

Schrodinger Picture of Quantum Gravitational Collapse

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The functional Schrodinger equation is used to study the quantum collapse of a gravitating, spherical domain wall and a massless scalar field coupled to the metric. The approach includes backreaction of pre-Hawking radiation on the gravitational collapse. Truncating the degrees of freedom to a minisuperspace leads to an integro-differential Schrodinger equation. We define a “black hole” operator and find its eigenstates. The black hole operator does not commute with the Hamiltonian, leading to an energy-black holeness uncertainty relation. We discuss energy eigenstates and also obtain a partial differential equation for the time-dependent gravitational collapse problem.

I. INTRODUCTION

Gravitational collapse is expected to result in the formation of a black hole, which then evaporates by Hawking radiation [1]. During this process, any information contained in the initial state is lost, since black hole evaporation leads to thermal radiation which is uncorrelated with the initial state. If the initial state is chosen to be a pure quantum state, the final state would be described by a density matrix and would not be a pure state. The evolution of a pure quantum state into a mixed state violates the tenets of quantum mechanics (*e.g.* [2]). These problems of black hole formation, evaporation and information loss have been central to discussions of combining general relativity and quantum mechanics over the last several decades.

A number of approaches have been made to resolve the issues that arise in combining black holes and quantum field theory, including more sophisticated calculations of the emitted radiation, modifications of quantum mechanics, modifications of quantum field theory, lower dimensional calculations, loop gravity, and string theory (*e.g.* for some reviews see [2, 3, 4, 5, 6]). Depending on the approach and the new ingredient in the calculation, the conclusions have varied, and no consensus has emerged so far. The issues have become ever more pressing with discussion of black hole formation in highly energetic collisions in LHC [7, 8, 9] and by ultra-high energy cosmic rays colliding with the atmosphere [10].

In contrast to earlier studies in which a black hole spacetime is adopted as an arena for quantum field theory, we wish to study the temporal development of an initial state that does not contain a black hole. From the laws of quantum mechanics, the initial state will evolve unitarily in time. If unitary evolution leads to black hole formation then, as has been described by Hawking and others [1, 2], unitarity will be lost.

The 3+1 dimensional analysis in Ref. [11] used the functional Schrodinger formalism to study the temporal evolution of a scalar field in the *classical* background of a gravitating, collapsing, spherical domain wall. It was found that the collapse was accompanied by quantum radiation of scalar particles, called “pre-Hawking” radiation, with a non-thermal spectrum. The analyses in

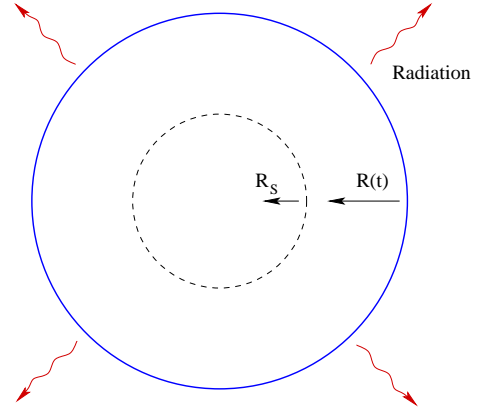


FIG. 1: A picture of the system under consideration. A spherical domain wall with radius $R(t)$ collapses towards its Schwarzschild radius, R_S , and emits scalar radiation in the process. Due to the emission, R_S also decreases.

Refs. [11, 12] were clearly limited by their use of the semiclassical approximation, in which the backreaction of the scalar radiation on the gravitational collapse spacetime is ignored. The purpose of the present work is to attempt to remedy this shortcoming within the functional Schrodinger formalism. The aim of the present work is to provide a framework to study quantum gravitational collapse beyond the semiclassical approximation. The framework includes backreaction of the scalar radiation on the collapse dynamics (see Fig. 1).

The specifics of the domain wall plus scalar field system are not expected to be very important for the questions we are interested in. For example, the Hamiltonian for a collapsing wall that is close to forming a black hole reduces to the form for an ultra-relativistic particle [11], which is expected to be true for any form of matter. So the choice of a collapsing domain wall does not affect the near-horizon dynamics. The scalar field can also be thought of as representing one degree of freedom of a photon or a graviton. One assumed property that is special though, is the assumed masslessness of the scalar field. If the scalar field has a mass and/or carries a global or gauge charge, we do expect some differences that we discuss briefly in Sec. VIII.

In general, the Schrodinger formulation yields a functional differential equation for the wavefunctional, $\Psi[g_{\mu\nu}, X^\mu, \Phi, t]$, where $g_{\mu\nu}$ is the metric, X^μ the domain wall position, and Φ the scalar field. We will truncate this problem to minisuperspace, considering only spherical domain walls described by a radial function, $R(t)$, the metric fixed to the Schwarzschild form though with variable Schwarzschild radius, R_S , and Φ decomposed into modes with mode coefficients $\{a_k\}$. Then the wavefunctional gets replaced by an ordinary wavefunction, $\Psi(R, \{a_k\}, t)$. We are interested in finding the time evolution of this minisuperspace wavefunction, starting from black hole free initial conditions, and in determining if there is a breakdown of the evolution at some time.

In any treatment of the gravitational collapse problem, it is necessary to choose a time coordinate since the collapse, by its very definition, is an evolution problem. In the current analysis, we have chosen to work with the Schwarzschild time coordinate. However, there is the possibility that we may miss some portion of spacetime due to this choice of slicing. If this were the case, we would expect some sickness (*e.g.* geodesic incompleteness) to show up in the temporal evolution of the wavefunction. If the temporal evolution remains well-behaved at all times, the evolution is no different from that of an ordinary star for which Schwarzschild coordinates are valid and commonly used. In other words, the Schrodinger formulation simply evolves the system forward in time, without assuming the presence or absence of an event horizon in the future. This is different from other approaches where quantum field theory is used on a spacetime background, where the entire spacetime must be assumed at the onset and then sliced using some coordinate system. However, it would be worthwhile to rework gravitational collapse in the Schrodinger formalism with a different choice of time coordinate.

The backreaction problem, even for classical point charges, is notorious for its non-local nature, since the trajectory depends on the radiative losses over the entire past. However, this problem only occurs in a perturbative treatment of a point charge, since there is then a zeroth order trajectory due to which there is radiation, and then backreaction to first order in some coupling constant, then a first order correction to the trajectory, then radiation and backreaction to second order, *ad infinitum*. Instead, if the classical point charge is replaced by a regular solution to some field theory, as in a 't Hooft-Polyakov magnetic monopole, a classical solution of the field theory will include the full dynamics and radiation of the point (magnetic) charge. Similarly, the wavefunction for the spherical wall and radiation in the Schrodinger formalism, includes the full dynamics of the wall and the radiation and non-locality is absent for this reason. However, there are two other reasons that make the analysis difficult and lead to non-locality. The first is that the pre-Hawking scalar radiation not only affects the dynamics of the collapse but also contributes to the precise form of the metric. In the minisuperspace approach we

have adopted, a form of the metric needs to be specified. This is chosen to be of the Schwarzschild form and thus amounts to the assumption that the energy-momentum tensor in the pre-Hawking radiation only causes negligible departures from the minisuperspace. We expect this approximation to be justified for large collapsing mass where we know that the energy-momentum density of the radiation is comparatively small. This has also been explicitly verified in the semiclassical calculation [13]. Also, the present approach is similar to that taken in calculating radiation backreaction on an accelerating charge in electrodynamics where the backreaction is taken to affect the dynamics of the charge but the effects on the Coulomb electric field of the charge is not considered. The second factor that complicates the analysis is the non-linear nature of gravity. In particular, the minisuperspace Hamiltonian depends on the mass of the collapsing object, and this is itself related to the Hamiltonian. In order to isolate the Hamiltonian, we need to invert the momentum operator and this leads to an integro-differential form for the Hamiltonian. Solutions of the corresponding integro-differential Schrodinger equation are hard to find in general but we are able to transform the problem to a purely differential equation in the “incipient limit” where the collapse approaches black hole formation. We discuss energy eigenstates in Sec. VI and the time-dependent gravitational collapse problem in Sec. VII.

In order to solve the gravitational collapse problem, we need to solve the Schrodinger equation to obtain a wavefunction that describes the collapsing wall and radiation. The first task, however, is to specify an initial value for the wavefunction, such that the initial state itself does not contain a black hole. This means that we need to specify a criterion for deciding if a given wavefunction is black hole free. To answer this question, we propose a “black hole operator” in Sec. V. Eigenvalues of the black hole operator signify the “black holeness” of the state. Interestingly we find that the black hole operator does not commute with the Hamiltonian, implying an uncertainty relation between energy and black holeness.

We start by discussing the classical Hamiltonian (Sec. II), which is then promoted to a quantum Hamiltonian in Sec. III. We then discuss explicit representations of an important operator in Sec. IV and use it to define a black hole free state in Sec. V. We then discuss energy eigenstates in Sec. VI and the gravitational collapse problem in Sec. VII. We discuss our results and conclude in Sec. VIII. A discussion of Hermiticity of certain operators can be found in Appendix A.

II. CLASSICAL HAMILTONIAN

The action contains the Einstein-Hilbert, massless scalar field, and Nambu-Goto terms

$$S = \int d^4x \sqrt{-g} \left[-\frac{\mathcal{R}}{16\pi G} + \frac{1}{2}(\partial_\mu \Phi)^2 \right] - \sigma \int d^3\xi \sqrt{-\gamma} \quad (1)$$

where σ is the wall tension and the domain wall world-volume metric is given by

$$\gamma_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \quad (2)$$

We will restrict our attention to spherical symmetry, in which case the form of the line element for the domain wall alone is [14]

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) dt^2 + \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \quad r > R(t) \quad (3)$$

where, $R_S = 2GM$ is the Schwarzschild radius in terms of the mass, M , of the wall, and $d\Omega^2$ is the usual angular line element. In the interior of the spherical domain wall, the line element is flat, as expected by Birkhoff's theorem,

$$ds^2 = -dT^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad r < R(t) \quad (4)$$

$$\frac{dT}{d\tau} = \left[1 + \left(\frac{dR}{d\tau} \right)^2 \right]^{1/2}, \quad \frac{dt}{d\tau} = \frac{1}{B} \left[B + \left(\frac{dR}{d\tau} \right)^2 \right]^{1/2} \quad (5)$$

$$B \equiv 1 - \frac{R_S}{R} \quad (6)$$

We will consider the case when the mass of the domain wall is large compared to the energy-momentum contribution of the scalar field and so it is a good approximation to ignore the scalar field when writing the metric.

Our next goal is to find the Hamiltonian for the wall-scalar field system. In Ref. [11] we have found that the mass of the wall can be written as

$$M = 4\pi\sigma R^2 \left[\frac{1}{\sqrt{1 - R_T^2}} - 2\pi G\sigma R \right] \quad (7)$$

or more suggestively

$$M = \frac{\tilde{M}}{\sqrt{1 - R_T^2}} - \frac{G\tilde{M}^2}{2R} \quad (8)$$

where $\tilde{M} \equiv \sigma 4\pi R^2$. The first term is the Lorentz boosted energy contribution while the second is the gravitational binding energy. Using this expression for the mass and the relation between T and t in Eq. (5) we find

$$H_{\text{wall}} = 4\pi\sigma B^{3/2} R^2 \left[\frac{1}{\sqrt{B^2 - \dot{R}^2}} - \frac{2\pi G\sigma R}{\sqrt{B^2 - (1 - B)\dot{R}^2}} \right] \quad (9)$$

where overdots denote derivatives with respect to t .

The form of the wall Hamiltonian simplifies in the in-cipient limit ($B \rightarrow 0$). Then the canonical momentum is given by

$$\Pi \approx \frac{4\pi\mu R^2 \dot{R}}{\sqrt{B}\sqrt{B^2 - \dot{R}^2}} \quad (10)$$

where $\mu \equiv \sigma(1 - 2\pi G\sigma R_S)$, leading to

$$H_{\text{wall}} \approx \frac{4\pi\mu B^{3/2} R^2}{\sqrt{B^2 - \dot{R}^2}} \quad (11)$$

$$= [(B\Pi)^2 + B(4\pi\mu R^2)^2]^{1/2} \quad (12)$$

which has the form of the energy of a relativistic particle, $\sqrt{p^2 + m^2}$, with a position dependent mass. In the limit $B \rightarrow 0$, the mass term can be neglected – the wall is ultra-relativistic – and hence

$$H_{\text{wall}} \approx -B\Pi \quad (13)$$

where we have chosen the negative sign appropriate for describing a collapsing wall.

Next we introduce the scalar field Φ . Even when we include the scalar field, we will continue to use the metric in the Ipser-Sikivie form described above. This assumes that the dominant effect of backreaction on the metric is to change the wall energy which enters the metric via R_S . This is not rigorously true since the scalar field also contributes a non-vanishing energy-momentum density. However, we assume that this contribution is small compared to the energy in the wall. In the conventional case of evaporation from an existing black hole, this corresponds to the assumption that Hawking radiation causes the black hole to evaporate and lose mass, but the black hole metric remains Schwarzschild to a very good approximation.

The scalar field, Φ , is decomposed into a complete set of basis functions denoted by $\{f_k(r)\}$

$$\Phi = \sum_k a_k(t) f_k(r) \quad (14)$$

The exact form of the functions $f_k(r)$ will not be important for us. We will be interested in the wavefunction for the mode coefficients $\{a_k\}$.

The Hamiltonian for the scalar field modes is found by inserting the scalar field mode decomposition and the background metric into the action

$$S_\Phi = \int d^4x \sqrt{-g} \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \quad (15)$$

The Hamiltonian for the scalar field modes takes the form of coupled simple harmonic oscillators with R -dependent masses and couplings due to the non-trivial metric. In the regime $R \sim R_S$, for a normal mode denoted by b , the Hamiltonian is [11]

$$H_b = \left(1 - \frac{R_S}{R} \right) \frac{\pi^2}{2m} + \frac{K}{2} b^2 \quad (16)$$

where π is the momentum conjugate to b , and m and K are approximate constants whose precise values are not important for us. The reason m and K are only approximately constant is because they depend on R_S and, with backreaction included, R_S changes (slowly) with time. By treating m and K as constant, we are assuming that the dominant coupling between b and R is due to the $(1 - R_S/R)$ factor.

Hence the total Hamiltonian for the wall and the normal modes of the scalar field in the incipient limit is

$$H = H_{\text{wall}} + \sum_{\text{modes}} H_b = -B\Pi + \sum_{\text{modes}} \left\{ B \frac{\pi^2}{2m} + \frac{Kb^2}{2} \right\} \quad (17)$$

III. QUANTUM HAMILTONIAN

The wall Hamiltonian given by Eq. (9) is especially complicated because H_{wall} also enters the right-hand side through B . In the incipient limit, however, H_{wall} simplifies to the form Eq. (13) and is amenable to analysis. Now, in Eq. (13), we replace

$$R_S \rightarrow 2GH_{\text{wall}}$$

to get

$$H_{\text{wall}} = - \left(1 - \frac{2GH_{\text{wall}}}{R} \right) \Pi$$

and therefore

$$H_{\text{wall}} = - \left(1 - \frac{2G\Pi}{R} \right)^{-1} \Pi$$

allowing us to identify

$$B = \frac{1}{1 - 2G\Pi/R} \quad (18)$$

To quantize, we promote classical quantities to quantum operators. For the momentum operators we take [19]

$$\hat{\Pi} = -i \frac{\partial}{\partial R}, \quad \hat{\pi} = -i \frac{\partial}{\partial b} \quad (19)$$

The classical Hamiltonian gets promoted to an operator and is obtained by replacing classical variables by quantum operators and making sure that the end result is Hermitian. From the form of the classical Hamiltonian in Eq. (17), Hermiticity can be obtained if we choose the quantum operators corresponding to $B\Pi$ and B to be Hermitian. This is achieved by using

$$\hat{B}^{-1} = 1 - G \left(\frac{1}{R} \hat{\Pi} + \hat{\Pi} \frac{1}{R} \right) \quad (20)$$

and

$$\begin{aligned} \hat{B} &= \left\{ 1 - G \left(\frac{1}{R} \hat{\Pi} + \hat{\Pi} \frac{1}{R} \right) \right\}^{-1} \\ &= \sum_{n=0}^{\infty} \left\{ G \left(\frac{1}{R} \hat{\Pi} + \hat{\Pi} \frac{1}{R} \right) \right\}^n \end{aligned} \quad (21)$$

where $\hat{\Pi}$ is defined in Eq. (19). The Hamiltonian operator is

$$\hat{H} = -\frac{1}{2}(\hat{B}\hat{\Pi} + \hat{\Pi}\hat{B}) + \sum_{\text{modes}} \left\{ \hat{B} \frac{\hat{\pi}^2}{2m} + \frac{K}{2} b^2 \right\} \quad (22)$$

The Hermiticity of the Hamiltonian depends crucially on the Hermiticity of \hat{B} and we discuss this further in Appendix A.

Note that \hat{B} contains the inverse of the derivative operator and hence is really an integral operator. In Sec. IV we will explicitly find the integral representation of \hat{B} .

With the Hamiltonian in Eq. (22), we need to solve the Schrodinger equation

$$\hat{H}\Psi = i \frac{\partial \Psi}{\partial t} \quad (23)$$

where the minisuperspace wavefunction depends on b , R and t : $\Psi = \Psi(b, R, t)$.

We are mostly interested in the time evolution problem, where the initial wavefunction describes a collapsing wavepacket for the wall and the scalar field is in its ground state. Alternately we could study the stationary problem and seek eigenstates of the Hamiltonian

$$\Psi = e^{-iEt} \psi(b, R) \quad (24)$$

This leads to the eigenvalue problem

$$\hat{H}\psi = E\psi \quad (25)$$

We shall discuss the stationary problem further in Sec. VI but before doing that we find explicit expressions for the operators \hat{B} and \hat{B}^{-1} , and discuss the interpretation of the wavefunction in terms of a black hole free state.

IV. \hat{B} AND \hat{B}^{-1}

Using

$$[R, \Pi] = i \quad (26)$$

we find

$$\hat{B}^{-1} = 1 - i \frac{G}{R^2} - \frac{2G}{R} \hat{\Pi} \quad (27)$$

Let us define χ by

$$\Psi = \hat{B}^{-1} \chi \quad (28)$$

Then it is simple to find Ψ in terms of χ using Eq. (27)

$$\Psi = \left(1 - i\frac{G}{R^2}\right)\chi + i\frac{2G}{R}\partial_R\chi \quad (29)$$

We can also solve this differential equation to find χ in terms of Ψ

$$\chi = \hat{B}\Psi = -\frac{i\sqrt{R}}{2G}e^{iR^2/4G} \int^R dR' \sqrt{R'}e^{-iR'^2/4G}\Psi(R') \quad (30)$$

where the integral operator is indefinite. In other words, this gives us an explicit integral form for the \hat{B} operator

$$\hat{B}(\cdot) = -\frac{i\sqrt{R}}{2G}e^{iR^2/4G} \int^R dR' \sqrt{R'}e^{-iR'^2/4G}(\cdot) \quad (31)$$

The inverse operator can similarly be written as a differential operator

$$\hat{B}^{-1}(\cdot) = +i\frac{2G}{\sqrt{R}}e^{+iR^2/4G}\partial_R \left[\frac{e^{-iR^2/4G}}{\sqrt{R}}(\cdot) \right] \quad (32)$$

For notational convenience, we define

$$\alpha(R) \equiv \frac{e^{-iR^2/4G}}{\sqrt{R}} \quad (33)$$

and then

$$\hat{B}(\cdot) = -\frac{i}{2G}\alpha^{-1}(R) \int^R dR' R' \alpha(R')(\cdot) \quad (34)$$

and

$$\hat{B}^{-1}(\cdot) = +i\frac{2G}{R}\alpha^{-1}(R)\partial_R[\alpha(R)(\cdot)] \quad (35)$$

It can be checked explicitly that $\hat{B}^{-1}\hat{B} = 1 = \hat{B}\hat{B}^{-1}$.

V. BLACK HOLE FREE STATE

Since we wish to study the formation of a black hole starting with a state that does not contain a black hole, it is important for us to define what we mean by a state that is “black hole free”. At the classical level we can define a black hole free state by the condition $B > 0$ or $B^{-1} > 0$. We can lift these conditions to the quantum level by defining the operator

$$\mathcal{B} = 1 - \frac{G}{R}\hat{H}_{\text{wall}} - G\hat{H}_{\text{wall}}\frac{1}{R} \quad (36)$$

A “no black hole” or “black hole free” state would only have overlap with eigenfunctions of \mathcal{B} whose eigenvalues lie in the interval $(0, \infty)$. This choice of a black hole operator is not unique. Another possibility is $R - 2G\hat{H}_{\text{wall}}$.

In the incipient limit, we replace \mathcal{B} by \hat{B} . We will only be able to find eigenstates in this limit. Let us now find these states by solving the eigenvalue problem

$$\hat{B}\xi_\beta = \beta\xi_\beta \quad (37)$$

or, equivalently,

$$\hat{B}^{-1}\xi_\beta = \frac{1}{\beta}\xi_\beta \quad (38)$$

This corresponds to the differential equation (see Eq. (29))

$$\left(1 - i\frac{G}{R^2}\right)\xi_\beta + i\frac{2G}{R}\partial_R\xi_\beta = \frac{1}{\beta}\xi_\beta \quad (39)$$

The equation is solved to find the eigenfunctions

$$\xi_\beta = A\sqrt{R}e^{i(1-\beta^{-1})R^2/4G} \quad (40)$$

and the solution holds for a continuum of β . Eigenstates with $\beta > 0$ are black hole free.

The overlap of the wavefunction with an eigenstate of \hat{B}^{-1} is given by the coefficient

$$a_\beta \equiv \langle \xi_\beta | \psi \rangle \quad (41)$$

If $a_\beta = 0$ for all $\beta \leq 0$, then the state ψ is black hole free. The eigenvalue β can be said to quantify the “black holeness” of a state.

A problem with using \hat{B} as the black hole operator is that the eigenstates ξ_β are not orthonormal because R lies in the semi-infinite interval $(0, \infty)$. This is a familiar problem: \hat{B}^{-1} contains the operator $\hat{\Pi}$ which is like the radial momentum operator in quantum mechanics, and hence has no self-adjoint extension [15]. The problem may be traced back to our approximation, $\mathcal{B} \rightarrow \hat{B}$, or equivalently in Eq. (13).

We now consider if there are simultaneous eigenstates of the black hole operator and the Hamiltonian. Using the expressions for \hat{B}^{-1} (Eq. (32)) and the Hamiltonian (Eq. (22)) one easily sees that

$$[\hat{B}^{-1}, \hat{H}] \neq 0 \quad (42)$$

This observation implies an uncertainty relation between energy and black holeness – if we know the energy of a state precisely then there is uncertainty in its black holeness and if we know that an object is a black hole, its energy must not be precisely known. This ties in nicely with the usual understanding of a black hole not as a pure state but as a thermal state. While the value of the black holeness is uncertain for an energy eigenstate, it may still be possible to say if a particular energy eigenstate is black hole free because such a state is defined not by a single value of β but by the semi-infinite interval $(0, \infty)$. In other words, the state of being black hole free is considerably weaker than a state of definite black holeness, and an energy eigenstate may be black hole free even if its black holeness is uncertain.

The commutator in Eq. (42) can be evaluated explicitly

$$[\hat{B}^{-1}, \hat{H}] = G\hat{B} \left(\frac{1}{R^3} - \frac{i}{R^2}\hat{\Pi} \right) + G \left(\frac{1}{R^3} - \frac{i}{R^2}\hat{\Pi} \right) \hat{B}$$

The right-hand side is a complicated operator but can be roughly estimated in the incipient limit using $-\hat{B}\hat{\Pi} \sim M$, the mass of the collapsing object. Then

$$[\hat{B}^{-1}, \hat{H}] \sim \frac{i}{R_S} \quad (43)$$

as may also be expected on dimensional grounds.

In the next section, we will discuss eigenstates of the Hamiltonian. These are stationary states and hence cannot resolve the gravitational collapse problem, which requires solving the time-dependent problem in which the initial state is black hole free. We will consider the gravitational collapse problem in further detail in Sec. VII.

VI. STATIONARY STATES

We now consider the Schrodinger equation, Eq. (25), for a single eigenmode of the scalar field [11, 12],

$$\left[-\frac{1}{2}(\hat{B}\hat{\Pi} + \hat{\Pi}\hat{B}) + \left\{ \hat{B} \frac{\hat{\pi}^2}{2m} + \frac{K}{2}b^2 \right\} \right] \psi = E\psi \quad (44)$$

This is an integro-differential operator since \hat{B} is an integral operator.

We first act on Eq. (44) by \hat{B}^{-1} on the left and use

$$[\hat{B}^{-1}, \hat{\Pi}] = -\frac{2G}{R^3} + i\frac{2G}{R^2}\hat{\Pi} \quad (45)$$

This brings Eq. (44) to the form

$$\begin{aligned} \left[1 - \frac{2G}{R} \left(E - \frac{Kb^2}{2} \right) \right] \hat{\Pi}\psi = \\ \left[\frac{\hat{\pi}^2}{2m} + \left(\frac{Kb^2}{2} - E \right) \left(1 - i\frac{G}{R^2} \right) \right] \psi \\ - \frac{2G}{R^3} \left[1 - iR\hat{\Pi} \right] \hat{B}\psi \end{aligned} \quad (46)$$

To simplify the last term we use Eq. (27) in the form

$$\hat{\Pi} = \frac{R}{2G} \left[-\hat{B}^{-1} + 1 - i\frac{G}{R^2} \right] \quad (47)$$

Hence

$$\frac{2G}{R^3} \left[1 - iR\hat{\Pi} \right] \hat{B}\psi = \frac{i}{R}\psi - \frac{i}{R} \left(1 + \frac{iG}{R^2} \right) \hat{B}\psi \quad (48)$$

Therefore the Schrodinger equation becomes

$$\begin{aligned} \left[1 - \frac{2G}{R} \left(E - \frac{Kb^2}{2} \right) \right] \hat{\Pi}\psi = \\ \left[\frac{\hat{\pi}^2}{2m} + \left(\frac{Kb^2}{2} - E \right) \left(1 - i\frac{G}{R^2} \right) - \frac{i}{R} \right. \\ \left. + \frac{i}{R} \left(1 + i\frac{G}{R^2} \right) \hat{B} \right] \psi \end{aligned} \quad (49)$$

In the incipient limit ($B \rightarrow 0$) we expect the last term to be small, say compared to the second last term ($-i/R$) and we drop it. Thus we obtain the following differential equation

$$\begin{aligned} \left[1 - \frac{2G}{R} \left(E - \frac{Kb^2}{2} \right) \right] \partial_R \psi = \\ -\frac{i}{2m} \partial_b^2 \psi + i \left(\frac{Kb^2}{2} - E \right) \left(1 - i\frac{G}{R^2} \right) \psi + \frac{\psi}{R} \end{aligned} \quad (50)$$

The right-hand side of Eq. (50) contains the G/R^2 and $1/R$ terms arising due to the commutators of operators. The equation becomes intuitively obvious under approximations that allow us to ignore these terms. The first of these terms is small if the size of the spherical wall is assumed to be much larger than the Planck scale. The second term, $1/R$, is much smaller than E since we have assumed that the mass is much larger than the Planck mass. With these approximations, the equation reduces to

$$\begin{aligned} \left[1 - \frac{2G}{R} \left(E - \frac{Kb^2}{2} \right) \right] i\partial_R \psi = \\ \left[E - \left(-\frac{1}{2m} \partial_b^2 + \frac{Kb^2}{2} \right) \right] \psi \end{aligned} \quad (51)$$

Now the right-hand side is simply the total energy minus the simple harmonic oscillator Hamiltonian for b , and the left-hand side is the usual time evolution operator if R is viewed as a time coordinate. Also, as expected, the equation has a singularity. However, somewhat unexpectedly, the singularity occurs at

$$R = 2G \left(E - \frac{Kb^2}{2} \right)$$

and not at $2G(E - E_b)$ where the total energy in mode b , E_b , includes both kinetic and potential terms. The reason that only the potential term enters the location of the singularity can be traced back to Eq. (44), where it is clear that b interacts with the metric only via the kinetic term. Multiplying that equation by \hat{B}^{-1} transfers the interaction to be only between the $Kb^2/2$ term and the wall momentum, $\hat{\Pi}$, since \hat{B}^{-1} includes a term with $\hat{\Pi}$ (Eq. (27)).

In the limit

$$R \rightarrow 2G \left(E - \frac{Kb^2}{2} \right)$$

the leading order behavior can be found by equating the right-hand side of Eq. (50) to zero. An example of a non-singular function that satisfies the differential equation to leading order in the above limit is

$$\psi \sim \exp \left[\pm i \sqrt{\frac{mR}{G} \left(1 + i\frac{G}{R^2} \right)} b \right] \quad (52)$$

A more complete solution would require the wavefunction to extend away from the singular curve in the (R, b)

plane. We cannot exclude that the wavefunction could be badly behaved in a solution which is required to satisfy certain boundary conditions.

VII. GRAVITATIONAL COLLAPSE

The gravitational collapse problem can now be defined. We want to solve the time-dependent integro-differential equation

$$\left[-\frac{1}{2}(\hat{B}\hat{\Pi} + \hat{\Pi}\hat{B}) + \left\{ \hat{B}\frac{\hat{\pi}^2}{2m} + \frac{K}{2}b^2 \right\} \right] \Psi = i\frac{\partial\Psi}{\partial t} \quad (53)$$

This integro-differential problem can be converted to a differential problem exactly as for the stationary states in the previous section, except that E must be replaced by $i\partial_t$. The differential equation analogous to Eq. (50) is

$$\left[1 - \frac{2G}{R} \left(i\partial_t - \frac{Kb^2}{2} \right) \right] \partial_R \Psi = -\frac{i}{2m} \partial_b^2 \Psi + i \left(\frac{Kb^2}{2} - i\partial_t \right) \left(1 - i\frac{G}{R^2} \right) \Psi + \frac{\Psi}{R} \quad (54)$$

Under assumptions similar to those discussed in the previous section, we get an equation analogous to Eq. (51)

$$\left[1 - \frac{2G}{R} \left(i\partial_t - \frac{Kb^2}{2} \right) \right] i\partial_R \Psi = \left[i\partial_t - \left(-\frac{1}{2m} \partial_b^2 + \frac{Kb^2}{2} \right) \right] \Psi \quad (55)$$

Any solution of the time-dependent simple harmonic oscillator problem will make the right-hand side vanish and will be an R -independent solution of this equation. However, since the equation was derived assuming a large collapsing mass, such a solution, in which all the energy resides in the scalar radiation, cannot be taken too literally. Also, the solution does not resolve the time-dependent gravitational collapse problem. For that, we need to choose the initial Ψ such that it represents a gravitationally collapsing object which is black hole free *i.e.* $a_\beta = 0$ for all $\beta \leq 0$ (see Eq. (41)). With evolution, the coefficients a_β will change and we are interested in finding out if the system remains black hole free. The solution to this problem will also allow us to track the evolution of the harmonic oscillator and hence the transfer of energy via radiation from the wall to the scalar field during quantum collapse.

Eq. (55) (or (54)) is a partial differential equation in three variables and contains mixed t and R derivatives. It can also be singular at certain points. These features make it hard to analyze. We hope to return to Eq. (55) in future, perhaps using numerical techniques. An alternative would be to consider a linear superposition of stationary states that match the initial conditions.

VIII. DISCUSSION

We have set up a minisuperspace version of the Schrodinger formalism for studying quantum gravitational collapse of a spherical domain wall in the presence of a massless scalar field coupled to the metric. The description automatically includes backreaction of the quantum radiation on the quantum dynamics of the domain wall. Although the passage to minisuperspace involves a drastic truncation of the system degrees of freedom, it still retains those that are relevant to describe black hole formation and evaporation. A clear advantage of the present approach is that the action for the system greatly simplifies in the interesting limit of an incipient black hole and raises the hope that a solution, even to the notorious back-reaction problem, may be within reach.

In the incipient limit and in the approximation that the collapsing mass is large (see the discussion in Sec. I), the Hamiltonian is given by Eq. (22). A striking feature of the Hamiltonian is that it involves the operator \hat{B} which is an integral operator, for which we are also able to find an explicit representation (Eq. (34)).

Our analysis in the incipient limit should be sufficient to study the problem of quantum collapse and pre-Hawking radiation. However, unlike in the semiclassical approximation where the wall radius takes on a definite value, the wavefunction is defined over the entire range of possible wall radii, including very large radii. In a fuller treatment of the problem, it may become necessary to extend the Hamiltonian that we have found in the incipient limit to large values of the radius. The precise extension is not expected to be important, as long as the incipient limit of the extended Hamiltonian matches the Hamiltonian found here.

Once we have the minisuperspace Schrodinger equation, we need the relevant solutions to it that represent a solution to the gravitational collapse problem. Such a solution would contain the fate of matter that is collapsing towards forming a black hole. However, the initial conditions have to be chosen to be “black hole free” and some measure of “black holeness” has to be defined. We have discussed some possible black hole operators, an example of which is the operator \mathcal{B} that coincides with \hat{B} in the incipient limit. We find eigenstates of \hat{B} and the eigenvalue β may be used as a measure of black holeness. States with $\beta > 0$ can be said to be black hole free. However, the black hole operator does not commute with the Hamiltonian, thus implying an energy-black holeness uncertainty relation.

It is interesting to note that the eigenfunction ξ_β of the operator \hat{B} oscillates infinitely fast as $\beta \rightarrow 0$ which corresponds to the domain wall tending to the event horizon (see Eq. (40)). Naively, if we start with a wavefunction that overlaps only with states with some finite range with $\beta > 0$, to get an overlap with states with $\beta < 0$, it would seem that we would need to go through the infinitely oscillating state at $\beta = 0$. Quantum mechanically, though, this is not clear. An analogy might help clarify this situ-

ation. In the case of a non-relativistic Schrodinger particle, our initial state may be a wavepacket having overlap with only positive momentum eigenstates e^{ikx} with $k > 0$. Suppose at some later time, we find that there is non-zero overlap with a negative momentum eigenstate, e^{ipx} with $p < 0$. Classically this would mean that the particle's momentum has reversed and so, at some point in time, the particle had to pass through zero momentum. In quantum mechanics, however, this is not essential since the particle need not even have a well-defined momentum at intermediate times. So it seems that the singular state at $\beta = 0$ cannot be used to argue, at least straightforwardly, that there is an obstruction to the formation of a black hole.

While we have limited ourselves in this paper to a massless scalar field, it is interesting to consider how the analysis might change if the scalar field has a non-vanishing mass. In that case, the spectrum of eigenmodes of the scalar field would contain bound states in addition to scattering states. Gravitational collapse of the domain wall would then populate both the scattering and bound states, transferring energy from the wall to pre-Hawking radiation and also to an atmosphere of self-gravitating scalar particles. Presumably, the bound states will result in a boson star, though the details are not clear since boson stars as solutions of free or interacting scalar field theory are themselves known to have instabilities if their mass is large [16, 17, 18].

In Sec. VI we have discussed eigenstates of the Hamiltonian and in Sec. VII we have set up a differential equation that describes the gravitational collapse problem. The differential equation contains mixed t and R derivatives, and it may also become singular along curves in the (R, b) plane. These characteristics make the equation hard enough that we have postponed attempts at its solution for future work. In its solution, though, might lie answers to some of the questions we have raised in the introduction, and we may be able to see if the unitary evolution contained in the Schrodinger equation is self-limiting. On the other hand, it may be that the Schrodinger equation always yields unitary evolution, and since it is expected that black holes violate unitarity, it may be impossible to get to the black hole state as suggested by the semiclassical calculation.

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APPENDIX A: HERMITICITY

The Hamiltonian in the incipient limit contains the momentum operator but the variable R lies in the interval $(0, \infty)$. Thus $\hat{\Pi}$ resembles the radial momentum operator which is known to have problems with self-adjointness [15]. For us, however, $\hat{\Pi}$ only arises in the incipient approximation and there are issues if we blindly use this Hamiltonian over the entire range of variables. Here we consider the Hermiticity of $\hat{\Pi}$ and \hat{B} , listing the boundary conditions on the wavefunctions necessary to ensure Hermiticity.

For the momentum operator we have

$$\int_0^\infty dR f^\dagger \hat{\Pi} g = \int_0^\infty dR (\hat{\Pi} f)^\dagger g - i[f^\dagger g]_0^\infty \quad (\text{A1})$$

The boundary term vanishes if the functions $f(R)$ and $g(R)$ vanish at $R = 0$ and as $R \rightarrow \infty$. So $\hat{\Pi}^\dagger = \hat{\Pi}$ when acting (only) on this set of functions.

Next we find $(\hat{B}^{-1})^\dagger = \hat{B}^{-1}$ by using the Hermiticity of $1/R$ and $\hat{\Pi}$. This gives

$$\begin{aligned} \int_0^\infty dR f^\dagger \hat{B}^{-1} g &= \int_0^\infty dR f^\dagger \left(1 - \frac{G}{R} \hat{\Pi} - G \hat{\Pi} \frac{1}{R}\right) g \\ &= \int_0^\infty dR (\hat{B}^{-1} f)^\dagger g + i2G[f^\dagger g/R]_0^\infty \end{aligned}$$

and so \hat{B}^{-1} is also Hermitian provided f and g vanish sufficiently rapidly at the origin and do not grow at infinity.

Next we find \hat{B}^\dagger .

$$\begin{aligned} \int_0^\infty dR f^\dagger \hat{B} g &= \int_0^\infty dR (\hat{B}^{-1} \hat{B} f)^\dagger \hat{B} g \\ &= \int_0^\infty dR (\hat{B} f)^\dagger \hat{B}^{-1} \hat{B} g - i2G[(\hat{B} f)^\dagger (\hat{B} g)/R]_0^\infty \\ &= \int_0^\infty dR (\hat{B} f)^\dagger g - i2G[(\hat{B} f)^\dagger (\hat{B} g)/R]_0^\infty \end{aligned}$$

Therefore $\hat{B}^\dagger = \hat{B}$ on functions for which the boundary term vanishes.

If acting on a space of wavefunctions such that the boundary terms are not zero, the operator \hat{B} will not be Hermitian. This need not invalidate the formalism we have developed since the explicit expression for \hat{B} is only really valid in the incipient limit, and does not remain valid at $R = 0$ and $R \rightarrow \infty$. The Hamiltonian we have found would also need to be extended beyond the domain of an incipient black hole.

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