{2}\eta_{ij}u^i(x)u^j(x)$) there always locally exist functions $F^{(s)}_n(u)$, $1\le n\le L$, such that

$$\frac{\partial^2\psi_n}{\partial u^j\partial u^p}\eta^{jr}\frac{\partial h_s}{\partial u^r}=\frac{\partial F^{(s)}_n}{\partial u^p} \tag{13.24}$$

and there always locally exists a function $h_{s+1}(u(x))$ such that

$$\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\frac{\partial^2\psi_m}{\partial u^j\partial u^k}F^{(s)}_n(u)=\frac{\partial^2 h_{s+1}}{\partial u^j\partial u^k}. \tag{13.25}$$

Above we have already proved that this statement is true for $s=1$ (in this case, in particular, $F^{(1)}_n=F_n$, $F^{(2)}_n=G_n$). It can be proved in just the same way that if this statement is true for $s=K\ge 1$, then it is true also for $s=K+1$ (see (13.13)–(13.18) and (13.21), (13.22)). Thus we have proved that for each $s\ge 1$ the corresponding flow of the integrable hierarchy (13.3) has the form of a local system of hydrodynamic type

$$u^i_{t_s}=\sum_{m=1}^{L}\sum_{n=1}^{L}\mu^{mn}\eta^{ip}F^{(s)}_n(u)\frac{\partial^2\psi_m}{\partial u^p\partial u^k}u^k_x. \tag{13.26}$$

All the flows in the hierarchy (13.3) are commuting integrable bi-Hamiltonian systems of hydrodynamic type with an infinite family of local integrals in involution with respect to both Poisson brackets:

$$u^i_{t_s} = M_1^{ij} \frac{\delta H_s}{\delta u^j(x)} = \{u^i(x), H_s\}_1, \quad H_s = \int h_s(u(x))dx, \tag{13.27}$$

$$u^i_{t_s} = M_2^{ij} \frac{\delta H_{s+1}}{\delta u^j(x)} = \{u^i(x), H_{s+1}\}_2, \quad H_{s+1} = \int h_{s+1}(u(x))dx, \tag{13.28}$$

$$\{H_p, H_r\}_1 = 0, \quad \{H_p, H_r\}_2 = 0, \tag{13.29}$$

and the densities $h_s(u(x))$ of the Hamiltonians are related by the recursion relations (13.24), (13.25), which are always solvable in our case.

14 Locality and integrability of Hamiltonian systems with nonlocal Poisson brackets of hydrodynamic type

Let the functions $\psi_n(u)$, $1 \le n \le L$, be a solution of the integrable system (4.8), (4.9) of nonlinear equations describing all flat torsionless submanifolds in pseudo-Euclidean spaces; in particular, in this case the nonlocal operator M_1^{ij} given by the formula (12.2) is Hamiltonian and compatible with the constant Hamiltonian operator M_2^{ij} (12.3).

Consider the Hamiltonian system

$$u^i_t = M_1^{ij} \frac{\delta H}{\delta u^j(x)} = \{u^i(x), H\}_1 \tag{14.1}$$

with an arbitrary Hamiltonian of hydrodynamic type

$$H = \int h(u(x))dx. \tag{14.2}$$

Ferapontov proved in [12] that a Hamiltonian system with a nonlocal Hamiltonian operator of hydrodynamic type (8.1) and with a Hamiltonian of hydrodynamic type (14.2) is local if and only if the Hamiltonian is an integral of all the structural flows of the nonlocal Hamiltonian operator. This statement is also true for Hamiltonian operators of the form (9.3), and moreover, it is always true for any weakly nonlocal Hamiltonian operators (see [14]). We prove that for the nonlocal Hamiltonian operators M_1^{ij} (12.2) given by solutions of the integrable system (4.8), (4.9) this condition on the Hamiltonians is sufficient for integrability, i.e., all the corresponding local Hamiltonian systems (14.1), (14.2) are integrable bi-Hamiltonian systems.

Lemma 14.1 [6] *The system* (14.1), (14.2) *is local if and only if the density* $h(u(x))$ *of the Hamiltonian satisfies the linear equations*

$$\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial^2 h}{\partial u^r \partial u^p} = \frac{\partial^2 \psi_n}{\partial u^j \partial u^p} \eta^{jr} \frac{\partial^2 h}{\partial u^r \partial u^s}, \quad 1 \le n \le L. \tag{14.3}$$

Consider a system (14.1), (14.2):

$$u^i_t = M_1^{ij} \frac{\delta H}{\delta u^j(x)} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \left(\eta^{jr} \frac{\partial^2 \psi_n}{\partial u^r \partial u^s} u^s_x \frac{\partial h}{\partial u^j}\right). \tag{14.4}$$

The system (14.4) is local if and only if there locally exist functions $P_n(u)$, $1 \le n \le L$, such that

$$\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial h}{\partial u^r} = \frac{\partial P_n}{\partial u^s}, \tag{14.5}$$

i.e., if and only if the consistency relation

$$\frac{\partial}{\partial u^p} \left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^s} \eta^{jr} \frac{\partial h}{\partial u^r} \right) = \frac{\partial}{\partial u^s} \left(\frac{\partial^2 \psi_n}{\partial u^j \partial u^p} \eta^{jr} \frac{\partial h}{\partial u^r} \right) \tag{14.6}$$

is satisfied. Then the system (14.4) takes a local form

$$u^i_t = M_1^{ij} \frac{\delta H}{\delta u^j(x)} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \eta^{ip} \frac{\partial^2 \psi_m}{\partial u^p \partial u^k} P_n(u) u^k_x. \tag{14.7}$$

The consistency relation (14.6) is equivalent to the linear equations (14.3).

Theorem 14.1 [6] *If the functions $\psi_n(u)$, $1 \le n \le L$, are a solution of the integrable system* (4.8), (4.9) *and the corresponding Hamiltonian system* (14.1), (14.2) *is local, i.e., the density $h(u(x))$ of the Hamiltonian satisfies the linear equations* (14.3)*, then this Hamiltonian system is integrable and bi-Hamiltonian.*

Proof. In this case the system (14.1), (14.2) takes the form (14.7), (14.5). Let us prove that there always locally exists a function $f(u)$ such that

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} P_n(u) = \frac{\partial^2 f}{\partial u^j \partial u^k}. \tag{14.8}$$

Indeed, we have

$$\begin{aligned} &\frac{\partial}{\partial u^i} \left(\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} P_n(u) \right) = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^3 \psi_m}{\partial u^i \partial u^j \partial u^k} P_n(u) + \\ &+ \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} \frac{\partial P_n}{\partial u^i} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^3 \psi_m}{\partial u^i \partial u^j \partial u^k} P_n(u) + \\ &+ \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} \frac{\partial^2 \psi_n}{\partial u^i \partial u^p} \eta^{pr} \frac{\partial h}{\partial u^r} = \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^3 \psi_m}{\partial u^i \partial u^j \partial u^k} P_n(u) + \\ &+ \sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^i} \frac{\partial^2 \psi_n}{\partial u^k \partial u^p} \eta^{pr} \frac{\partial h}{\partial u^r}, \end{aligned} \tag{14.9}$$

where we have used the relation (4.8). Consequently, by virtue of symmetry with respect to the indices i and j, we obtain

$$\frac{\partial}{\partial u^i} \left(\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} P_n(u) \right) = \frac{\partial}{\partial u^j} \left(\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^i \partial u^k} P_n(u) \right), \tag{14.10}$$

i.e., there locally exist functions $b_k(u)$, $1 \le k \le N$, such that

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} P_n(u) = \frac{\partial b_k}{\partial u^j}. \tag{14.11}$$

By virtue of symmetry with respect to the indices j and k, we obtain

$$\frac{\partial b_k}{\partial u^j} = \frac{\partial b_j}{\partial u^k}, \tag{14.12}$$

i.e., there locally exists a function $f(u)$ such that

$$b_k(u) = \frac{\partial f}{\partial u^k}. \tag{14.13}$$

Thus we have

$$\sum_{m=1}^{L} \sum_{n=1}^{L} \mu^{mn} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} P_n(u) = \frac{\partial b_k}{\partial u^j} = \frac{\partial^2 f}{\partial u^j \partial u^k}. \tag{14.14}$$

Consequently, the system (14.1), (14.2) in the case under consideration can be presented in the form

$$u_t^i = \eta^{ij} \frac{\partial^2 f}{\partial u^j \partial u^k} u_x^k = M_2^{ij} \frac{\delta F}{\delta u^j(x)} = \{u^i(x), F\}_2, \quad F = \int f(u) dx, \tag{14.15}$$

i.e., it is an integrable bi-Hamiltonian system with the compatible Hamiltonian operators M_1^{ij} (12.2) and M_2^{ij} (12.3).

Thus we have an integrable description (4.8), (4.9) and (14.3) of the class of local Hamiltonian systems of the form (14.1), (14.2) (we have proved that each of these local Hamiltonian systems is integrable and bi-Hamiltonian).

15 Systems of integrals in involution generated by arbitrary flat torsionless submanifolds in pseudo-Euclidean spaces

The nonlinear equations of the form (4.9) and (14.3) are of independent interest. They play an important role and have a very natural interpretation.

Lemma 15.1 [6] *The nonlinear equations* (4.9) *are equivalent to the condition that the integrals*

$$\Psi_n = \int \psi_n(u(x)) dx, \quad 1 \leq n \leq L, \tag{15.1}$$

are in involution with respect to the Poisson bracket defined by the constant Hamiltonian operator M_2^{ij} (12.3)*, i.e., the condition*

$$\{\Psi_n, \Psi_m\}_2 = 0, \quad 1 \leq n, m \leq L. \tag{15.2}$$

Proof. Indeed, we have

$$\{\Psi_n, \Psi_m\}_2 = \int \frac{\partial \psi_n}{\partial u^i} \eta^{ij} \frac{d}{dx} \frac{\partial \psi_m}{\partial u^j} dx = \int \frac{\partial \psi_n}{\partial u^i} \eta^{ij} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} u_x^k dx. \tag{15.3}$$

Consequently, the integrals are in involution, i.e.,

$$\{\Psi_n, \Psi_m\}_2 = 0, \tag{15.4}$$

if and only if there exists a function $S_{nm}(u)$ such that

$$\frac{\partial \psi_n}{\partial u^i} \eta^{ij} \frac{\partial^2 \psi_m}{\partial u^j \partial u^k} = \frac{\partial S_{nm}}{\partial u^k}, \tag{15.5}$$

i.e., if and only if the consistency relation

$$\frac{\partial}{\partial u^l}\left(\frac{\partial \psi_n}{\partial u^i}\eta^{ij}\frac{\partial^2 \psi_m}{\partial u^j \partial u^k}\right) = \frac{\partial}{\partial u^k}\left(\frac{\partial \psi_n}{\partial u^i}\eta^{ij}\frac{\partial^2 \psi_m}{\partial u^j \partial u^l}\right) \tag{15.6}$$

is satisfied. The consistency relation (15.6) is equivalent to the equations (4.9).

Likewise, the equations (14.3) are equivalent to the condition

$$\{\Psi_n, H\}_2 = 0, \qquad H = \int h(u(x))dx. \tag{15.7}$$

We note that the equations (5.10) are equivalent to the condition that L integrals are in involution with respect to an arbitrary Dubrovin–Novikov bracket (a nondegenerate local Poisson bracket of hydrodynamic type).

Theorem 15.1 *If the functions $\psi_n(u)$, $1 \le n \le L$, are a solution of the integrable system* (4.8), (4.9) *of nonlinear equations describing all flat torsionless submanifolds in pseudo-Euclidean spaces, then the integrals $\Psi_n = \int \psi_n(u(x))dx$* (15.1) *are in involution with respect to both the Poisson brackets given by the nonlocal Hamiltonian operator M_1^{ij}* (12.2) *and the constant Hamiltonian operator M_2^{ij}* (12.3)*, and therefore the integrals are in involution with respect to the corresponding pencil of Poisson brackets*:

$$\{\Psi_n, \Psi_m\}_1 = 0, \quad \{\Psi_n, \Psi_m\}_2 = 0, \quad 1 \le n, m \le L. \tag{15.8}$$

Theorem 15.2 [6] *If the functions $\psi_n(u)$, $1 \le n \le L$, are a solution of the integrable system* (4.8), (4.9)*, then the corresponding Hamiltonian system* (14.1), (14.2) *is local if and only if it is generated by a family of $L+1$ integrals in involution with respect to the Poisson bracket defined by the constant Hamiltonian operator M_2^{ij}* (12.3); *namely,*

$$\Psi_n = \int \psi_n(u(x))dx, 1 \le n \le L,\ H = \int h(u(x))dx,\ \ \{\Psi_n, \Psi_m\}_2 = 0,\ \{\Psi_n, H\}_2 = 0,\ \ 1 \le n, m \le L. \tag{15.9}$$

Moreover, in this case the system (14.1), (14.2) *is an integrable bi-Hamiltonian system and*

$$\{\Psi_n, \Psi_m\}_1 = 0,\ \{\Psi_n, H\}_1 = 0, \quad 1 \le n, m \le L. \tag{15.10}$$

16 Systems of integrals in involution generated by the associativity equations

An important special class of integrals in involution is generated by the associativity equations of two-dimensional topological quantum field theory (the WDVV equations).

Theorem 16.1 [6], [15] *A function $\Phi(u^1, ..., u^N)$ generates a family of N integrals in involution with respect to the Poisson bracket defined by the constant Hamiltonian operator M_2^{ij}* (12.3)*, namely, integrals whose densities are the first-order partial derivatives of the function* (*the potential*) $\Phi(u)$

$$I_n = \int \frac{\partial \Phi}{\partial u^n}(u(x))dx, \quad \{I_n, I_m\}_2 = 0, \quad 1 \le n, m \le N, \tag{16.1}$$

if and only if the function $\Phi(u)$ is a solution of the associativity equations (2.16) *of two-dimensional topological quantum field theory* (*the WDVV equations*).

Such special families of integrals in involution (whose densities are the first-order partial derivatives of a potential $\Phi(u)$ and the potential $\Phi(u)$ itself) with respect to special nonlocal Poisson brackets and with respect to pencils of compatible Poisson brackets generated by the associativity equations will be also considered further in Section 18. Note that we have constructed an infinite family of integrals in involution with respect to both local and nonlocal Poisson brackets generated by an arbitrary flat torsionless submanifold in a pseudo-Euclidean space, in particular, by an arbitrary Frobenius manifold, in Section 13.

17 Associativity equations and nonlocal Poisson brackets of hydrodynamic type

Since the associativity equations (2.16) are a natural reduction (see Section 6) of the integrable system (4.8), (4.9) of nonlinear equations describing all flat torsionless submanifolds in pseudo-Euclidean spaces and generating all nonlocal Hamiltonian operators of hydrodynamic type with flat metrics, each solution $\Phi(u)$ of the associativity equations (2.16), which are known to be consistent and integrable by the inverse scattering method and possess a rich set of nontrivial solutions (see [1]), defines a nonlocal Hamiltonian operator of hydrodynamic type with a flat metric

$$L^{ij} = \eta^{ij} \frac{d}{dx} + \sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \eta^{ip} \eta^{jr} \frac{\partial^3 \Phi}{\partial u^p \partial u^m \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^3 \Phi}{\partial u^r \partial u^n \partial u^s} u^s_x \tag{17.1}$$

and even a pencil of compatible Hamiltonian operators

$$L^{ij}_{\lambda_1, \lambda_2} = \lambda_1 \eta^{ij} \frac{d}{dx} + \lambda_2 \sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \eta^{ip} \eta^{jr} \frac{\partial^3 \Phi}{\partial u^p \partial u^m \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^3 \Phi}{\partial u^r \partial u^n \partial u^s} u^s_x, \tag{17.2}$$

where λ_1 and λ_2 are arbitrary constants. In particular, if $\Phi(u)$ is an arbitrary solution of the associativity equations (2.16), then the operator

$$L^{ij}_{0,1} = \sum_{m=1}^{N} \sum_{n=1}^{N} \eta^{mn} \eta^{ip} \eta^{jr} \frac{\partial^3 \Phi}{\partial u^p \partial u^m \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \circ \frac{\partial^3 \Phi}{\partial u^r \partial u^n \partial u^s} u^s_x \tag{17.3}$$

is a Hamiltonian operator compatible with the constant Hamiltonian operator

$$L^{ij}_{1,0} = \eta^{ij} \frac{d}{dx}. \tag{17.4}$$

The converse is also true.

Theorem 17.1 *The nonlocal operator $L^{ij}_{0,1}$* (17.3) *is Hamiltonian if and only if the function $\Phi(u)$ is a solution of the associativity equations* (2.16).

Therefore, for each solution of the associativity equations (2.16) (in particular, for each Frobenius manifold) we obtain the corresponding natural pencil of compatible Poisson structures (local and nonlocal) and the corresponding natural integrable hierarchies (see Section 13).

Thus, for each Frobenius manifold there are a very natural nonlocal Hamiltonian operator of the form (17.1), a pencil of compatible Hamiltonian operators (17.2) and very natural integrable hierarchies connected to the Frobenius manifold.

We have considered the nonlocal Hamiltonian operators of the form (8.3) with flat metrics and came to the associativity equations defining the affinors of such operators. A statement that is in some sense the converse is also true, namely, if all the affinors $w_n(u)$ of a nonlocal Hamiltonian operator (8.3) with $L = N$ are defined by an arbitrary solution $\Phi(u)$ of the associativity equations (2.16) by the formula

$$(w_n)^i_j(u) = \zeta^{is}\xi^r_j \frac{\partial^3 \Phi}{\partial u^n \partial u^s \partial u^r},$$

where ζ^{is}, ξ^r_j are arbitrary nondegenerate constant matrices, then the metric of this Hamiltonian operator must be flat. But, in general, it is not necessarily that this metric will be constant in the local coordinates under consideration.

The structural flows (see [12], [14]) of the nonlocal Hamiltonian operator (17.1) have the form:

$$u^i_{t_n} = \eta^{is} \frac{\partial^3 \Phi}{\partial u^s \partial u^n \partial u^k} u^k_x. \tag{17.5}$$

These systems are integrable bi-Hamiltonian systems of hydrodynamic type and coincide with the primary part of the Dubrovin hierarchy constructed by any solution of the associativity equations in [1]. The condition of commutation for the structural flows (17.5) is also equivalent to the associativity equations (2.16).

Theorem 17.2 *For an arbitrary solution $\Phi(u)$ of the associativity equations* (2.16)*, each structural flow* (17.5) *generates an integrable hierarchy of hydrodynamic type with the recursion operator given by the compatible Hamiltonian operators* (17.3) *and* (17.4) (*see the recursion operator* (13.1) *in a more general case*); *each of these integrable hierarchies is local and bi-Hamiltonian with respect to the compatible Hamiltonian operators* (17.3) *and* (17.4).

A great number of concrete examples of Frobenius manifolds and solutions of the associativity equations is given in Dubrovin's paper [1]. Consider here only one simple example from [1] as an illustration. Let $N = 3$ and the metric η_{ij} be antidiagonal

$$(\eta_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{17.6}$$

and the function $\Phi(u)$ has the form

$$\Phi(u) = \frac{1}{2}(u^1)^2 u^3 + \frac{1}{2} u^1 (u^2)^2 + f(u^2, u^3).$$

In this case e_1 is the unit in the Frobenius algebra (2.14), (2.15), and the associativity equations (2.16) for the function $\Phi(u)$ are equivalent to the following remarkable integrable Dubrovin equation for the function $f(u^2, u^3)$:

$$\frac{\partial^3 f}{\partial (u^3)^3} = \left(\frac{\partial^3 f}{\partial (u^2)^2 \partial u^3} \right)^2 - \frac{\partial^3 f}{\partial (u^2)^3} \frac{\partial^3 f}{\partial u^2 \partial (u^3)^2}. \tag{17.7}$$

This equation is connected to quantum cohomology of projective plane and classical problems of enumerative geometry (see [16]). In particular, all nontrivial polynomial solutions of the equation (17.7) that satisfy the requirement of the quasihomogeneity and locally define a structure of Frobenius manifold are described by Dubrovin in [1]:

$$f = \frac{1}{4}(u^2)^2(u^3)^2 + \frac{1}{60}(u^3)^5, \quad f = \frac{1}{6}(u^2)^3 u^3 + \frac{1}{6}(u^2)^2(u^3)^3 + \frac{1}{210}(u^3)^7, \tag{17.8}$$

$$f = \frac{1}{6}(u^2)^3(u^3)^2 + \frac{1}{20}(u^2)^2(u^3)^5 + \frac{1}{3960}(u^3)^{11}. \tag{17.9}$$

As is shown by the author in [17] (see also [18]), the equation (17.7) is equivalent to the integrable nondiagonalizable homogeneous system of hydrodynamic type

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_{u^3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & 2b & -a \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}_{u^2}, \tag{17.10}$$

$$a = \frac{\partial^3 f}{\partial (u^2)^3}, \quad b = \frac{\partial^3 f}{\partial (u^2)^2 \partial u^3}, \quad c = \frac{\partial^3 f}{\partial u^2 \partial (u^3)^2}. \tag{17.11}$$

In this case the affinors of the nonlocal Hamiltonian operator (17.1) have the form:

$$(w_1)^i_j(u) = \delta^i_j, \quad (w_2)^i_j(u) = \begin{pmatrix} 0 & b & c \\ 1 & a & b \\ 0 & 1 & 0 \end{pmatrix}, \quad (w_3)^i_j(u) = \begin{pmatrix} 0 & c & b^2 - ac \\ 0 & b & c \\ 1 & 0 & 0 \end{pmatrix}. \tag{17.12}$$

For concrete solutions of the associativity equation (17.7), in particular, for (17.8) and (17.9), the corresponding linear systems (7.1), (7.2) giving explicit realizations of the corresponding Frobenius manifolds as potential flat torsionless submanifolds in pseudo-Euclidean spaces can be solved in special functions; we shall give the explicit realizations in a separate paper.

18 Associativity equations and special integrals in involution with respect to nonlocal Poisson brackets of hydrodynamic type and Poisson pencils

If the function $\Phi(u^1, \ldots, u^N)$ is an arbitrary solution of the associativity equations (2.16), then the operator $L^{ij}_{0,1}$ (17.3) is a Hamiltonian operator, and we can consider the corresponding Poisson bracket and integrals in involution with respect to this Poisson bracket.

Theorem 18.1 [15] *If the function $\Phi(u^1, \ldots, u^N)$ satisfies the associativity equations* (2.16)*, then the functionals $I_n = \int (\partial \Phi / \partial u^n) dx$, $n = 1, \ldots, N$,* (16.1) *are in involution with respect to the Poisson bracket given by the nonlocal Hamiltonian operator $L^{ij}_{0,1}$* (17.3)*, i.e.,*

$$\{I_n, I_m\}_1 = 0, \qquad n, m = 1, \ldots, N, \tag{18.1}$$

where

$$\{u^i(x), u^j(y)\}_1 = \eta^{mn} \eta^{ip} \eta^{jr} \frac{\partial^3 \Phi}{\partial u^p \partial u^m \partial u^k} u^k_x \left(\frac{d}{dx} \right)^{-1} \left(\frac{\partial^3 \Phi}{\partial u^r \partial u^n \partial u^s} u^s_x \delta(x - y) \right). \tag{18.2}$$

The Poisson brackets $\{u^i(x), u^j(y)\}_2 = \eta^{ij} \delta'(x - y)$ given by the constant Hamiltonian operator $L^{ij}_{1,0}$ (17.4) and $\{u^i(x), u^j(y)\}_1$ given by the nonlocal Hamiltonian operator $L^{ij}_{0,1}$ (17.3) are compatible and form a pencil $\lambda_1 \{u^i(x), u^j(y)\}_1 + \lambda_2 \{u^i(x), u^j(y)\}_2$ of Poisson brackets, where λ_1 and λ_2 are arbitrary constants, so that for any constants λ_1 and λ_2 the bracket

$$\{u^i(x), u^j(y)\}_{\lambda_1, \lambda_2} = \lambda_1 \{u^i(x), u^j(y)\}_1 + \lambda_2 \{u^i(x), u^j(y)\}_2$$

is a Poisson bracket.

Corollary 18.1 [15] *If the function $\Phi(u^1,\dots,u^N)$ satisfies the associativity equations* (2.16)*, then the functionals $I_n = \int(\partial\Phi/\partial u^n)dx$, $n=1,\dots,N$,* (16.1) *are in involution with respect to the pencil of Poisson brackets $\{u^i(x),u^j(y)\}_{\lambda_1,\lambda_2} = \lambda_1\{u^i(x),u^j(y)\}_1+\lambda_2\{u^i(x),u^j(y)\}_2$, where λ_1 and λ_2 are arbitrary constants.*

We consider also the functional F whose density is the potential $\Phi(u^1(x),\dots,u^N(x))$ itself:

$$F = \int \Phi(u^1(x),\dots,u^N(x))dx. \tag{18.3}$$

Theorem 18.2 [15] *A function $\Phi(u^1,...,u^N)$ generates a family of $N+1$ integrals in involution with respect to the constant Poisson bracket $\{u^i(x),u^j(y)\}_2 = \eta^{ij}\delta'(x-y)$, namely, the functional F* (18.3) *and the functionals $I_n = \int(\partial\Phi/\partial u^n)dx$, $n=1,\dots,N$,* (16.1)*, $\{I_n,I_m\}_2 = 0$, $1\le n,m\le N$, $\{I_n,F\}_2=0$, $1\le n\le N$, if and only if the function $\Phi(u^1,\dots,u^N)$ satisfies the equations*

$$\frac{\partial^2\Phi}{\partial u^k\partial u^i}\eta^{ij}\frac{\partial^3\Phi}{\partial u^j\partial u^n\partial u^l} = \frac{\partial^2\Phi}{\partial u^l\partial u^i}\eta^{ij}\frac{\partial^3\Phi}{\partial u^j\partial u^n\partial u^k}. \tag{18.4}$$

The equations (18.4) have arised in a different context in the author's papers [19]–[21] and play an important role in the theory of compatible Poisson brackets of hydrodynamic type, the theory of the associativity equations and the theory of Frobenius manifolds.

Theorem 18.3 [15] *If the function $\Phi(u^1,\dots,u^N)$ satisfies the equations* (18.4)*, then the functional F* (18.3) *and the functionals $I_n = \int(\partial\Phi/\partial u^n)dx$, $n=1,\dots,N$,* (16.1) *are in involution with respect to the Poisson bracket $\{u^i(x),u^j(y)\}_1$ given by the nonlocal Hamiltonian operator $L^{ij}_{0,1}$* (17.3)*.*

Corollary 18.2 [15] *If the function $\Phi(u^1,\dots,u^N)$ satisfies the equations* (18.4)*, then the functional F* (18.3) *and the functionals $I_n = \int(\partial\Phi/\partial u^n)dx$, $n=1,\dots,N$,* (16.1) *are in involution with respect to the pencil of Poisson brackets $\{u^i(x),u^j(y)\}_{\lambda_1,\lambda_2} = \lambda_1\{u^i(x),u^j(y)\}_1+\lambda_2\{u^i(x),u^j(y)\}_2$, where λ_1 and λ_2 are arbitrary constants.*

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