# Functional Integrals in Affine Quantum Gravity*

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**Abstract**

A sketch of a recent approach to quantum gravity is presented which involves several unconventional aspects. The basic ingredients include: (1) Affine kinematical variables; (2) Affine coherent states; (3) Projection operator approach for quantum constraints; (4) Continuous-time regularized functional integral representation without/with constraints; and (5) Hard core picture of nonrenormalizability. Emphasis is given to the functional integral expressions.

## 1 Introduction

This paper offers an introduction to the program of Affine Quantum Gravity (AQG) and its use of functional integrals. It is important at the outset to remark that this program is not string theory nor is it loop quantum gravity, the two most commonly studied approaches to quantum gravity at the present time. Although many aspects of this approach are still to be developed, AQG seems to the author to be more natural than most traditional views, and, moreover, it lies closer to classical (Einstein) gravity as well. Some general references for this paper are [1, 2, 3].

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## 2 Affine Kinematical Variables

### Metric positivity

A fundamental requirement of AQG is the strict positivity of the spatial metric. For the classical metric, this property means that for any nonvanishing set $\{u^a\}$ of real numbers and any nonvanishing, nonnegative test function, $f(x) \geq 0$, that

$$\textstyle\int f(x)u^a g_{ab}(x)u^b d^3x > 0 \;, \tag{1}$$

where $1 \leq a, b \leq 3$. We also insist that this inequality holds when the classical metric field $g_{ab}(x)$ is replaced with the $3 \times 3$ operator metric field $\hat{g}_{ab}(x)$.

### Affine commutation relations

Since the canonical commutation relations are in conflict with the requirement of metric positivity, our initial step involves replacing the classical ADM canonical momentum $\pi^{ab}(x)$ with the classical mixed-index momentum $\pi^a_b(x) \equiv \pi^{ac}(x)g_{cb}(x)$. We refer to $\pi^a_b(x)$ as the "momentric" tensor being a combination of the canonical *momen*tum and the canonical me*tric*. Besides the metric being promoted to an operator $\hat{g}_{ab}(x)$, we also promote the classical momentric tensor to an operator field $\hat{\pi}^a_b(x)$; this pair of operators form the basic kinematical affine operator fields, and all operators of interest are given as functions of this fundamental pair. The basic kinematical operators are chosen so that they satisfy the following set of *affine commutation relations* (in units where $\hbar = 1$, which are normally used throughout):

$$\begin{aligned}
[\hat{\pi}^a_b(x),\, \hat{\pi}^c_d(y)] &= \tfrac{1}{2} i[\delta^c_b \hat{\pi}^a_d(x) - \delta^a_d \hat{\pi}^c_b(x)]\,\delta(x,y) \;, \\
[\hat{g}_{ab}(x),\, \hat{\pi}^c_d(y)] &= \tfrac{1}{2} i[\delta^c_a \hat{g}_{bd}(x) + \delta^c_b \hat{g}_{ad}(x)]\,\delta(x,y) \;, \\
[\hat{g}_{ab}(x),\, \hat{g}_{cd}(y)] &= 0 \;.
\end{aligned} \tag{2}$$

These commutation relations arise as the transcription into operators of equivalent Poisson brackets for the corresponding classical fields, namely, the spatial metric $g_{ab}(x)$ and the momentric field $\pi^c_d(x) \equiv \pi^{cb}(x)g_{bd}(x)$, along with the usual Poisson brackets between the canonical metric field $g_{ab}(x)$ and the canonical momentum field $\pi^{cd}(x)$.

The virtue of the affine variables and their associated commutation relations is evident in the relation

$$e^{i\int \gamma^a_b(y)\hat{\pi}^b_a(y)\,d^3y}\,\hat{g}_{cd}(x)\,e^{-i\int \gamma^a_b(y)\hat{\pi}^b_a(y)\,d^3y} = \{e^{\gamma(x)/2}\}^e_c\,\hat{g}_{ef}(x)\,\{e^{\gamma(x)/2}\}^f_d\,. \tag{3}$$

This algebraic relation confirms that suitable transformations by the momentric field preserve metric positivity.

# 3 Affine Coherent States

It is noteworthy that the algebra generated by $\hat{g}_{ab}$ and $\hat{\pi}^a_b$ closes. These operators form the generators of the *affine group* whose elements may be defined by

$$U[\pi,\gamma] \equiv e^{i\int \pi^{ab}(y)\hat{g}_{ab}(y)\,d^3y}\,e^{-i\int \gamma^a_b(y)\hat{\pi}^b_a(y)\,d^3y}\,, \tag{4}$$

e.g., for all real, smooth $c$-number functions $\pi^{ab}$ and $\gamma^a_b$ of compact support. Since we assume that the smeared $\hat{g}_{ab}$ and $\hat{\pi}^a_b$ fields are self-adjoint operators, it follows that $U[\pi,\gamma]$ are unitary operators for all $\pi$ and $\gamma$, and moreover, these unitary operators are strongly continuous in the label fields $\pi$ and $\gamma$.

To define a representation of the basic operators it suffices to choose a fiducial vector and thereby to introduce a set of affine coherent states, i.e., coherent states formed with the help of the affine group. We choose $|\eta\rangle$ as a normalized fiducial vector in the original Hilbert space $\mathfrak{H}$, and we consider a set of unit vectors each of which is given by

$$|\pi,\gamma\rangle \equiv e^{i\int \pi^{ab}(x)\,\hat{g}_{ab}(x)\,d^3x}\,e^{-i\int \gamma^d_c(x)\,\hat{\pi}^c_d(x)\,d^3x}\,|\eta\rangle\,. \tag{5}$$

As $\pi$ and $\gamma$ range over the space of smooth functions of compact support, such vectors form the desired set of coherent states. The specific representation of the kinematical operators is fixed once the vector $|\eta\rangle$ has been chosen. As minimum requirements on $|\eta\rangle$ we impose

$$\langle\eta|\hat{\pi}^a_b(x)|\eta\rangle = 0\,, \tag{6}$$

$$\langle\eta|\hat{g}_{ab}(x)|\eta\rangle = \tilde{g}_{ab}(x)\,, \tag{7}$$

where $\tilde{g}_{ab}(x)$ is a metric that determines the topology of the underlying space-like surface. As algebraic consequences of these conditions, it follows that

$$\langle\pi,\gamma|\hat{g}_{ab}(x)|\pi,\gamma\rangle = \{e^{\gamma(x)/2}\}_a^c\,\tilde{g}_{cd}(x)\,\{e^{\gamma(x)/2}\}_b^d \equiv g_{ab}(x)\;, \tag{8}$$

$$\langle\pi,\gamma|\hat{\pi}_c^a(x)|\pi,\gamma\rangle = \pi^{ab}(x)g_{bc}(x) \equiv \pi_c^a(x)\;. \tag{9}$$

These expectations are not gauge invariant since they are taken in the original Hilbert space where the constraints are not fulfilled.

By definition, the coherent states span the original, or kinematical, Hilbert space $\mathfrak{H}$, and thus we can characterize the coherent states themselves by giving their overlap with an arbitrary coherent state. In so doing, we choose the fiducial vector $|\eta\rangle$ so that the overlap is given by

$$\langle\pi'',\gamma''|\pi',\gamma'\rangle = \exp\Big[-2\int b(x)\,d^3\!x \\ \times\ln\Big(\frac{\det\{\frac{1}{2}[g''^{ab}(x)+g'^{ab}(x)]+\frac{1}{2}ib(x)^{-1}[\pi''^{ab}(x)-\pi'^{ab}(x)]\}}{\{\det[g''^{ab}(x)]\,\det[g'^{ab}(x)]\}^{1/2}}\Big)\Big] \tag{10}$$

where $b(x)$, $0<b(x)<\infty$, is a scalar density which is discussed below.

Additionally, we observe that $\gamma''$ and $\gamma'$ do *not* appear in the explicit functional form given in (10). In particular, the smooth matrix $\gamma$ has been replaced by the smooth matrix $g$ which is defined at every point by

$$g(x) \equiv e^{\gamma(x)/2}\,\tilde{g}(x)\,e^{\gamma(x)^T/2} \equiv \{g_{ab}(x)\}\;, \tag{11}$$

where $T$ denotes transpose, and the matrix $\tilde{g}(x) \equiv \{\tilde{g}_{ab}(x)\}$ is given by (7). The map $\gamma\rightarrow g$ is clearly many-to-one since $\gamma$ has nine independent variables at each point while $g$, which is symmetric, has only six. In view of this functional dependence we may denote the given functional in (10) by $\langle\pi'',g''|\pi',g'\rangle$, and henceforth we adopt this notation. In particular, we note that (8) and (9) become

$$\langle\pi,g|\hat{g}_{ab}(x)|\pi,g\rangle \equiv g_{ab}(x)\;, \tag{12}$$

$$\langle\pi,g|\hat{\pi}_c^a(x)|\pi,g\rangle = \pi^{ab}(x)g_{bc}(x) \equiv \pi_c^a(x)\;, \tag{13}$$

which show that the meaning of the labels $\pi$ and $g$ is that of *mean* values rather than sharp eigenvalues.

In addition, we observe that the coherent state overlap function (10) is a continuous function that can serve as a reproducing kernel for a reproducing kernel Hilbert space which provides a representation of the original Hilbert space $\mathfrak{H}$ by continuous functions of $\pi$ and $g$. For details of such spaces, see [4].

# 4 Projection Operator Approach for Quantum Constraints

Classically, constraints are either: (i) first class, for which the Lagrange multipliers are undetermined and must be chosen to find a solution; or (ii) second class, for which the Lagrange multipliers are fixed by the equations of motion.

The Dirac approach to the quantization of constraints requires quantization before reduction. Thus the constraints are first promoted to self-adjoint operators,

$$\phi_\alpha(p,q) \to \Phi_\alpha(P,Q) \;, \tag{14}$$

for all $\alpha$, and then the physical Hilbert space $\mathfrak{H}_{phys}$ is defined by those vectors $|\psi\rangle_{phys}$ for which

$$\Phi_\alpha(P,Q)|\psi\rangle_{phys} = 0 \tag{15}$$

for all $\alpha$. This procedure works for a limited set of classical first class constraint systems, but it does not work in general and especially not for second class constraints.

The projection operator approach to quantum constraints involves a slight relaxation of the Dirac procedure. Instead of insisting that (15) holds exactly, we introduce a projection operator $\mathbb{E}$ defined by

$$\mathbb{E} = \mathbb{E}(\Sigma_\alpha \Phi_\alpha^2 \leq \delta(\hbar)^2) \;, \tag{16}$$

where $\delta(\hbar)$ is a positive *regularization parameter* and we have assumed that $\Sigma_\alpha \Phi_\alpha^2$ is self adjoint. This relation means that $\mathbb{E}$ projects onto the spectral range of the self-adjoint operator $\Sigma_\alpha \Phi_\alpha^2$ in the interval $[0, \delta(\hbar)^2]$, and then $\mathfrak{H}_{phys} = \mathbb{E}\mathfrak{H}$. As a final step, the parameter $\delta(\hbar)$ is reduced as much as required, and, in particular, when some second-class constraints are involved, $\delta(\hbar)$ ultimately remains strictly positive. This general procedure treats all constraints simultaneously and treats them all on an equal basis; for details see [5].

A few examples illustrate how the projection operator method works. If $\Sigma_\alpha \Phi_\alpha^2 = J_1^2 + J_2^2 + J_3^2$, the Casimir operator of $su(2)$, then $0 \le \delta(\hbar)^2 < 3\hbar^2/4$ works for this first class example. If $\Sigma_\alpha \Phi_\alpha^2 = P^2 + Q^2$, where $[Q, P] = i\hbar \mathbb{1}$, then $\hbar \le \delta(\hbar)^2 < 3\hbar$ covers this second class example. If the single constraint $\Phi = Q$, an operator whose zero lies in the continuous spectrum, then it is convenient to take an appropriate form limit of the projection operator as $\delta \to 0$; see [5]. The projection operator scheme can also deal with irregular constraints such as $\Phi = Q^3$, and even mixed examples with regular and irregular constraints such as $\Phi = Q^3(1-Q)$, etc.; see [6].

It is also of interest that the desired projection operator has a general, time-ordered integral representation (see [7]) given by

$$\mathbb{E} = \mathbb{E}\big(\Sigma_\alpha \Phi_\alpha^2 \le \delta(\hbar)^2\big) = \int \mathsf{T} e^{-i\int \lambda^\alpha(t)\Phi_\alpha \, dt} \, \mathcal{D}R(\lambda) \, . \tag{17}$$

The weak measure $R$ depends on the number of Lagrange multipliers, the time interval, and the regularization parameter $\delta(\hbar)^2$. The measure $R$ does *not* depend on the constraint operators, and thus this relation is an operator identity, holding for any set of operators $\{\Phi_\alpha\}$. The time-ordered integral representation for $\mathbb{E}$ given in (17) can be used in path-integral representations as will become clear below.

# 5 Continuous-time Regularized Functional Integral Representation without/with Constraints

It is pedagocially useful to reexpress the coherent-state overlap function by means of a functional integral. This process can be aided by the fact that the expression (10) is analytic in the variable $g''^{ab}(x) + ib(x)^{-1}\pi''^{ab}(x)$ up to a factor. As a consequence, the coherent-state overlap function satisfies a complex polarization condition, which leads to a second-order differential operator that annihilates it. This fact can be used to generate a functional integral representation of the form

$$\langle\pi'',g''|\pi',g'\rangle = \exp\Big[-2\int b(x)\,d^3\!x$$
$$\times\ln\Big(\frac{\det\{\frac{1}{2}[g''^{ab}(x)+g'^{ab}(x)]+\frac{1}{2}ib(x)^{-1}[\pi''^{ab}(x)-\pi'^{ab}(x)]\}}{\{\det[g''^{ab}(x)]\,\det[g'^{ab}(x)]\}^{1/2}}\Big)\Big]$$
$$=\lim_{\nu\to\infty}\overline{\mathcal{N}}_\nu\int\exp[-i\textstyle\int g_{ab}\dot{\pi}^{ab}\,d^3\!x\,dt]$$
$$\times\exp\{-(1/2\nu)\textstyle\int[b(x)^{-1}g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da}+b(x)g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]\,d^3\!x\,dt\}$$
$$\times[\Pi_{x,t}\,\Pi_{a\le b}\,d\pi^{ab}(x,t)\,dg_{ab}(x,t)]\;. \tag{18}$$

Because of the way the new independent variable $t$ appears in the right-hand term of this equation, it is natural to interpret $t$, $0\le t\le T$, $T>0$, as coordinate "time". The fields on the right-hand side all depend on space and time, i.e., $g_{ab}=g_{ab}(x,t)$, $\dot{g}_{ab}=\partial g_{ab}(x,t)/\partial t$, etc., and, importantly, the integration domain of the formal measure is strictly limited to the domain where $\{g_{ab}(x,t)\}$ is a positive-definite matrix for all $x$ and $t$. For the boundary conditions, we have $\pi'^{ab}(x)\equiv\pi^{ab}(x,0)$, $g'_{ab}(x)\equiv g_{ab}(x,0)$, as well as $\pi''^{ab}(x)\equiv\pi^{ab}(x,T)$, $g''_{ab}(x)\equiv g_{ab}(x,T)$ for all $x$. Observe that the right-hand term holds for any $T$, $0<T<\infty$, while the left-hand and middle terms are independent of $T$ altogether.

In like manner, we can incorporate the constraints into a functional integral by using an appropriate form of the integral representation (17). The resultant expression has a functional integral representation given by

$$\langle\pi'',g''|\mathbb{E}|\pi',g'\rangle=\int\langle\pi'',g''|\mathbf{T}\,e^{-i\int[N^a\mathcal{H}_a+N\mathcal{H}]\,d^3\!x\,dt}|\pi',g'\rangle\,\mathcal{D}R(N^a,N)$$
$$=\lim_{\nu\to\infty}\overline{\mathcal{N}}_\nu\int e^{-i\int[g_{ab}\dot{\pi}^{ab}+N^aH_a+NH]\,d^3\!x\,dt}$$
$$\times\exp\{-(1/2\nu)\textstyle\int[b(x)^{-1}g_{ab}g_{cd}\dot{\pi}^{bc}\dot{\pi}^{da}+b(x)g^{ab}g^{cd}\dot{g}_{bc}\dot{g}_{da}]\,d^3\!x\,dt\}$$
$$\times[\Pi_{x,t}\,\Pi_{a\le b}\,d\pi^{ab}(x,t)\,dg_{ab}(x,t)]\,\mathcal{D}R(N^a,N)\;. \tag{19}$$

Despite the general appearance of (19), we emphasize once again that this representation has been based on the affine commutation relations and *not* on any canonical commutation relations.

The expression $\langle\pi'',g''|\mathbb{E}|\pi',g'\rangle$ denotes the coherent-state matrix elements of the projection operator $\mathbb{E}$ which projects onto a subspace of the original Hilbert space on which the quantum constraints are fulfilled in a regularized fashion. Furthermore, the expression $\langle\pi'', g''|\mathbb{E}|\pi', g'\rangle$ is another continuous functional that can be used as a reproducing kernel and thus used directly to generate the reproducing kernel physical Hilbert space on which the quantum constraints are fulfilled in a regularized manner. Observe that $N^a$ and $N$ denote Lagrange multiplier fields (classically interpreted as the shift and lapse), while $H_a$ and $H$ denote phase-space symbols (since $\hbar \neq 0$) associated with the quantum diffeomorphism and Hamiltonian constraint field operators, respectively. Up to a surface term, therefore, the phase factor in the functional integral represents the canonical action for general relativity.

# 6 Hard-core Picture of Nonrenormalizability

Nonrenormalizable quantum field theories involve an infinite number of distinct counterterms when approached by a regularized, renormalized perturbation analysis. Focusing on scalar field theories, a qualitative Euclidean functional integral formulation is given by

$$S_\lambda(h) = \mathcal{N}_\lambda \int e^{\int h\phi\, d^n x - W_o(\phi) - \lambda V(\phi)}\, \mathcal{D}\phi\ , \tag{20}$$

where $W_o(\phi) \geq 0$ denotes the free action and $V(\phi) \geq 0$ the interaction term. If $\lambda = 0$, the support of the integral is determined by $W_o(\phi)$; when $\lambda > 0$, the support is determined by $W_o(\phi) + \lambda V(\phi)$. Formally, as $\lambda \to 0$, $S_\lambda(h) \to S_0(h)$, the functional integral for the free theory. However, it may happen that

$$\lim_{\lambda\to 0} S_\lambda(h) = S_0'(h) \neq S_0(h)\ , \tag{21}$$

where $S_0'(h)$ defines a so-called *pseudofree* theory. Such behavior arises formally if $V(\phi)$ acts partially as a *hard core*, projecting out certain fields that are not restored to the support of the free theory as $\lambda \to 0$ [8].

It is noteworthy that there exist highly idealized nonrenormalizable model quantum field theories with exactly the behavior described; see [9]. Such examples involve counterterms not suggested by a renormalized perturbation analysis. It is our belief that these soluble models strongly suggest that nonrenormalizable $\varphi^4_n$, $n \geq 5$, models can be understood by the same mechanism, and that they too can be properly formulated by the incorporation of a limited number of counterterms distinct from those suggested by a perturbation treatment. Although technically more complicated, we see no fundamental obstacle in dealing with quantum gravity on the basis of an analogous hard-core interpretation.

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