

Inspiralling compact binaries in quasi-elliptical orbits: The complete third post-Newtonian energy flux

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Abstract

The instantaneous contributions to the third post-Newtonian (3PN) gravitational wave luminosity from the inspiral phase of a binary system of compact objects moving in a quasi elliptical orbit is computed using the multipolar post-Minkowskian wave generation formalism. The necessary inputs for this calculation include the 3PN accurate mass quadrupole moment for general orbits and the mass octupole and current quadrupole moments at 2PN. Using the recently obtained 3PN quasi-Keplerian representation of elliptical orbits the flux is averaged over the binary's orbit. Supplementing this by the important hereditary contributions arising from tails, tails-of-tails and tails squared terms calculated in a previous paper, the complete 3PN energy flux is obtained. The final result presented in this paper would be needed for the construction of ready-to-use templates for binaries moving on non-circular orbits, a plausible class of sources not only for the space based detectors like LISA but also for the ground based ones.

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I. INTRODUCTION

Inspiralling compact binaries, one of the prototype sources for laser interferometric gravitational wave (GW) detectors, are usually modelled as moving in quasi-circular orbits. This is justified since gravitational radiation reaction, under which it inspirals, circularizes the orbit towards the late stages of inspiral [1, 2]. This late phase of inspiral and the ensuing merger phase offer promises for the GW interferometric detectors. The recently discovered double pulsar system [3, 4] has an eccentricity as low as 0.088 consistent with the circular orbit assumption for the late inspiral and pre-merger phases, believed to be reasonable enough for most of the binary systems made of neutron stars or black holes (BHs).

The theoretical modelling of the binary’s phase evolution to a very high precision is called the phasing formula. This is the basic theoretical ingredient used in the construction of search templates for GW using matched filtering [5]. The two key inputs required for the construction of templates for binaries moving in quasi-circular orbits in the adiabatic approximation are the orbital energy and the GW luminosity (energy flux). These are computed using a cocktail of approximation schemes in general relativity. The schemes include the multipole decomposition, the post-Minkowskian expansion of the gravitational field or non-linearity expansion in Newton’s constant G , the post-Newtonian expansion in v/c , and the far-zone expansion in powers of $1/R$, where R is the distance from the source (see [6] for a recent review).

Though the garden variety binary sources of GWs for terrestrial laser interferometric GW detectors are those moving in quasi-circular orbits, there is an increased recent interest in inspiralling binaries moving in *quasi-eccentric* orbits. Astrophysical scenarios currently exist which lead to binaries with non-zero eccentricity in the GW detector bandwidth, both terrestrial and space-based. For instance, inner binaries of hierarchical triplets undergoing Kozai oscillations [7] could not only merge due to gravitational radiation reaction but a good fraction ($\sim 30\%$) of them will have eccentricity greater than about 0.1 as they enter the sensitivity band of advanced ground based interferometers [8]. Almost all the above systems possess eccentricities below 0.2 at 40 Hz and below 0.02 at 200 Hz. The population of stellar mass binaries in globular clusters is expected to have a thermal distribution of eccentricities [9]. In a study on the growth of intermediate BHs [10] in globular clusters it was found that the binaries have eccentricities between 0.1 and 0.2 in the LISA bandwidth. Though, supermassive black hole binaries are powerful GW sources for LISA, it is not yet conclusive if they would be in quasi-circular or quasi-eccentric orbits [11]. If a Kozai mechanism is at work, these supermassive BH binaries could be in highly eccentric orbits and merge within the Hubble time [12]. Sources of the kind discussed above provide the prime motivation to investigate higher post-Newtonian order modelling for quasi-eccentric binaries.

The GW energy flux or luminosity from a system of two point masses in elliptic motion was first computed by Peters and Mathews at Newtonian order [1, 2]. The post-Newtonian (PN) corrections to the gravitational wave flux at 1PN and 1.5PN were provided in [13, 14, 15, 16, 17] and used to study the associated evolution of orbital elements using the 1PN “quasi-Keplerian” representation of the binary’s orbit [18]. Gopakumar and Iyer [19, 20] further extended these results to 2PN order using the generalized quasi-Keplerian representation developed in Ref. [21, 22, 23]. The results for the energy flux and waveform presented in [19] was in perfect agreement with those obtained by Will and Wiseman using a different formalism [24]. Recently, Damour, Gopakumar and Iyer [25] discussed an analytic method for constructing high accuracy templates for the GW signals from the inspiral phase of compact binaries moving on quasi-elliptical orbits. They used an improved “method of variation of constants” to combine the three time scales involved in the elliptical orbit case, namely, orbital period, periastron precession and radiation reaction time scales,

without making the usual approximation of treating the radiative time scale as an adiabatic process.

The generation problem for gravitational waves at any PN order requires the solution to two independent problems. The first relates to the equation of motion of the binary and the second to the far zone fluxes of energy, angular momentum and linear momentum. The latter requires the computation of the relativistic mass and current multipole moments to appropriate PN orders. The 3PN equations of motion (EOM) required to handle gravitational wave phasing turned out to be technically very involved due to the issues related to the self-field regularization using Riesz or Hadamard regularizations [26, 27]. Only by a deeper understanding of the origin of the ambiguities in Hadamard regularization, and the use of dimensional regularization has the problem been uniquely resolved [28, 29] and provided the value of the ambiguity parameter ω_s [26] or equivalently λ [27]. We thus have in hand the requisite 3PN EOM for compact binaries moving in general orbits. The computation of the GW luminosity at 3PN or $(v/c)^6$ beyond the leading Einstein quadrupole formula crucially requires the computation of the 3PN accurate mass quadrupole moment. For its completion the same technique as in the EOM was successfully applied, namely, to compute using Hadamard’s regularization all the terms except a few terms parametrized by ambiguity parameters (which turn out to be three, denoted ξ , κ and ζ) [30, 31], and then to determine the value of these parameters by computing the difference between the dimensional and Hadamard regularizations [31, 32, 33, 34]. These works thus provide the fully determined 3PN accurate mass quadrupole for general orbits – the other important ingredient to compute the 3PN accurate energy and angular momentum fluxes for inspiralling compact binaries moving in general non-circular orbits. The 3.5PN phasing of inspiralling compact binaries moving in quasi-circular orbits is now complete and available for use in GW data analysis [32, 35]. Note that the 3PN contribution to the energy flux comes not only from the “instantaneous” terms discussed in this paper but also includes “hereditary” contributions arising from tails, tails of tails and tail-squared terms. A semi-analytical scheme is proposed and discussed in detail in a companion paper [36]¹ to evaluate these history-dependent contributions.

In this paper, for binaries moving in elliptical orbits, we compute all the instantaneous contributions to the 3PN accurate GW energy flux. The orbital average of this flux will be obtained using the 3PN quasi-Keplerian parametrization of the binary’s orbital motion recently constructed by Memmesheimer, Gopakumar and Schäfer [37]. We shall supplement these by contributions from the hereditary terms computed in Paper I. The final expression will represent gravitational waves from a binary evolving negligibly under gravitational radiation reaction, including precisely up to 3PN order the effects of eccentricity and periastron precession during epochs of inspiral when the orbital parameters are essentially constant over a few orbital revolutions. It also represents the first step towards the discussion of the *quasi-elliptical* case: the evolution of the binary in an elliptical orbit under gravitational radiation reaction. The present work extends the circular orbit results at 2.5PN [38] and 3PN [30, 32] to the elliptical orbit case. Further, it extends earlier works on instantaneous contributions for binaries moving in elliptical orbits at 1PN [14, 15] and 2PN [19] to 3PN order. Similarly, Paper I extends hereditary contributions at 1.5PN [16] to 2.5PN order and 3PN, where the 3PN hereditary contributions comprise the tails of tails and are extensions of Refs. [39, 40] for circular orbits to the elliptical orbit case.

In Sect. II we begin with the structure of the far-zone flux of energy, use expressions relating the radiative moments to the source moments and decompose the energy flux expression into its instantaneous and hereditary parts. Section III lists all the requisite multipole moments in standard harmonic coordinates for binaries moving in general (non-circular) orbits. Section IV

¹ Hereafter Ref. [36] will be called Paper I.

introduces the 3PN equations of motion which are necessary to handle the time derivatives of the moments. Section V discusses the computation of the instantaneous terms in the energy flux and Sect. VI recasts the flux in modified harmonic coordinates (without logarithms at 3PN order) and Arnowitt, Deser and Misner (ADM) coordinates. Section VII summarises the 3PN quasi-Keplerian representation required to average the flux expression over an orbit. Section VIII exhibits the orbital average of the energy flux in modified harmonic coordinates and ADM coordinates, and finally provides an expression of the complete energy flux in terms of gauge invariant variables.

II. THE FAR ZONE FLUX OF ENERGY

In this section, we discuss the computation of the 3PN accurate energy flux for general isolated sources. Starting from the expression for the far zone flux in terms of the radiative multipole moments and using the relations connecting the radiative multipole moments to the source moments, we write the resultant structure of the GW energy flux.

Following Thorne [41], the expression for the 3PN accurate far zone energy flux $\mathcal{F} \equiv (d\mathcal{E}/dt)^{\text{GW}}$ in terms of symmetric trace-free (STF) radiative multipole moments reads as²

$$\begin{aligned} \mathcal{F} = & \frac{G}{c^5} \left\{ \frac{1}{5} U_{ij}^{(1)} U_{ij}^{(1)} \right. \\ & + \frac{1}{c^2} \left[\frac{1}{189} U_{ijk}^{(1)} U_{ijk}^{(1)} + \frac{16}{45} V_{ij}^{(1)} V_{ij}^{(1)} \right] + \frac{1}{c^4} \left[\frac{1}{9072} U_{ijkm}^{(1)} U_{ijkm}^{(1)} + \frac{1}{84} V_{ijk}^{(1)} V_{ijk}^{(1)} \right] \\ & \left. + \frac{1}{c^6} \left[\frac{1}{594000} U_{ijkmn}^{(1)} U_{ijkmn}^{(1)} + \frac{4}{14175} V_{ijkm}^{(1)} V_{ijkm}^{(1)} \right] + \mathcal{O}(8) \right\}. \end{aligned} \quad (2.1)$$

In the above U_L and V_L (where $L = i_1 i_2 \dots i_l$ represents a multi-index composed of l spatial indices) are the mass-type and current-type radiative multipole moments respectively and $U_L^{(l)}$ and $V_L^{(l)}$ denote their l^{th} time derivatives. The moments are functions of retarded time $U \equiv T - R/c$ in radiative coordinates.

In the multipolar-post-Minkowskian (MPM) formalism, the radiative moments U_L and V_L can be re-expressed in terms of the source moments to an accuracy sufficient for the computation of the energy flux. For the flux to be complete up to 3PN approximation, one must compute the mass type radiative quadrupole U_{ij} to 3PN accuracy, mass octupole U_{ijk} and current quadrupole V_{ij} to 2PN accuracy, mass hexadecapole U_{ijkm} and current octupole V_{ijk} to 1PN accuracy and finally U_{ijkmn} and V_{ijkm} to Newtonian accuracy.

The relations connecting the different radiative moments U_L and V_L to the corresponding source moments I_L and J_L are given below. For the 3PN mass quadrupole moment we have [38, 39, 40, 42]

$$\begin{aligned} U_{ij}(U) = & I_{ij}^{(2)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{2} \right] I_{ij}^{(4)}(U - \tau) \\ & + \frac{G}{c^5} \left\{ -\frac{2}{7} \int_0^{+\infty} d\tau I_{a(i}^{(3)}(U - \tau) I_{j)a}^{(3)}(U - \tau) \right. \\ & \left. + \frac{1}{7} I_{a(i}^{(5)} I_{j)a} - \frac{5}{7} I_{a(i}^{(4)} I_{j)a}^{(1)} - \frac{2}{7} I_{a(i}^{(3)} I_{j)a}^{(2)} + \frac{1}{3} \varepsilon_{ab(i} I_{j)a}^{(4)} J_b \right\} \end{aligned}$$

² The shorthand $\mathcal{O}(n)$ is used throughout and indicates that the post-Newtonian remainder is of order of $\mathcal{O}(c^{-n})$.

$$\begin{aligned}
& +4 \left[W^{(2)} I_{ij} - W^{(1)} I_{ij}^{(1)} \right]^{(2)} \Big\} \\
& + 2 \left(\frac{GM}{c^3} \right)^2 \int_0^{+\infty} d\tau I_{ij}^{(5)}(U - \tau) \left[\ln^2 \left(\frac{c\tau}{2r_0} \right) + \frac{57}{70} \ln \left(\frac{c\tau}{2r_0} \right) + \frac{124627}{44100} \right] + O(7), \quad (2.2)
\end{aligned}$$

where the bracket $\langle \rangle$ surrounding indices denotes the STF projection, and ε_{abi} is the usual Levi-Civita symbol such that $\varepsilon_{123} = +1$. The I_L 's and J_L 's are the mass and current-type source moments (and $I_L^{(p)}$, $J_L^{(p)}$ denote their p^{th} time derivatives), and W is the monopole corresponding to the set of ‘‘gauge’’ moments W_L , using the same definitions as in [30]. In the above formula, M (which is in factor of the tail integral at 1.5PN order and the tail-of-tail integral at 3PN) is the total ADM mass of the source. The non-linear memory integral at 2.5PN is a time anti-derivative and will become instantaneous in the energy flux. The moments needed at 2PN order include only the dominant tails and are

$$U_{ijk}(U) = I_{ijk}^{(3)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{97}{60} \right] I_{ijk}^{(5)}(U - \tau) + O(5), \quad (2.3a)$$

$$V_{ij}(U) = J_{ij}^{(2)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{7}{6} \right] J_{ij}^{(4)}(U - \tau) + O(5). \quad (2.3b)$$

For all the other moments required in the computation we need only the leading order accuracy in the relation between radiative and source moments, so that

$$U_L(U) = I_L^{(l)}(U) + O(3), \quad (2.4a)$$

$$V_L(U) = J_L^{(l)}(U) + O(3). \quad (2.4b)$$

The constant length r_0 scaling the logarithm is the one introduced in the general MPM formalism and has been chosen here to match with the choice made in the computation of tails-of-tails in [40]. It is a freely specifiable constant, entering the relation between the retarded time $U = T - R/c$ in radiative coordinates and the corresponding retarded time $t_{\text{H}} - r_{\text{H}}/c$ in harmonic coordinates (where r_{H} is the distance of the source in harmonic coordinates). More precisely we have

$$U = t_{\text{H}} - \frac{r_{\text{H}}}{c} - \frac{2GM}{c^3} \ln \left(\frac{r_{\text{H}}}{r_0} \right) + O(5). \quad (2.5)$$

From Eqs (2.2)-(2.4), it is clear that the radiative moments have two distinct contributions. One part depends on the moments only at the retarded time, $U = T - R/c$; this part is referred to as the ‘‘instantaneous contribution’’, and forms the subject matter of the present paper. The second part on the other hand depends on the dynamics of the system in its entire past, *i.e.* at any $U - \tau < U$, and is referred to as the ‘‘hereditary contribution’’. Equally important but requiring a different treatment, the hereditary contribution is dealt with in Paper I as mentioned earlier. We are thus allowed to write down explicitly the different kinds of contributions to the far zone energy flux up to 3PN. We have,

$$\mathcal{F} = \mathcal{F}_{\text{inst}} + \mathcal{F}_{\text{hered}}, \quad (2.6)$$

where the instantaneous contribution of interest in this paper is explicitly given by

$$\mathcal{F}_{\text{inst}} = \frac{G}{c^5} \left\{ \frac{1}{5} I_{ij}^{(3)} I_{ij}^{(3)} \right\}$$

$$\begin{aligned}
& + \frac{1}{c^2} \left[\frac{1}{189} I_{ijk}^{(4)} I_{ijk}^{(4)} + \frac{16}{45} J_{ij}^{(3)} J_{ij}^{(3)} \right] + \frac{1}{c^4} \left[\frac{1}{9072} I_{ijkm}^{(5)} I_{ijkm}^{(5)} + \frac{1}{84} J_{ijk}^{(4)} J_{ijk}^{(4)} \right] \\
& + \frac{G}{c^5} \left[\frac{8}{5} I_{ij}^{(3)} \left(I_{ij} W^{(5)} + 2I_{ij}^{(1)} W^{(4)} - 2I_{ij}^{(3)} W^{(2)} - I_{ij}^{(4)} W^{(1)} \right) \right. \\
& \quad \left. + \frac{2}{5} I_{ij}^{(3)} \left(-\frac{4}{7} I_{ai}^{(5)} I_{aj}^{(1)} - I_{ai}^{(4)} I_{aj}^{(2)} - \frac{2}{7} I_{ai}^{(3)} I_{aj}^{(3)} + \frac{1}{7} I_{ai}^{(6)} I_{aj} \right) + \frac{2}{15} \varepsilon_{abi} I_{aj}^{(5)} J_b I_{ij}^{(3)} \right] \\
& + \frac{1}{c^6} \left[\frac{1}{594000} I_{ijkmn}^{(6)} I_{ijkmn}^{(6)} + \frac{4}{14175} J_{ijkm}^{(5)} J_{ijkm}^{(5)} \right] + \mathcal{O}(8) \}. \tag{2.7}
\end{aligned}$$

The hereditary contribution is given in Sect. III A of Paper I. We recall that it is decomposed as

$$\mathcal{F}_{\text{hered}} = \mathcal{F}_{\text{tail}} + \mathcal{F}_{\text{tail}(\text{tail})} + \mathcal{F}_{(\text{tail})^2}. \tag{2.8}$$

The quadratic-order (proportional to G^2) tails are given by

$$\begin{aligned}
\mathcal{F}_{\text{tail}} = & \frac{4G^2 M}{5c^8} I_{ij}^{(3)}(U) \int_0^{+\infty} d\tau I_{ij}^{(5)}(U - \tau) \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{11}{12} \right] \\
& + \frac{4G^2 M}{189c^{10}} I_{ijk}^{(4)}(U) \int_0^{+\infty} d\tau I_{ijk}^{(6)}(U - \tau) \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{97}{60} \right] \\
& + \frac{64G^2 M}{45c^{10}} J_{ij}^{(3)}(U) \int_0^{+\infty} d\tau J_{ij}^{(5)}(U - \tau) \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{7}{6} \right], \tag{2.9}
\end{aligned}$$

and the cubic-order tails (proportional to G^3) read

$$\mathcal{F}_{\text{tail}(\text{tail})} = \frac{4G^3 M^2}{5c^{11}} I_{ij}^{(3)}(U) \int_0^{+\infty} d\tau I_{ij}^{(6)}(U - \tau) \left[\ln^2 \left(\frac{c\tau}{2r_0} \right) + \frac{57}{70} \ln \left(\frac{c\tau}{2r_0} \right) + \frac{124627}{44100} \right], \tag{2.10a}$$

$$\mathcal{F}_{(\text{tail})^2} = \frac{4G^3 M^2}{5c^{11}} \left(\int_0^{+\infty} d\tau I_{ij}^{(5)}(U - \tau) \left[\ln \left(\frac{c\tau}{2r_0} \right) + \frac{11}{12} \right] \right)^2. \tag{2.10b}$$

All the tail contributions are thoroughly computed in Paper I and we shall use those results to obtain the complete GW energy flux in Sec. VIII.

III. THE MULTIPOLE MOMENTS OF COMPACT BINARY SYSTEMS

We provide, in this Section, the requisite multipole moments needed for the computation of the 3PN accurate energy flux for compact binaries in the *standard harmonic* coordinate system. By standard harmonic coordinates we refer to the specific coordinate system which has been used consistently in previous works [27, 29, 30, 31, 32, 34, 35, 43]. We recall that these coordinates contain some *logarithms* at the 3PN level both in the equations of motion of the binary [27, 43] and in their multipole moments [30, 31, 32]. Later, we shall also define some *modified harmonic* coordinates which do not involve such logarithms at the 3PN order.

The multipole moments are generalisations to non-circular orbits of the expressions available in Ref. [30] for circular orbits. They are computed by implementing the detailed method described in Ref. [31]. Though algebraically long and involved, the procedure is fairly algorithmic as explained in [30, 31]. We thus skip all those details of computations and list the final results we need. The

3PN mass quadrupole I_{ij} is already given in Ref. [31] and its expression (valid in the frame of the center of mass) is

$$I_{ij} = \nu m \left\{ \left[A - \frac{24}{7} \frac{\nu}{c^5} \frac{G^2 m^2}{r^2} \dot{r} \right] x_{<i} x_{j>} + B \frac{r^2}{c^2} v_{<i} v_{j>} + 2 \left[C \frac{r \dot{r}}{c^2} + \frac{24}{7} \frac{\nu}{c^5} \frac{G^2 m^2}{r} \right] x_{<i} v_{j>} \right\}, \quad (3.1)$$

where the coefficients, up to 3PN order, are

$$\begin{aligned} A = & 1 + \frac{1}{c^2} \left[v^2 \left(\frac{29}{42} - \frac{29\nu}{14} \right) + \frac{Gm}{r} \left(-\frac{5}{7} + \frac{8}{7}\nu \right) \right] \\ & + \frac{1}{c^4} \left[\frac{Gm}{r} v^2 \left(\frac{2021}{756} - \frac{5947}{756}\nu - \frac{4883}{756}v^2 \right) \right. \\ & \quad + \frac{G^2 m^2}{r^2} \left(-\frac{355}{252} - \frac{953}{126}\nu + \frac{337}{252}v^2 \right) \\ & \quad + v^4 \left(\frac{253}{504} - \frac{1835}{504}\nu + \frac{3545}{504}v^2 \right) \\ & \quad \left. + \frac{Gm}{r} \dot{r}^2 \left(-\frac{131}{756} + \frac{907}{756}\nu - \frac{1273}{756}v^2 \right) \right] \\ & + \frac{1}{c^6} \left[v^6 \left(\frac{4561}{11088} - \frac{7993}{1584}\nu + \frac{117067}{5544}v^2 - \frac{328663}{11088}v^3 \right) \right. \\ & \quad + v^4 \frac{Gm}{r} \left(\frac{307}{77} - \frac{94475}{4158}\nu + \frac{218411}{8316}v^2 + \frac{299857}{8316}v^3 \right) \\ & \quad + \frac{G^3 m^3}{r^3} \left(\frac{6285233}{207900} + \frac{15502}{385}\nu - \frac{3632}{693}v^2 + \frac{13289}{8316}v^3 \right. \\ & \quad \quad \left. - \frac{428}{105} \ln\left(\frac{r}{r_0}\right) - \frac{44}{3}\nu \ln\left(\frac{r}{r'_0}\right) \right) \\ & \quad + \frac{G^2 m^2}{r^2} \dot{r}^2 \left(-\frac{8539}{20790} + \frac{52153}{4158}\nu - \frac{4652}{231}v^2 - \frac{54121}{5544}v^3 \right) \\ & \quad + \frac{Gm}{r} \dot{r}^4 \left(\frac{2}{99} - \frac{1745}{2772}\nu + \frac{16319}{5544}v^2 - \frac{311}{99}v^3 \right) \\ & \quad + \frac{G^2 m^2}{r^2} v^2 \left(\frac{187183}{83160} - \frac{605419}{16632}\nu + \frac{434909}{16632}v^2 - \frac{37369}{2772}v^3 \right) \\ & \quad \left. + \frac{Gm}{r} v^2 \dot{r}^2 \left(-\frac{757}{5544} + \frac{5545}{8316}\nu - \frac{98311}{16632}v^2 + \frac{153407}{8316}v^3 \right) \right], \quad (3.2a) \end{aligned}$$

$$\begin{aligned} B = & \frac{11}{21} - \frac{11}{7}\nu \\ & + \frac{1}{c^2} \left[\frac{Gm}{r} \left(\frac{106}{27} - \frac{335}{189}\nu - \frac{985}{189}v^2 \right) \right. \\ & \quad + v^2 \left(\frac{41}{126} - \frac{337}{126}\nu + \frac{733}{126}v^2 \right) + \dot{r}^2 \left(\frac{5}{63} - \frac{25}{63}\nu + \frac{25}{63}v^2 \right) \left. \right] \\ & + \frac{1}{c^4} \left[v^4 \left(\frac{1369}{5544} - \frac{19351}{5544}\nu + \frac{45421}{2772}v^2 - \frac{139999}{5544}v^3 \right) \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{G^2 m^2}{r^2} \left(-\frac{40716}{1925} - \frac{10762}{2079} \nu + \frac{62576}{2079} \nu^2 - \frac{24314}{2079} \nu^3 \right. \\
& \quad \left. + \frac{428}{105} \ln\left(\frac{r}{r_0}\right) \right) \\
& + \frac{Gm}{r} \dot{r}^2 \left(\frac{79}{77} - \frac{5807}{1386} \nu + \frac{515}{1386} \nu^2 + \frac{8245}{693} \nu^3 \right) \\
& + \frac{Gm}{r} v^2 \left(\frac{587}{154} - \frac{67933}{4158} \nu + \frac{25660}{2079} \nu^2 + \frac{129781}{4158} \nu^3 \right) \\
& + v^2 \dot{r}^2 \left(\frac{115}{1386} - \frac{1135}{1386} \nu + \frac{1795}{693} \nu^2 - \frac{3445}{1386} \nu^3 \right) \Bigg], \tag{3.2b}
\end{aligned}$$

$$\begin{aligned}
C &= -\frac{2}{7} + \frac{6}{7} \nu \\
& + \frac{1}{c^2} \left[v^2 \left(-\frac{13}{63} + \frac{101}{63} \nu - \frac{209}{63} \nu^2 \right) \right. \\
& \quad \left. + \frac{Gm}{r} \left(-\frac{155}{108} + \frac{4057}{756} \nu + \frac{209}{108} \nu^2 \right) \right] \\
& + \frac{1}{c^4} \left[\frac{Gm}{r} v^2 \left(-\frac{2839}{1386} + \frac{237893}{16632} \nu - \frac{188063}{8316} \nu^2 - \frac{58565}{4158} \nu^3 \right) \right. \\
& \quad + \frac{G^2 m^2}{r^2} \left(-\frac{12587}{41580} + \frac{406333}{16632} \nu - \frac{2713}{396} \nu^2 + \frac{4441}{2772} \nu^3 \right) \\
& \quad + v^4 \left(-\frac{457}{2772} + \frac{6103}{2772} \nu - \frac{13693}{1386} \nu^2 + \frac{40687}{2772} \nu^3 \right) \\
& \quad \left. + \frac{Gm}{r} \dot{r}^2 \left(\frac{305}{5544} + \frac{3233}{5544} \nu - \frac{8611}{5544} \nu^2 - \frac{895}{154} \nu^3 \right) \right]. \tag{3.2c}
\end{aligned}$$

In the above equation r_0 is the length scale appearing in the definition of the source multipole moments [31] and is the same as in Eq. (2.5). On the other hand, the different constant r'_0 is related to two other length scales r'_1 and r'_2 (one for each particle) by $m \ln r'_0 = m_1 \ln r'_1 + m_2 \ln r'_2$, and is specific to the application of the formalism to point particle systems. It comes from regularizing the self-field of point particles in the standard harmonic coordinate system. It is very important to note that the two length scales r'_1 and r'_2 are the same as the two scales that appear in the final expression of the 3PN equations of motion in standard harmonic coordinates [27]. The requirement that these r'_1 and r'_2 should match with similar scales that appear in the equations of motion determine, using dimensional regularization, the values of the Hadamard's regularization constants ξ , κ and ζ that formerly appeared in the 3PN multipole moments [30, 31]. The regularization constants are thus determined and we have consistently replaced ξ , κ and ζ by their values known from [32, 34]. The constants r'_1 , r'_2 and hence r'_0 are “unphysical” in the sense that they can be arbitrarily changed by a coordinate transformation of the “bulk” metric outside the particles [27], or, more appropriately (when considering the renormalisation which follows the dimensional regularization), by some shifts of the particles' world lines [29, 34].

The 2PN mass octupole and current quadrupole moments for general orbits are the other non-

trivial moments required. They are given by

$$\begin{aligned}
I_{ijk} = & \nu m \sqrt{1-4\nu} \left\{ x_{\langle ij k \rangle} \left[-1 + \frac{1}{c^2} \left[\frac{Gm}{r} \left(\frac{5}{6} - \frac{13\nu}{6} \right) + v^2 \left(-\frac{5}{6} + \frac{19}{6} \nu \right) \right] \right. \right. \\
& + \frac{1}{c^4} \left[v^4 \left(-\frac{257}{440} + \frac{7319}{1320} \nu - \frac{5501}{440} \nu^2 \right) + \frac{G^2 m^2}{r^2} \left(\frac{47}{33} + \frac{1591}{132} \nu - \frac{235}{66} \nu^2 \right) \right. \\
& + \left. \left. \frac{Gm}{r} \dot{r}^2 \left(\frac{247}{1320} - \frac{531}{440} \nu + \frac{1347}{440} \nu^2 \right) + \frac{Gm}{r} v^2 \left(-\frac{3853}{1320} + \frac{14257}{1320} \nu + \frac{17371}{1320} \nu^2 \right) \right] \right] \\
& + x_{\langle ij \nu k \rangle} \frac{r \dot{r}}{c^2} \left[1 - 2\nu + \frac{1}{c^2} \left[\frac{Gm}{r} \left(\frac{2461}{660} - \frac{8689}{660} \nu - \frac{1389}{220} \nu^2 \right) \right. \right. \\
& + \left. \left. v^2 \left(\frac{13}{22} - \frac{107}{22} \nu + \frac{102}{11} \nu^2 \right) \right] \right] + x_{\langle i \nu j k \rangle} \frac{r^2}{c^2} \left[-1 + 2\nu + \frac{1}{c^2} \left[v^2 \left(-\frac{61}{110} + \frac{519}{110} \nu - \frac{504}{55} \nu^2 \right) \right. \right. \\
& + \left. \left. \dot{r}^2 \left(\frac{1}{11} - \frac{4}{11} \nu + \frac{3}{11} \nu^2 \right) + \frac{Gm}{r} \left(-\frac{1949}{330} - \frac{62}{165} \nu + \frac{483}{55} \nu^2 \right) \right] \right] \\
& + \left. v_{\langle ij k \rangle} \frac{\dot{r} r^3}{c^4} \left(-\frac{13}{55} + \frac{52}{55} \nu - \frac{39}{55} \nu^2 \right) \right\} + O(6), \tag{3.3a}
\end{aligned}$$

$$\begin{aligned}
J_{ij} = & m \nu \sqrt{1-4\nu} \left\{ \varepsilon_{ab\langle i} x_{j \rangle a} v_b \left[-1 + \frac{1}{c^2} \left[\frac{Gm}{r} \left(-\frac{27}{14} - \frac{15}{7} \nu \right) + v^2 \left(-\frac{13}{28} + \frac{17}{7} \nu \right) \right] \right. \right. \\
& + \frac{1}{c^4} \left[v^4 \left(-\frac{29}{84} + \frac{11}{3} \nu - \frac{505}{56} \nu^2 \right) + \frac{G^2 m^2}{r^2} \left(\frac{43}{252} + \frac{1543}{126} \nu - \frac{293}{84} \nu^2 \right) \right. \\
& + \left. \left. \frac{Gm}{r} \dot{r}^2 \left(\frac{5}{252} + \frac{241}{252} \nu + \frac{335}{84} \nu^2 \right) + \frac{Gm}{r} v^2 \left(-\frac{671}{252} + \frac{1297}{126} \nu + \frac{121}{12} \nu^2 \right) \right] \right] \\
& + \varepsilon_{ab\langle i} v_{j \rangle b} x_a \frac{r \dot{r}}{c^2} \left[-\frac{5}{28} + \frac{5}{14} \nu + \frac{1}{c^2} \left[v^2 \left(-\frac{25}{168} + \frac{25}{24} \nu - \frac{25}{14} \nu^2 \right) \right. \right. \\
& + \left. \left. \frac{Gm}{r} \left(-\frac{103}{63} - \frac{337}{126} \nu + \frac{173}{84} \nu^2 \right) \right] \right] \right\} + O(6). \tag{3.3b}
\end{aligned}$$

In the above and what follows, $x_{ijk\dots} \equiv x_i x_j x_k \dots$ and $v_{ijk\dots} \equiv v_i v_j v_k \dots$, and the brackets $\langle \rangle$ denote the STF projection. The 1PN moments read as

$$\begin{aligned}
I_{ijkl} = & \nu m \left\{ x_{\langle ijkl \rangle} \left[1 - 3\nu + \frac{1}{c^2} \left[\left(\frac{103}{110} - \frac{147}{22} \nu + \frac{279}{22} \nu^2 \right) v^2 \right. \right. \right. \\
& - \left. \left. \left(\frac{10}{11} - \frac{61}{11} \nu + \frac{105}{11} \nu^2 \right) \frac{Gm}{r} \right] \right] - \frac{72}{55} v_{\langle i} x_{jkl \rangle} \frac{r \dot{r}}{c^2} (1 - 5\nu + 5\nu^2) \right. \\
& + \left. \frac{78}{55} v_{\langle ij} x_{kl \rangle} \frac{r^2}{c^2} (1 - 5\nu + 5\nu^2) \right\} + O(4), \tag{3.4a}
\end{aligned}$$

$$\begin{aligned}
J_{ijk} = & \nu m \varepsilon_{ab\langle i} \left\{ x_{j \rangle a} v_b \left[1 - 3\nu \right. \right. \\
& + \frac{1}{c^2} \left[\left(\frac{41}{90} - \frac{77}{18} \nu + \frac{185}{18} \nu^2 \right) v^2 + \left(\frac{14}{9} - \frac{16}{9} \nu - \frac{86}{9} \nu^2 \right) \frac{Gm}{r} \right] \right] \\
& + \left. \frac{7}{45} v_{jk \rangle b} x_a \frac{r^2}{c^2} (1 - 5\nu + 5\nu^2) + \frac{2}{9} x_{j \underline{a}} v_{k \rangle b} \frac{r \dot{r}}{c^2} (1 - 5\nu + 5\nu^2) \right\} + O(4). \tag{3.4b}
\end{aligned}$$

[The underlined index \underline{a} means that it should be excluded from the STF projection.] Finally we also need

$$I_{ijklm} = -\nu m \sqrt{1-4\nu} (1-2\nu) x_{<i j k l m>} + O(2), \quad (3.5a)$$

$$J_{ijkl} = -\nu m \sqrt{1-4\nu} (1-2\nu) \varepsilon_{ab<i} x_{jkl>a} v_b + O(2), \quad (3.5b)$$

as well as W , the monopole corresponding to the gauge moments W_L , and which is given by

$$W = \frac{1}{3} \nu m r \dot{r} + O(2). \quad (3.6)$$

IV. THE EQUATIONS OF MOTION OF COMPACT BINARY SYSTEMS

A. The equations of motion in standard harmonic coordinates

The computation of the flux will involve the time derivatives of the latter source moments. The 3PN accurate flux requires the 3PN equations of motion for compact binaries which are now complete [27, 28, 29, 44, 45]. For the present work, where the multipole moments are computed in standard harmonic coordinates and reduced to the centre of mass (CM) frame, we require the 3PN accurate equation of motion (or acceleration) in the CM frame associated with the standard harmonic gauge. This was computed in [43] and given as

$$a^i = \frac{dv^i}{dt} = -\frac{Gm}{r^2} [(1+P)n^i + Qv^i] + O(7), \quad (4.1)$$

where the coefficients P and Q are:

$$\begin{aligned} P = & \frac{1}{c^2} \left\{ -\frac{3\dot{r}^2 \nu}{2} + v^2 + 3\nu v^2 - \frac{Gm}{r} (4+2\nu) \right\} \\ & + \frac{1}{c^4} \left\{ \frac{15\dot{r}^4 \nu}{8} - \frac{45\dot{r}^4 \nu^2}{8} - \frac{9\dot{r}^2 \nu v^2}{2} + 6\dot{r}^2 v^2 v^2 + 3\nu v^4 - 4v^2 v^4 \right. \\ & \quad \left. + \frac{Gm}{r} \left(-2\dot{r}^2 - 25\dot{r}^2 \nu - 2\dot{r}^2 v^2 - \frac{13\nu v^2}{2} + 2v^2 v^2 \right) \right. \\ & \quad \left. + \frac{G^2 m^2}{r^2} \left(9 + \frac{87\nu}{4} \right) \right\} \\ & + \frac{1}{c^5} \left\{ -\frac{24\dot{r} \nu v^2 Gm}{5r} - \frac{136\dot{r} \nu G^2 m^2}{15r^2} \right\} \\ & + \frac{1}{c^6} \left\{ -\frac{35\dot{r}^6 \nu}{16} + \frac{175\dot{r}^6 \nu^2}{16} - \frac{175\dot{r}^6 \nu^3}{16} + \frac{15\dot{r}^4 \nu v^2}{2} \right. \\ & \quad - \frac{135\dot{r}^4 \nu^2 v^2}{4} + \frac{255\dot{r}^4 \nu^3 v^2}{8} - \frac{15\dot{r}^2 \nu v^4}{2} + \frac{237\dot{r}^2 \nu^2 v^4}{8} \\ & \quad - \frac{45\dot{r}^2 \nu^3 v^4}{2} + \frac{11\nu v^6}{4} - \frac{49v^2 v^6}{4} + 13\nu^3 v^6 \\ & \quad \left. + \frac{Gm}{r} \left(79\dot{r}^4 \nu - \frac{69\dot{r}^4 \nu^2}{2} - 30\dot{r}^4 \nu^3 - 121\dot{r}^2 \nu v^2 + 16\dot{r}^2 \nu^2 v^2 \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + 20 \dot{r}^2 \nu^3 v^2 + \frac{75 \nu v^4}{4} + 8 \nu^2 v^4 - 10 \nu^3 v^4 \Big) \\
& + \frac{G^2 m^2}{r^2} \left(\dot{r}^2 + \frac{32573 \dot{r}^2 \nu}{168} + \frac{11 \dot{r}^2 \nu^2}{8} - 7 \dot{r}^2 \nu^3 + \frac{615 \dot{r}^2 \nu \pi^2}{64} - \frac{26987 \nu v^2}{840} \right. \\
& \quad \left. + \nu^3 v^2 - \frac{123 \nu \pi^2 v^2}{64} - 110 \dot{r}^2 \nu \ln\left(\frac{r}{r'_0}\right) + 22 \nu v^2 \ln\left(\frac{r}{r'_0}\right) \right) \\
& + \frac{G^3 m^3}{r^3} \left(-16 - \frac{437 \nu}{4} - \frac{71 \nu^2}{2} + \frac{41 \nu \pi^2}{16} \right) \Big\} , \tag{4.2a}
\end{aligned}$$

$$\begin{aligned}
Q &= \frac{1}{c^2} \left\{ -4 \dot{r} + 2 \dot{r} \nu \right\} \\
& + \frac{1}{c^4} \left\{ \frac{9 \dot{r}^3 \nu}{2} + 3 \dot{r}^3 \nu^2 - \frac{15 \dot{r} \nu v^2}{2} - 2 \dot{r} \nu^2 v^2 \right. \\
& \quad \left. + \frac{Gm}{r} \left(2 \dot{r} + \frac{41 \dot{r} \nu}{2} + 4 \dot{r} \nu^2 \right) \right\} \\
& + \frac{1}{c^5} \left\{ \frac{8 \nu v^2 Gm}{5 r} + \frac{24 \nu G^2 m^2}{5 r^2} \right\} \\
& + \frac{1}{c^6} \left\{ -\frac{45 \dot{r}^5 \nu}{8} + 15 \dot{r}^5 \nu^2 + \frac{15 \dot{r}^5 \nu^3}{4} + 12 \dot{r}^3 \nu v^2 \right. \\
& \quad - \frac{111 \dot{r}^3 \nu^2 v^2}{4} - 12 \dot{r}^3 \nu^3 v^2 - \frac{65 \dot{r} \nu v^4}{8} + 19 \dot{r} \nu^2 v^4 + 6 \dot{r} \nu^3 v^4 \\
& \quad + \frac{Gm}{r} \left(\frac{329 \dot{r}^3 \nu}{6} + \frac{59 \dot{r}^3 \nu^2}{2} + 18 \dot{r}^3 \nu^3 - 15 \dot{r} \nu v^2 - 27 \dot{r} \nu^2 v^2 - 10 \dot{r} \nu^3 v^2 \right) \\
& \quad \left. + \frac{G^2 m^2}{r^2} \left(-4 \dot{r} - \frac{18169 \dot{r} \nu}{840} + 25 \dot{r} \nu^2 + 8 \dot{r} \nu^3 - \frac{123 \dot{r} \nu \pi^2}{32} \right. \right. \\
& \quad \left. \left. + 44 \dot{r} \nu \ln\left(\frac{r}{r'_0}\right) \right) \right\} . \tag{4.2b}
\end{aligned}$$

Recall that there was initially a regularization ambiguity constant denoted λ in [27], which has been replaced here by its uniquely determined value $\lambda = -\frac{1987}{3080}$ [29]. On the other hand the constant r'_0 is the *same* as the one in the 3PN quadrupole moment (3.1)–(3.2).

B. The modified harmonic coordinates (without logarithms)

The standard harmonic (hereafter SH) coordinate system used till now is useful for analytical algebraic checks, but contains gauge-dependent logarithmic terms that are not very convenient in numerical calculations. More importantly, in the presence of the logarithmic terms the simple generalized quasi-Keplerian representation (reviewed in Sect. VII) is not possible impeding the process of averaging the flux over the orbital period. Consequently, it is useful to have the expression for the energy flux in a modified harmonic (MH) coordinate system without logarithms like the one explicitly used in [46] (we shall alternatively use ADM type coordinates which are also free of such logarithms at 3PN). This will require us to re-express the instantaneous expressions for the energy flux [given by Eqs. (5.2) below] in terms of corresponding variables in the MH or ADM coordinate systems. We provide in this section the definition of the MH coordinate system.

Consider the coordinate transformation $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x)$ which removes the logarithms $\ln(r/r'_0)$ at the level of the equations of motion as discussed in Ref. [27]. It is given by

$$\varepsilon_{\mu} = \frac{22}{3} \frac{G^2 m_1 m_2}{c^6} \partial_{\mu} \left[\frac{G m_2}{r_1} \ln\left(\frac{r}{r'_1}\right) + \frac{G m_1}{r_2} \ln\left(\frac{r}{r'_2}\right) \right], \quad (4.3)$$

where $r_1 = |\mathbf{x} - \mathbf{y}_1|$ and $r_2 = |\mathbf{x} - \mathbf{y}_2|$ are the distances to the two particles with trajectories $y_1^i(t)$ and $y_2^i(t)$, and where $r = |\mathbf{y}_1 - \mathbf{y}_2|$ is their relative distance. Following [29] the logarithms can be equivalently removed by the shifts (sometimes also called the ‘‘contact’’ transformations) of the particle world-lines induced by the change of coordinates, namely

$$y_1^i = y_1^i + \xi_1^i, \quad (4.4a)$$

$$y_2^i = y_2^i + \xi_2^i. \quad (4.4b)$$

The equalities here are *functional* relations, *i.e.* the two sides of the equations are evaluated at the *same* coordinate time say, t . The spatial shifts ξ_1^i and ξ_2^i of the two world-lines are related to the coordinate transformation restricted to the world-lines [denoted $\varepsilon_1^{\mu}(t) \equiv \varepsilon^{\mu}(t, y_1(t))$ and $\varepsilon_2^{\mu}(t) \equiv \varepsilon^{\mu}(t, y_2(t))$] by

$$\xi_1^i = \varepsilon_1^i - \frac{v_1^i}{c} \varepsilon_1^0 + \mathcal{O}(\varepsilon^2), \quad (4.5a)$$

$$\xi_2^i = \varepsilon_2^i - \frac{v_2^i}{c} \varepsilon_2^0 + \mathcal{O}(\varepsilon^2), \quad (4.5b)$$

where $v_1^i = dy_1^i/dt$ and $v_2^i = dy_2^i/dt$ are the coordinate velocities. The latter relations are valid at linear order in ε_1^{μ} and ε_2^{μ} . Now the coordinate transformation (4.3) is at 3PN order so we have $\varepsilon^0 = \mathcal{O}(7)$ and $\varepsilon^i = \mathcal{O}(6)$. Hence, we see from (4.5) that at the 3PN order the shifts simply agree with the spatial components of the coordinate transformation,

$$\xi_1^i = \varepsilon_1^i + \mathcal{O}(8), \quad (4.6a)$$

$$\xi_2^i = \varepsilon_2^i + \mathcal{O}(8). \quad (4.6b)$$

They are readily obtained from Eq. (4.3) as

$$\xi_1^i = -\frac{22}{3} \frac{G^3 m_1^2 m_2}{c^6 r^2} n^i \ln\left(\frac{r}{r'_1}\right) + \mathcal{O}(8), \quad (4.7a)$$

$$\xi_2^i = \frac{22}{3} \frac{G^3 m_1 m_2^2}{c^6 r^2} n^i \ln\left(\frac{r}{r'_2}\right) + \mathcal{O}(8). \quad (4.7b)$$

Under the shifts of world-lines the accelerations of the particles are changed by the amounts $\delta_{\xi} a_1^i$ and $\delta_{\xi} a_2^i$ (*i.e.* such that the functional equalities $a_{1,2}^i = a_{1,2}^i + \delta_{\xi} a_{1,2}^i$ hold) given by

$$\delta_{\xi} a_1^i = \frac{d^2 \xi_1^i}{dt^2} - (\xi_1^j - \xi_2^j) \partial_{ij} \left(\frac{G m_2}{r} \right) + \mathcal{O}(8), \quad (4.8a)$$

$$\delta_{\xi} a_2^i = \frac{d^2 \xi_2^i}{dt^2} + (\xi_1^j - \xi_2^j) \partial_{ij} \left(\frac{G m_1}{r} \right) + \mathcal{O}(8). \quad (4.8b)$$

where the second terms come from re-expressing the gravitational force – gradient of the Newtonian potential – in terms of the new trajectories (4.4). The relative acceleration $a^i \equiv a_1^i - a_2^i$ is changed by the amount

$$\delta_\xi a^i = \frac{d^2 \xi_{12}^i}{dt^2} - \xi_{12}^j \partial_{ij} \left(\frac{Gm}{r} \right) + O(8), \quad (4.9)$$

where $m \equiv m_1 + m_2$ and $\xi_{12}^i \equiv \xi_1^i - \xi_2^i$.³ An easy calculation shows that the change in the relative acceleration associated with the shifts (4.7) is

$$\begin{aligned} \delta_\xi a^i = \frac{G^3 m^3 v}{c^6 r^4} & \left\{ \left[(-110\dot{r}^2 + 22v^2) n^i + 44\dot{r} v^i \right] \ln \left(\frac{r}{r'_0} \right) \right. \\ & \left. + \left(\frac{176}{3} \dot{r}^2 - \frac{22}{3} v^2 + \frac{22}{3} \frac{Gm}{r} \right) n^i - \frac{44}{3} \dot{r} v^i \right\} + O(8). \end{aligned} \quad (4.10)$$

Adding the above shift to the expression for the relative acceleration in SH coordinates as given by Eqs. (4.1)–(4.2), yields the expression for the acceleration in MH coordinates. Since $a^{i'} = a^i + \delta_\xi a^i$ is a functional identity, the resulting MH acceleration is obtained as a function of the “dummy” variables denoted v^2 , \dot{r} and r . Evidently these variables are to be interpreted as the natural variables describing the binary motion in MH coordinates.⁴ As expected, the logarithms in Eq. (4.10) exactly cancel the logarithms in the SH acceleration (4.1)–(4.2). Some 3PN coefficients in the EOM are also modified and the final result agrees with that displayed in Ref. [46] (see also [6]).

For completeness we note also that the above shifts will modify the 3PN conserved energy of the binary (associated with the conservative part of the 3PN equations of motion) by the amount

$$\delta_\xi E = -m_1 v_1^i \frac{d\xi_1^i}{dt} - m_2 v_2^i \frac{d\xi_2^i}{dt} + \xi_{12}^i \partial_i \left(\frac{Gm_1 m_2}{r} \right) + O(8). \quad (4.11)$$

For the case at hand with the shifts (4.7) and in the center-of-mass frame we find

$$\delta_\xi E = \frac{22}{3} \frac{G^3 m^4 v^2}{c^6 r^3} \left\{ \left[\frac{Gm}{r} - 3\dot{r}^2 + v^2 \right] \ln \left(\frac{r}{r'_0} \right) + \dot{r}^2 \right\} + O(8). \quad (4.12)$$

Comparing with the 3PN energy in SH coordinates as given by Eq. (4.8) in [43], we see that the logarithms $\ln(r/r'_0)$ are also cancelled in the expression for the energy by going to the MH coordinates.

V. THE INSTANTANEOUS PART OF THE 3PN ENERGY FLUX

Using the multipole moments given in Eqs. (3.1)–(3.5), one computes the required time derivatives with the help of the equations of motion (4.1) and obtains the instantaneous part of the energy flux as defined by Eq. (2.7). Here we are working in SH coordinates, in which the equations of motion are given by Eqs. (4.1)–(4.2). In the next section we consider the case of alternative coordinate systems. The hereditary part computed in Paper I will be added after the process of averaging

³ This means that $x_{\text{MH}}^i = x_{\text{SH}}^i + \xi_{12}^i$.

⁴ To avoid making the notation too heavy we do not add a subscript MH on the variables v^2 , \dot{r} and r . In the following our notation may not always be completely consistent but should be clear from the context.

over one orbit (this contribution is the same in all alternative coordinate systems considered in this paper). Though lengthy the computation of the different parts constituting the instantaneous terms in the energy flux at 3PN order is straightforward.⁵ We write the result as

$$\mathcal{F}_{\text{inst}} = \mathcal{F}_{\text{inst}}^{\text{N}} + \mathcal{F}_{\text{inst}}^{\text{1PN}} + \mathcal{F}_{\text{inst}}^{\text{2PN}} + \mathcal{F}_{\text{inst}}^{\text{2.5PN}} + \mathcal{F}_{\text{inst}}^{\text{3PN}} + \mathcal{O}(7), \quad (5.1)$$

and find that the various PN pieces are given by

$$\mathcal{F}_{\text{inst}}^{\text{N}} = \frac{32 G^3 m^4 v^2}{5 c^5 r^4} \left\{ v^2 - \frac{11}{12} \dot{r}^2 \right\}, \quad (5.2a)$$

$$\begin{aligned} \mathcal{F}_{\text{inst}}^{\text{1PN}} = & \frac{32 G^3 m^4 v^2}{5 c^7 r^4} \left\{ v^4 \left(\frac{785}{336} - \frac{71}{28} v \right) + \dot{r}^2 v^2 \left(-\frac{1487}{168} + \frac{58}{7} v \right) \right. \\ & + \frac{G m}{r} v^2 \left(-\frac{170}{21} + \frac{10}{21} v \right) + \dot{r}^4 \left(\frac{687}{112} - \frac{155}{28} v \right) \\ & \left. + \frac{G m}{r} \dot{r}^2 \left(\frac{367}{42} - \frac{5}{14} v \right) + \frac{G^2 m^2}{r^2} \left(\frac{1}{21} - \frac{4}{21} v \right) \right\}, \end{aligned} \quad (5.2b)$$

$$\begin{aligned} \mathcal{F}_{\text{inst}}^{\text{2PN}} = & \frac{32 G^3 m^4 v^2}{5 c^9 r^4} \left\{ v^6 \left(\frac{47}{14} - \frac{5497}{504} v + \frac{2215}{252} v^2 \right) \right. \\ & + \dot{r}^2 v^4 \left(-\frac{573}{56} + \frac{1713}{28} v - \frac{1573}{42} v^2 \right) \\ & + \frac{G m}{r} v^4 \left(-\frac{247}{14} + \frac{5237}{252} v - \frac{199}{36} v^2 \right) \\ & + \dot{r}^4 v^2 \left(\frac{1009}{84} - \frac{5069}{56} v + \frac{631}{14} v^2 \right) \\ & + \frac{G m}{r} \dot{r}^2 v^2 \left(\frac{4987}{84} - \frac{8513}{84} v + \frac{2165}{84} v^2 \right) \\ & + \frac{G^2 m^2}{r^2} v^2 \left(\frac{281473}{9072} + \frac{2273}{252} v + \frac{13}{27} v^2 \right) \\ & + \dot{r}^6 \left(-\frac{2501}{504} + \frac{10117}{252} v - \frac{2101}{126} v^2 \right) \\ & + \frac{G m}{r} \dot{r}^4 \left(-\frac{5585}{126} + \frac{60971}{756} v - \frac{7145}{378} v^2 \right) \\ & + \frac{G^2 m^2}{r^2} \dot{r}^2 \left(-\frac{106319}{3024} - \frac{1633}{504} v - \frac{16}{9} v^2 \right) \\ & \left. + \frac{G^3 m^3}{r^3} \left(-\frac{253}{378} + \frac{19}{7} v - \frac{4}{27} v^2 \right) \right\}, \end{aligned} \quad (5.2c)$$

$$\mathcal{F}_{\text{inst}}^{\text{2.5PN}} = \frac{32 G^3 m^4 v^2}{5 c^{10} r^4} \left\{ \dot{r} v \left(-\frac{12349}{210} \frac{G m}{r} v^4 + \frac{4524}{35} \frac{G m}{r} v^2 \dot{r}^2 - \frac{2753}{126} \frac{G^2 m^2}{r^2} v^2 \right) \right.$$

⁵ In order to perform some independent checks on the long and involved algebra, we have found it expeditious to make two computations using the two harmonic coordinate systems: SH containing the (gauge dependent) log terms *à la* [43] and MH without log terms as in Refs. [6, 46].

$$\begin{aligned}
& \left. -\frac{985}{14} \frac{G m}{r} \dot{r}^4 + \frac{13981}{630} \frac{G^2 m^2}{r^2} \dot{r}^2 - \frac{1}{315} \frac{G^3 m^3}{r^3} \right\}, \tag{5.2d} \\
\mathcal{F}_{\text{inst}}^{3\text{PN}} = & \frac{32}{5} \frac{G^3 m^4 v^2}{c^{11} r^4} \left\{ v^8 \left(\frac{80315}{14784} - \frac{694427}{22176} v + \frac{604085}{11088} v^2 - \frac{16985}{462} v^3 \right) \right. \\
& + \dot{r}^2 v^6 \left(-\frac{31499}{1008} + \frac{1119913}{5544} v - \frac{44701}{132} v^2 + \frac{38725}{231} v^3 \right) \\
& + \frac{G m}{r} v^6 \left(-\frac{61669}{3696} + \frac{95321}{1008} v - \frac{955013}{11088} v^2 + \frac{47255}{1386} v^3 \right) \\
& + \dot{r}^4 v^4 \left(\frac{204349}{2464} - \frac{3522149}{7392} v + \frac{2354753}{3696} v^2 - \frac{109447}{462} v^3 \right) \\
& + \frac{G m}{r} \dot{r}^2 v^4 \left(\frac{136695}{1232} - \frac{202693}{336} v + \frac{744377}{1232} v^2 - \frac{931099}{5544} v^3 \right) \\
& + \frac{G^2 m^2}{r^2} v^4 \left(\frac{598614941}{2494800} - \frac{856}{35} \ln\left(\frac{r}{r_0}\right) \right. \\
& \left. + \left[\frac{39896}{2079} - \frac{369}{64} \pi^2 \right] v + \frac{1300907}{33264} v^2 - \frac{161783}{24948} v^3 \right) \\
& + \dot{r}^6 v^2 \left(-\frac{1005979}{11088} + \frac{2589599}{5544} v - \frac{1322141}{2772} v^2 + \frac{90455}{693} v^3 \right) \\
& + \frac{G m}{r} \dot{r}^4 v^2 \left(-\frac{715157}{3696} + \frac{35158037}{33264} v - \frac{3672143}{3696} v^2 + \frac{871025}{4158} v^3 \right) \\
& + \frac{G^2 m^2}{r^2} \dot{r}^2 v^2 \left(-\frac{35629009}{37800} + \frac{3424}{35} \ln\left(\frac{r}{r_0}\right) \right. \\
& \left. + \left[-\frac{150739}{1232} + \frac{861}{32} \pi^2 \right] v - \frac{453247}{1848} v^2 + \frac{496081}{8316} v^3 \right) \\
& + \frac{G^3 m^3}{r^3} v^2 \left(-\frac{24608492}{155925} + \frac{856}{105} \ln\left(\frac{r}{r_0}\right) \right. \\
& \left. + \left[-\frac{6356291}{22680} + \frac{44}{3} \ln\left(\frac{r}{r'_0}\right) + \frac{451}{64} \pi^2 \right] v + \frac{3725}{462} v^2 - \frac{841}{2268} v^3 \right) \\
& + \dot{r}^8 \left(\frac{1507925}{44352} - \frac{20365}{126} v + \frac{687305}{5544} v^2 - \frac{32755}{1386} v^3 \right) \\
& + \frac{G m}{r} \dot{r}^6 \left(\frac{5476951}{55440} - \frac{671765}{1232} v + \frac{5205019}{11088} v^2 - \frac{860477}{11088} v^3 \right) \\
& + \frac{G^2 m^2}{r^2} \dot{r}^4 \left(\frac{115627817}{166320} - \frac{214}{3} \ln\left(\frac{r}{r_0}\right) \right. \\
& \left. + \left[\frac{42671}{792} - \frac{697}{32} \pi^2 \right] v + \frac{1099355}{4752} v^2 - \frac{825331}{16632} v^3 \right) \\
& + \frac{G^3 m^3}{r^3} \dot{r}^2 \left(\frac{3202601}{23100} - \frac{1712}{315} \ln\left(\frac{r}{r_0}\right) \right. \\
& \left. + \left[\frac{6220199}{22680} - \frac{88}{9} \ln\left(\frac{r}{r'_0}\right) - \frac{1763}{192} \pi^2 \right] v + \frac{57577}{1848} v^2 - \frac{43018}{6237} v^3 \right)
\end{aligned}$$

$$+ \frac{G^4 m^4}{r^4} \left(\frac{37571}{8316} - \frac{14962}{891} v - \frac{3019}{594} v^2 - \frac{866}{6237} v^3 \right) \Bigg\}. \quad (5.2e)$$

The new results are the instantaneous terms at 2.5PN and 3PN orders. Up to 2PN order, all the terms match with those obtained in Refs. [1, 14, 19]. As one may notice, the 2.5PN terms in the above equation are all proportional to \dot{r} and hence are zero for the circular orbit case in agreement with the result of [38]. The \dot{r} dependence of the 2.5PN terms is important when we discuss their orbital average in Sect. VIII. The 3PN terms provide the generalization of the circular orbit results in Ref. [30]. As expected, the constant r_0 present in the expression of the mass quadrupole moment appears in the final expression for the 3PN flux (the presence of r'_0 is of a different type and is dealt with in the next Section). The dependence of the instantaneous terms on the scale r_0 should exactly cancel a similar contribution coming from the tail terms as determined in Paper I. This cancellation has already been checked for circular orbits in [30] and we shall prove this cancellation for quasi-elliptical orbits in Sect. VIII.

VI. THE 3PN ENERGY FLUX IN ALTERNATIVE COORDINATES

The dependence on r'_0 of the result (5.1)–(5.2) is due to our use of the SH coordinate system. For circular orbits, it was shown [30] that this r'_0 dependence disappears when the total flux is expressed in terms of the gauge invariant parameter $x = (Gm\omega/c^3)^{2/3}$ related to the GW frequency. In the general orbit case we shall transform away the dependence on r'_0 by going to different coordinate systems such as the MH coordinates studied in Sect. IV B. Subsequently we shall average the energy flux over an orbital period and exhibit alternative representations of the energy flux for elliptical orbits. In particular some of these are in terms of gauge invariant variables related to those suggested in Ref. [37].

A. The modified harmonic coordinates

We now provide the energy flux \mathcal{F} in the modified harmonic (MH) system avoiding the appearance of the logarithms $\ln(r/r'_0)$ at 3PN order and which has been introduced in Sect. IV B. First we notice that \mathcal{F} is a function of the “natural” variables r , \dot{r} and v^2 [see Eqs. (5.2)], and is a *scalar*, therefore it satisfies, under the shifts of these variables defined by (4.4),

$$\mathcal{F}[r, \dot{r}, v^2] = \mathcal{F}'[r', \dot{r}', v'^2]. \quad (6.1)$$

This means that we shall have the *functional* equality $\mathcal{F}' = \mathcal{F} + \delta_\xi \mathcal{F}$ in which

$$\delta_\xi \mathcal{F} = -\delta_\xi r \frac{\partial \mathcal{F}}{\partial r} - \delta_\xi \dot{r} \frac{\partial \mathcal{F}}{\partial \dot{r}} - \delta_\xi v^2 \frac{\partial \mathcal{F}}{\partial v^2} + O(\xi^2), \quad (6.2)$$

where

$$\left. \begin{aligned} \delta_\xi r &= n^i \xi_{12}^i \\ \delta_\xi \dot{r} &= n^i \frac{d\xi_{12}^i}{dt} + \frac{v^i - \dot{r} n^i}{r} \xi_{12}^i \\ \delta_\xi v^2 &= 2v^i \frac{d\xi_{12}^i}{dt} \end{aligned} \right\} + O(\xi^2). \quad (6.3)$$

(Recall $\xi_{12}^i \equiv \xi_1^i - \xi_2^i$.) Since the previous formulas are at linear order in the shifts and we are interested in the 3PN approximation, they are valid for any shifts at the 3PN order (case of the MH coordinates) and also at the 2PN order like the ones associated with the passage to ADM coordinates – in the latter case, the error will be at 4PN order.

In the case of the MH shifts, which start at the 3PN order, one can make an alternative computation of the modification of the energy flux. Indeed the only modification *vis à vis* the calculation in standard harmonic (SH) coordinates is the one related to the mass quadrupole moment which must be computed to 3PN accuracy. Under the shifts of the particles' trajectories $y_{1,2}^i$ as given by (4.4), the mass quadrupole moment I_{ij} , which equals $I_{ij} = m_1 y_1^{<ij>} + m_2 y_2^{<ij>} + \mathcal{O}(2)$ in the Newtonian approximation, is shifted by the amount ($1 \leftrightarrow 2$ meaning the same term but for the other particle)

$$\delta_\xi I_{ij} = 2m_1 y_1^{<i \xi_1^{j>} + 1 \leftrightarrow 2 + \mathcal{O}(8), \quad (6.4)$$

where the remainder $\mathcal{O}(8)$ comes from the 1PN corrections in the quadrupole moment coupled with the 3PN shifts. Using the explicit expressions of these shifts in (4.7) we find, in the center-of-mass frame,

$$\delta_\xi I_{ij} = -\frac{44}{3} \frac{G^3 m^4 v^2}{r^3} \ln\left(\frac{r}{r'_0}\right) x^{<ij>} + \mathcal{O}(8), \quad (6.5)$$

where r'_0 is given by $m \ln r'_0 = m_1 \ln r'_1 + m_2 \ln r'_2$. This modification of the quadrupole moment is seen to exactly cancel the $\ln(r/r'_0)$ dependence of the mass quadrupole moment in SH coordinates as given by (3.1)–(3.2). Thus in the MH gauge the r'_0 dependence of the mass quadrupole moment vanishes as expected. The rest of the expression of the moment remains exactly the same as in SH coordinates, Eqs. (3.1)–(3.2), and will not be repeated here.

Next we must take into account the fact that when computing the third time derivative of the quadrupole moment, needed in the expression of the flux, the acceleration in MH coordinates is modified. We get

$$\delta_\xi (\ddot{I}_{ij}) = \frac{d^3}{dt^3} (\delta_\xi I_{ij}) - 2m_1 \left[3v_1^{<i} \delta_\xi a_1^{j>} + y_1^{<i} \frac{d\delta_\xi a_1^{j>}}{dt} \right] + 1 \leftrightarrow 2 + \mathcal{O}(8), \quad (6.6)$$

where the dots mean the time derivative. The first term is the third time derivative of the direct modification of the quadrupole moment, Eqs. (6.4)–(6.5), and the extra terms come from the modification of the accelerations which are given by (4.8). On the other hand, all the other contributions coming from the higher multipole moments and their derivatives remain unchanged. We then find

$$\delta_\xi \mathcal{F} = -\frac{2G}{5c^5} \left[\ddot{I}_{ij} \delta_\xi (\ddot{I}_{ij}) + \mathcal{O}(8) \right]. \quad (6.7)$$

With the explicit expression of the shifts one finally obtains the modification of the 3PN energy flux in the MH coordinates as (thus, $\mathcal{F}_{\text{MH}} = \mathcal{F}_{\text{SH}} + \delta_\xi \mathcal{F}$)

$$\delta_\xi \mathcal{F} = -\frac{1408}{15} \frac{G^6 m^7 v^3}{c^{11} r^7} \left[\left(v^2 - \frac{2}{3} \dot{r}^2 \right) \ln\left(\frac{r}{r'_0}\right) - \frac{\dot{r}^2}{12} + \mathcal{O}(2) \right]. \quad (6.8)$$

[Of course the result agrees with the one we would obtain from using directly Eqs. (6.2)–(6.3).]

B. The ADM coordinates

Many related numerical relativity studies are in ADM (or ADM-type) coordinates and hence for future applications we wish to provide the explicit expression for the 3PN energy flux in ADM coordinates. To transform the energy flux we require the shift or contact transformation of the trajectories connecting the SH coordinates (with log terms) and the ADM coordinates. We recall that the ADM and SH coordinate systems agree at 1PN order inclusively, so that the contact transformation is composed of 2PN and 3PN terms. Hence the calculation is more involved than for the MH coordinates (for which only the modification of the quadrupole moment I_{ij} played a role), and we must come back to the general formulas (6.2)–(6.3). Note that the remainder $\mathcal{O}(\xi^2)$ in Eqs. (6.2)–(6.3) is of order 4PN which is still negligible in the transformation to ADM type coordinates.

The relative shift ξ_{12}^i linking SH and ADM coordinates, $x_{\text{ADM}}^i = x_{\text{SH}}^i + \xi_{12}^i$, is given in [43] as⁶

$$\begin{aligned} \xi_{12}^i = & \frac{Gm}{c^4} \left\{ \left[\frac{\dot{r}^2 v}{8} - \frac{5 v v^2}{8} + \frac{Gm}{r} \left(-\frac{1}{4} - 3v \right) \right] n^i + \frac{9 \dot{r} v}{4} v^i \right\} \\ & + \frac{Gm}{c^6} \left\{ \left[-\frac{\dot{r}^4 v}{16} + \frac{5 \dot{r}^4 v^2}{16} + \frac{5 \dot{r}^2 v v^2}{16} - \frac{15 \dot{r}^2 v^2 v^2}{16} - \frac{v v^4}{2} + \frac{11 v^2 v^4}{8} \right. \right. \\ & \quad + \frac{Gm}{r} \left(\frac{161 \dot{r}^2 v}{48} - \frac{5 \dot{r}^2 v^2}{2} - \frac{451 v v^2}{48} - \frac{3 v^2 v^2}{8} \right) \\ & \quad \left. \left. + \frac{G^2 m^2}{r^2} \left(\frac{2773 v}{280} + \frac{21 v \pi^2}{32} - \frac{22 v}{3} \ln \left(\frac{r}{r'_0} \right) \right) \right] n^i \right. \\ & \quad \left. + \left[-\frac{5 \dot{r}^3 v}{12} + \frac{29 \dot{r}^3 v^2}{24} + \frac{17 \dot{r} v v^2}{8} - \frac{21 \dot{r} v^2 v^2}{4} + \frac{Gm}{r} \left(\frac{43 \dot{r} v}{3} + 5 \dot{r} v^2 \right) \right] v^i \right\}, \quad (6.9) \end{aligned}$$

from which we deduce, applying Eqs. (6.3), the transformation of variables necessary to compute the ADM energy flux:

$$\begin{aligned} \delta_{\xi} r = & \frac{Gm}{c^4} \left\{ v^2 \left(\frac{5}{8} v \right) + \dot{r}^2 \left(-\frac{19}{8} v \right) + \frac{Gm}{r} \left(\frac{1}{4} + 3v \right) \right\} \\ & + \frac{Gmv}{c^6} \left\{ v^4 \left(\frac{1}{2} - \frac{11}{8} v \right) + \dot{r}^2 v^2 \left(-\frac{39}{16} + \frac{99}{16} v \right) \right. \\ & \quad + \dot{r}^4 \left(\frac{23}{48} - \frac{73}{48} v \right) + \frac{Gm}{r} v^2 \left(\frac{451}{48} + \frac{3}{8} v \right) \\ & \quad \left. + \frac{Gm}{r} \dot{r}^2 \left(-\frac{283}{16} - \frac{5}{2} v \right) + \frac{G^2 m^2}{r^2} \left(-\frac{2773}{280} + \frac{22}{3} \ln \left(\frac{r}{r'_0} \right) - \frac{21}{32} \pi^2 \right) \right\}, \quad (6.10a) \\ \delta_{\xi} \dot{r} = & \frac{Gm}{c^4 r} \dot{r} \left\{ v^2 \left(-\frac{19}{4} v \right) + \dot{r}^2 \left(\frac{19}{4} v \right) + \frac{Gm}{r} \left(-\frac{1}{4} + \frac{1}{2} v \right) \right\} \\ & + \frac{Gmv}{c^6 r} \dot{r} \left\{ v^4 \left(-\frac{39}{8} + \frac{99}{8} v \right) + \dot{r}^2 v^2 \left(\frac{163}{24} - \frac{443}{24} v \right) + \dot{r}^4 \left(-\frac{23}{12} + \frac{73}{12} v \right) \right\} \end{aligned}$$

⁶ For simplicity we use the same notation ξ_{12}^i as for the shift between the SH and MH coordinates.

$$\begin{aligned}
& + \frac{Gm}{r} v^2 \left(-\frac{1603}{48} - \frac{17}{4} \nu \right) + \frac{Gm}{r} \dot{r}^2 \left(\frac{1777}{48} + \frac{131}{24} \nu \right) \\
& + \frac{G^2 m^2}{r^2} \left(\frac{3121}{105} - \frac{44}{3} \ln \left(\frac{r}{r'_0} \right) + \frac{21}{16} \pi^2 - \frac{11}{4} \nu \right) \Big\} , \tag{6.10b}
\end{aligned}$$

$$\begin{aligned}
\delta_\xi v^2 = & \frac{Gm}{c^4 r} \left\{ v^4 \left(-\frac{13}{4} \nu \right) + \dot{r}^2 v^2 \left(\frac{5}{2} \nu \right) + \dot{r}^4 \left(\frac{3}{4} \nu \right) \right. \\
& \left. + \frac{Gm}{r} v^2 \left(\frac{1}{2} + \frac{21}{2} \nu \right) + \frac{Gm}{r} \dot{r}^2 \left(-1 - \frac{19}{2} \nu \right) \right\} \\
& + \frac{Gm\nu}{c^6 r} \left\{ v^6 \left(-\frac{13}{4} + \frac{31}{4} \nu \right) + \dot{r}^2 v^4 \left(\frac{31}{8} - \frac{75}{8} \nu \right) + \dot{r}^4 v^2 \left(-\frac{3}{2} \nu \right) + \dot{r}^6 \left(-\frac{5}{8} + \frac{25}{8} \nu \right) \right. \\
& + \frac{Gm}{r} v^4 \left(-\frac{9}{8} - \frac{25}{4} \nu \right) + \frac{Gm}{r} \dot{r}^2 v^2 \left(-\frac{131}{8} + \frac{121}{4} \nu \right) + \frac{Gm}{r} \dot{r}^4 \left(\frac{99}{4} - \frac{259}{12} \nu \right) \\
& + \frac{G^2 m^2}{r^2} v^2 \left(-\frac{3839}{420} + \frac{44}{3} \ln \left(\frac{r}{r'_0} \right) - \frac{21}{16} \pi^2 + \nu \right) \\
& \left. + \frac{G^2 m^2}{r^2} \dot{r}^2 \left(\frac{28807}{420} - 44 \ln \left(\frac{r}{r'_0} \right) + \frac{63}{16} \pi^2 - \frac{13}{2} \nu \right) \right\} . \tag{6.10c}
\end{aligned}$$

The above equations provide the 3PN generalization of Eq. (4.6) of [19]. They also incorporate the corrected transformation between ADM and harmonic coordinates at 2PN, as given in [25].

Using the latter expressions, one finds that the SH energy flux is changed by corrections at 2PN and 3PN relative orders given by (using a notation similar to that introduced above – *i.e.* in which the variables r , \dot{r}^2 and v^2 are considered as dummy variables⁷)

$$\begin{aligned}
\delta_\xi \mathcal{F} = & - \frac{G^4 m^5 \nu^2}{c^9 r^5} \left\{ \frac{184}{5} v^4 \nu - \frac{736}{5} \dot{r}^2 v^2 \nu \right. \\
& \left. + \frac{Gm}{r} v^2 \left(\frac{16}{5} + \frac{48}{5} \nu \right) + \frac{320}{3} \dot{r}^4 \nu + \frac{Gm}{r} \dot{r}^2 \left(-\frac{12}{5} - \frac{56}{15} \nu \right) \right\} \\
& - \frac{G^4 m^5 \nu^2}{c^{11} r^5} \left\{ v^6 \left(\frac{5886}{35} \nu - \frac{1616}{7} \nu^2 \right) + \dot{r}^2 v^4 \left(-\frac{129866}{105} \nu + \frac{21598}{15} \nu^2 \right) \right. \\
& + \frac{Gm}{r} v^4 \left(-\frac{22798}{105} \nu + \frac{7528}{35} \nu^2 \right) + \dot{r}^4 v^2 \left(\frac{689434}{315} \nu - \frac{714608}{315} \nu^2 \right) \\
& + \frac{Gm}{r} \dot{r}^2 v^2 \left(-\frac{936}{35} + \frac{16103}{21} \nu - \frac{14086}{21} \nu^2 \right) \\
& + \frac{G^2 m^2}{r^2} v^2 \left(-\frac{272}{7} + \left[-\frac{31856}{75} - \frac{42}{5} \pi^2 \right] \nu + \frac{96}{35} \nu^2 \right) \\
& + \dot{r}^6 \left(-\frac{116138}{105} \nu + \frac{110986}{105} \nu^2 \right) \\
& \left. + \frac{Gm}{r} \dot{r}^4 \left(\frac{328}{15} - \frac{198097}{315} \nu + \frac{143924}{315} \nu^2 \right) \right\}
\end{aligned}$$

⁷ $\mathcal{F}_{\text{ADM}} = \mathcal{F}_{\text{SH}} + \delta_\xi \mathcal{F}$.

$$\begin{aligned}
& + \frac{G^2 m^2}{r^2} \dot{r}^2 \left(\frac{1612}{35} + \left(\frac{673544}{1575} + \frac{28}{5} \pi^2 \right) v + \frac{828}{35} v^2 \right) \\
& + \frac{G^3 m^3}{r^3} \left(\frac{16}{35} + \frac{128}{35} v - \frac{768}{35} v^2 \right) \\
& + \frac{G^2 m^2}{r^2} v \left(\frac{1408}{15} v^2 - \frac{2816}{45} \dot{r}^2 \right) \ln \left(\frac{r}{r'_0} \right) \}. \tag{6.11}
\end{aligned}$$

The examination of the coefficient of $\ln(r/r'_0)$, given by the two last terms of (6.11), reveals that this coefficient is the same as in the contact transformation from SH to MH, given by (6.8). Therefore the contact transformation from SH to ADM exactly cancels out the logarithms of SH coordinates, and the final flux in ADM coordinates is free of $\ln r'_0$. This is consistent with the general understanding that the $\ln r'_0$ is a feature of a particular harmonic coordinate system and that ADM coordinates do not yield complications associated with such logarithms (the cancellation of the $\ln r'_0$ terms usually provides a useful internal check of the computations).

VII. THE GENERALIZED QUASI-KEPLERIAN REPRESENTATION

Before we discuss the calculation of the orbital average of the energy flux in Sect. VIII, we must summarize the 3PN generalized quasi-Keplerian representation of the binary motion recently obtained by Memmesheimer, Gopakumar and Schäfer [37]. Indeed, the main application of the present computation is the evolution of the orbital elements under GW radiation reaction to 3PN order. This requires one to average over an orbit the instantaneous expressions for the energy flux obtained in Sect. V. Averaging over an orbit is most conveniently accomplished by the use of an explicit solution of the equations of motion. The generalized quasi-Keplerian (QK) representation of the motion at 3PN order [37] constitutes an essential input for the computations to follow.

The QK representation was introduced by Damour and Deruelle [18] to discuss the problem of binary pulsar timing at 1PN order, where relativistic periastron precession first appears and complicates the simpler Keplerian picture. This elegant formulation also played an important role in our computation of the hereditary terms in Paper I where we provided a summary of it. The 2PN extension of this work in the ADM coordinates was next given in Refs. [21, 22, 23] and we shall now use the 3PN parametrization in ADM and MH coordinates [37].

The radial motion is given in parametric form as⁸

$$r = a_r (1 - e_r \cos u), \tag{7.1a}$$

$$\ell = u - e_t \sin u + f_t \sin V + g_t (V - u) + i_t \sin 2V + h_t \sin 3V, \tag{7.1b}$$

while the corresponding angular motion is

$$\frac{\phi - \phi_P}{K} = V + f_\phi \sin 2V + g_\phi \sin 3V + i_\phi \sin 4V + h_\phi \sin 5V. \tag{7.2}$$

The four angles V , u , ℓ and ϕ are respectively the true anomaly, the eccentric anomaly, the mean anomaly and the orbital phase (V , u and ℓ are measured from the periastron, and we denote by ϕ_P the value of ϕ at periastron). The mean anomaly is proportional to the time elapsed since the

⁸ For convenience in this paper we adapt somewhat the notation with respect to the one in Ref. [37].

instant t_P of passage at periastron,

$$\ell = n (t - t_P) , \quad (7.3)$$

where $n = 2\pi/P$ is the mean motion and P is the orbital period. The true anomaly V is given by

$$V = 2 \arctan \left[\left(\frac{1 + e_\phi}{1 - e_\phi} \right)^{1/2} \tan \frac{u}{2} \right]. \quad (7.4)$$

In the above a_r represents the semi-major axis of the orbit, and e_r, e_t, e_ϕ are three kinds of eccentricities, labelled after the coordinates t, r and ϕ , and which differ from each other starting at the 1PN order. The constant K is linked with the advance of periastron per orbital revolution, and is given by $K = \Phi/(2\pi)$ where Φ is the angle of return to the periastron. The notation $k \equiv K - 1$ for the relativistic precession is used in Paper I and will also be useful here. The orbital elements $f_t, f_\phi, g_t, g_\phi, \dots$ parametrize the 2PN and 3PN relativistic corrections, as will be clear from their expressions below. [More precisely, f_t, f_ϕ, g_t, g_ϕ are composed of 2PN and 3PN terms, but i_t, i_ϕ, h_t, h_ϕ start only at 3PN order.]

Crucial to the formalism are the explicit formulas for all the orbital elements and all the coefficients in Eqs. (7.1) above in terms of the 3PN conserved orbital energy E and angular momentum J (divided by the binary's reduced mass). Recall that the construction of a generalized quasi-Keplerian representation exploits the fact that the radial equation – which is given by Eq. (2.1a) in paper I – is a *polynomial* in $1/r$ (of seventh degree at 3PN order). Therefore the presence of logarithmic terms in the SH coordinates at 3PN order obstructs the construction of the QK parametrization (at least by this method) and Ref. [37] obtained it in coordinates avoiding such logarithms, namely the MH and ADM coordinates. In both ADM and MH coordinates the QK representation takes the same form given by Eqs. (7.1)–(7.2), but of course the equations linking the orbital elements to E and J are different. These have been obtained as post-Newtonian series up to 3PN order in Ref. [37]. Since they form the basis for our computation of the average energy flux, we provide the complete relations here.

For convenience in the present paper we introduce a PN parameter which is directly linked to the energy E and defined by

$$\varepsilon \equiv -\frac{2E}{c^2}. \quad (7.5)$$

[Recall that $E < 0$ for gravitationally bound orbits.] The equations to follow will then appear as PN expansions in terms of $\varepsilon = O(2)$. Also, we find useful to define, in place of the angular momentum J ,

$$j \equiv -\frac{2EJ^2}{(Gm)^2}. \quad (7.6)$$

We have $j = -2Eh^2$ in terms of the more usual definition $h \equiv J/(Gm)$. This parameter is at Newtonian order, $j = c^2\varepsilon h^2 = O(0)$. The point is now to give all the orbital elements as PN series in powers of ε with coefficients depending on j (and the dimensionless reduced mass ratio ν). In ADM coordinates these are given by [37]

$$\begin{aligned} n^{\text{ADM}} = & \frac{\varepsilon^{3/2} c^3}{Gm} \left\{ 1 + \frac{\varepsilon}{8} (-15 + \nu) + \frac{\varepsilon^2}{128} \left[555 + 30\nu + 11\nu^2 + \frac{192}{j^{1/2}} (-5 + 2\nu) \right] \right. \\ & \left. + \frac{\varepsilon^3}{3072} \left[-29385 - 4995\nu - 315\nu^2 + 135\nu^3 \right] \right\} \end{aligned}$$

$$- \frac{16}{j^{3/2}} \left(10080 + 123 \nu \pi^2 - 13952 \nu + 1440 \nu^2 \right) + \frac{5760}{j^{1/2}} (17 - 9 \nu + 2 \nu^2) \Big] \Big\}, \quad (7.7a)$$

$$K^{\text{ADM}} = 1 + \frac{3\varepsilon}{j} + \frac{\varepsilon^2}{4} \left[\frac{3}{j} (-5 + 2 \nu) + \frac{15}{j^2} (7 - 2 \nu) \right] + \frac{\varepsilon^3}{128} \left[\frac{24}{j} (5 - 5 \nu + 4 \nu^2) \right. \\ \left. - \frac{1}{j^2} (10080 - 13952 \nu + 123 \nu \pi^2 + 1440 \nu^2) \right. \\ \left. + \frac{5}{j^3} (7392 - 8000 \nu + 123 \nu \pi^2 + 336 \nu^2) \right], \quad (7.7b)$$

$$a_r^{\text{ADM}} = \frac{Gm}{\varepsilon c^2} \left\{ 1 + \frac{\varepsilon}{4} (-7 + \nu) + \frac{\varepsilon^2}{16} \left[1 + 10 \nu + \nu^2 + \frac{1}{j} (-68 + 44 \nu) \right] \right. \\ \left. + \frac{\varepsilon^3}{192} \left[3 - 9 \nu - 6 \nu^2 + 3 \nu^3 + \frac{1}{j} (864 + (-3 \pi^2 - 2212) \nu + 432 \nu^2) \right. \right. \\ \left. \left. + \frac{1}{j^2} (-6432 + (13488 - 240 \pi^2) \nu - 768 \nu^2) \right] \right\}, \quad (7.7c)$$

$$e_r^{\text{ADM}} = \left[1 - j + \frac{\varepsilon}{4} \left\{ 24 - 4 \nu + 5 j (-3 + \nu) \right\} \right. \\ \left. + \frac{\varepsilon^2}{8} \left\{ 52 + 2 \nu + 2 \nu^2 - j (80 - 55 \nu + 4 \nu^2) - \frac{8}{j} (-17 + 11 \nu) \right\} \right. \\ \left. + \frac{\varepsilon^3}{192} \left\{ -768 - 6 \nu \pi^2 - 344 \nu - 216 \nu^2 + 3 j (-1488 + 1556 \nu - 319 \nu^2 + 4 \nu^3) \right. \right. \\ \left. \left. - \frac{4}{j} (588 - 8212 \nu + 177 \nu \pi^2 + 480 \nu^2) + \frac{192}{j^2} (134 - 281 \nu + 5 \nu \pi^2 + 16 \nu^2) \right\} \right]^{1/2}, \quad (7.7d)$$

$$e_t^{\text{ADM}} = \left[1 - j + \frac{\varepsilon}{4} \left\{ -8 + 8 \nu - j (-17 + 7 \nu) \right\} \right. \\ \left. + \frac{\varepsilon^2}{8} \left\{ 8 + 4 \nu + 20 \nu^2 - j (112 - 47 \nu + 16 \nu^2) - 24 j^{1/2} (-5 + 2 \nu) \right. \right. \\ \left. \left. + \frac{4}{j} (17 - 11 \nu) - \frac{24}{j^{1/2}} (5 - 2 \nu) \right\} \right. \\ \left. + \frac{\varepsilon^3}{192} \left\{ 24 (-2 + 5 \nu) (-23 + 10 \nu + 4 \nu^2) - 15 j (-528 + 200 \nu - 77 \nu^2 + 24 \nu^3) \right. \right. \\ \left. \left. - 72 j^{1/2} (265 - 193 \nu + 46 \nu^2) - \frac{2}{j} (6732 + 117 \nu \pi^2 - 12508 \nu + 2004 \nu^2) \right. \right. \\ \left. \left. + \frac{2}{j^{1/2}} (16380 - 19964 \nu + 123 \nu \pi^2 + 3240 \nu^2) \right. \right. \\ \left. \left. - \frac{2}{j^{3/2}} (10080 + 123 \nu \pi^2 - 13952 \nu + 1440 \nu^2) + \frac{96}{j^2} (134 - 281 \nu + 5 \nu \pi^2 + 16 \nu^2) \right\} \right]^{1/2}, \quad (7.7e)$$

$$e_\phi^{\text{ADM}} = \left[1 - j + \frac{\varepsilon}{4} \left\{ 24 + j (-15 + \nu) \right\} \right]$$

$$\begin{aligned}
& + \frac{\varepsilon^2}{16} \left\{ -32 + 176\nu + 18\nu^2 - j(160 - 30\nu + 3\nu^2) + \frac{1}{j}(408 - 232\nu - 15\nu^2) \right\} \\
& + \frac{\varepsilon^3}{384} \left\{ -16032 + 2764\nu + 3\nu\pi^2 + 4536\nu^2 + 234\nu^3 \right. \\
& - 36j(248 - 80\nu + 13\nu^2 + \nu^3) - \frac{6}{j}(2456 - 26860\nu + 581\nu\pi^2 + 2689\nu^2 + 10\nu^3) \\
& \left. + \frac{3}{j^2}(27776 - 65436\nu + 1325\nu\pi^2 + 3440\nu^2 - 70\nu^3) \right\}^{1/2}, \tag{7.7f}
\end{aligned}$$

$$\begin{aligned}
f_t^{\text{ADM}} &= -\frac{\varepsilon^2}{8j^{1/2}} \left\{ (4 + \nu)\nu \sqrt{1-j} \right\} \\
& + \frac{\varepsilon^3}{192} \left\{ \frac{1}{j^{3/2}\sqrt{1-j}} (1728 - 4148\nu + 3\nu\pi^2 + 600\nu^2 + 33\nu^3) \right. \\
& + 3 \frac{j^{1/2}}{\sqrt{1-j}} \nu (-64 - 4\nu + 23\nu^2) \\
& \left. + \frac{1}{\sqrt{j(1-j)}} (-1728 + 4232\nu - 3\nu\pi^2 - 627\nu^2 - 105\nu^3) \right\}, \tag{7.7g}
\end{aligned}$$

$$\begin{aligned}
g_t^{\text{ADM}} &= \frac{3\varepsilon^2}{2} \left(\frac{5-2\nu}{j^{1/2}} \right) \\
& + \frac{\varepsilon^3}{192} \left\{ \frac{1}{j^{3/2}} (10080 + 123\nu\pi^2 - 13952\nu + 1440\nu^2) + \frac{1}{j^{1/2}} (-3420 + 1980\nu - 648\nu^2) \right\}, \tag{7.7h}
\end{aligned}$$

$$i_t^{\text{ADM}} = \frac{\varepsilon^3}{32} \nu \frac{1-j}{j^{3/2}} (23 + 12\nu + 6\nu^2), \tag{7.7i}$$

$$h_t^{\text{ADM}} = \frac{13\varepsilon^3}{192} \nu^3 \left(\frac{1-j}{j} \right)^{3/2}, \tag{7.7j}$$

$$\begin{aligned}
f_\phi^{\text{ADM}} &= \frac{\varepsilon^2}{8} \frac{1-j}{j^2} \nu (1-3\nu) \\
& + \frac{\varepsilon^3}{256} \left\{ \frac{4\nu}{j} (-11 - 40\nu + 24\nu^2) \right. \\
& + \frac{1}{j^2} (-256 + 1192\nu - 49\nu\pi^2 + 336\nu^2 - 80\nu^3) \\
& \left. + \frac{1}{j^3} (256 + 49\nu\pi^2 - 1076\nu - 384\nu^2 - 40\nu^3) \right\}, \tag{7.7k}
\end{aligned}$$

$$\begin{aligned}
g_\phi^{\text{ADM}} &= -\frac{3\varepsilon^2 \nu^2}{32 j^2} (1-j)^{3/2} \\
& + \frac{\varepsilon^3}{768} \sqrt{1-j} \left\{ -\frac{3}{j} \nu^2 (9 - 26\nu) - \frac{1}{j^2} \nu (220 + 3\pi^2 + 312\nu + 150\nu^2) \right. \\
& \left. + \frac{1}{j^3} \nu (220 + 3\pi^2 + 96\nu + 45\nu^2) \right\}, \tag{7.7l}
\end{aligned}$$

$$i_\phi^{\text{ADM}} = \frac{\varepsilon^3}{128} \frac{(1-j)^2}{j^3} \nu (5 + 28\nu + 10\nu^2), \tag{7.7m}$$

$$h_{\phi}^{\text{ADM}} = \frac{5 \varepsilon^3 v^3}{256 j^3} (1 - j)^{5/2}. \quad (7.7n)$$

The latter expressions are specific to the ADM coordinates and we want to give now the corresponding expressions in MH coordinates. However we recall first an important point related to the use of gauge invariant variables in the elliptical orbit case as stressed by Ref. [37]. Indeed Damour and Schäfer [21] showed that the functional form of n and $K = \Phi/(2\pi)$ as functions of gauge invariant variables like ε and j are identical in different coordinate systems. Hence the expressions in MH coordinates of these two parameters are the same as in ADM coordinates,

$$n \equiv n^{\text{MH}} = n^{\text{ADM}}, \quad (7.8a)$$

$$K \equiv K^{\text{MH}} = K^{\text{ADM}}. \quad (7.8b)$$

This prompted Ref. [37] to suggest the use of n and $k = K - 1$ as two gauge invariant variables in the general orbit case.⁹ In the present work we propose to use a variant of the former variables. Namely, instead of working with the mean motion n we shall systematically use the orbital frequency $\omega = K n$ as defined in a general context in Sect. II.A of Paper I, and define as a gauge invariant post-Newtonian parameter

$$x = \left(\frac{G m \omega}{c^3} \right)^{2/3}. \quad (7.9)$$

This choice constitutes the obvious generalization of the gauge invariant variable x used in the circular orbit case and will thus facilitate the straightforward reading out and check of the circular orbit limit. The parameter x is related to the energy and angular momentum variables ε and j up to 3PN order by

$$\begin{aligned} x = \varepsilon \left\{ 1 + \varepsilon \left(-\frac{5}{4} + \frac{1}{12}v + 2\frac{1}{j} \right) \right. \\ + \varepsilon^2 \left(\frac{5}{2} + \frac{5}{24}v + \frac{1}{18}v^2 + \frac{1}{j^{1/2}}(-5 + 2v) + \frac{1}{j} \left(-5 + \frac{7}{6}v \right) + \frac{1}{j^2} \left(\frac{33}{2} - 5v \right) \right) \\ + \varepsilon^3 \left(-\frac{235}{48} - \frac{25}{24}v - \frac{25}{576}v^2 + \frac{35}{1296}v^3 + \frac{1}{j} \left(\frac{35}{4} - \frac{5}{3}v + \frac{25}{36}v^2 \right) \right. \\ + \frac{1}{j^{1/2}} \left(\frac{145}{8} - \frac{235}{24}v + \frac{29}{12}v^2 \right) + \frac{1}{j^{3/2}} \left(-45 + \left(\frac{472}{9} - \frac{41}{96}\pi^2 \right) v - 5v^2 \right) \\ \left. \left. + \frac{1}{j^2} \left(-\frac{565}{8} + \left(\frac{1903}{24} - \frac{41}{64}\pi^2 \right) v - \frac{95}{12}v^2 \right) + \frac{1}{j^3} \left(\frac{529}{3} + \left(-\frac{610}{3} + \frac{205}{64}\pi^2 \right) v + \frac{35}{4}v^2 \right) \right) \right\}. \quad (7.10) \end{aligned}$$

The other orbital elements are not gauge invariant and therefore their expressions in MH coordinates differ at 2PN and 3PN orders from those in ADM coordinates. We conclude by giving here

⁹ Actually Ref. [37] used $x_{\text{MGS}} = (G m n/c^3)^{2/3}$ together with $k' = k/3$.

all the needed differences [37],

$$\begin{aligned}
a_r^{\text{MH}} - a_r^{\text{ADM}} = & G m \varepsilon \left(-\frac{5}{8} \nu + \frac{1}{j} \left(\frac{1}{4} + \frac{17}{4} \nu \right) \right) \\
& + G m \varepsilon^2 \left(\frac{1}{32} \nu + \frac{1}{32} \nu^2 + \frac{1}{j} \left(-\frac{1}{2} + \left(-\frac{11499}{560} + \frac{21}{32} \pi^2 \right) \nu + \frac{19}{4} \nu^2 \right) \right. \\
& \left. + \frac{1}{j^2} \left(\frac{3}{2} + \left(\frac{14501}{420} - \frac{21}{16} \pi^2 \right) \nu - 5 \nu^2 \right) \right), \tag{7.11a}
\end{aligned}$$

$$\begin{aligned}
e_r^{\text{MH}} - e_r^{\text{ADM}} = & \frac{\varepsilon^2}{\sqrt{1-j}} \left(\frac{1}{2} + \frac{73}{8} \nu - j \frac{5}{8} \nu + \frac{1}{j} \left(-\frac{1}{2} - \frac{17}{2} \nu \right) \right) \\
& + \frac{\varepsilon^3}{\sqrt{1-j}} \left(\frac{13}{16} + \left(-\frac{5237}{1680} + \frac{21}{32} \pi^2 \right) \nu + \frac{19}{16} \nu^2 + j \left(-\frac{143}{64} \nu + \frac{37}{64} \nu^2 \right) \right. \\
& \left. + \frac{1}{j} \left(\frac{13}{8} + \left(\frac{3667}{56} - \frac{105}{32} \pi^2 \right) \nu - \frac{51}{4} \nu^2 \right) + \frac{1}{j^2} \left(-3 + \left(-\frac{14501}{210} + \frac{21}{8} \pi^2 \right) \nu + 10 \nu^2 \right) \right), \tag{7.11b}
\end{aligned}$$

$$\begin{aligned}
e_t^{\text{MH}} - e_t^{\text{ADM}} = & \frac{\varepsilon^2}{\sqrt{1-j}} \left(\frac{1}{4} + \frac{17}{4} \nu \right) \left(1 - \frac{1}{j} \right) + \\
& \frac{\varepsilon^3}{\sqrt{1-j}} \left(-\frac{19}{32} - \frac{52}{3} \nu + \frac{225}{32} \nu^2 + \frac{1}{j} \left(\frac{29}{16} + \left(\frac{79039}{1680} - \frac{21}{16} \pi^2 \right) \nu - \frac{201}{16} \nu^2 \right) \right. \\
& \left. + \frac{1}{j^2} \left(-\frac{3}{2} + \left(-\frac{14501}{420} + \frac{21}{16} \pi^2 \right) \nu + 5 \nu^2 \right) \right), \tag{7.11c}
\end{aligned}$$

$$\begin{aligned}
e_\phi^{\text{MH}} - e_\phi^{\text{ADM}} = & \frac{\varepsilon^2}{\sqrt{1-j}} \left(-\frac{1}{4} - \frac{71}{16} \nu + j \frac{1}{32} \nu + \frac{1}{j} \left(\frac{1}{4} + \frac{141}{32} \nu \right) \right) \\
& + \frac{\varepsilon^3}{\sqrt{1-j}} \left(-\frac{13}{32} + \left(\frac{36511}{8960} - \frac{21}{128} \pi^2 \right) \nu - \frac{1723}{256} \nu^2 + j \left(\frac{17}{256} \nu + \frac{33}{256} \nu^2 \right) \right. \\
& \left. + \frac{1}{j} \left(-\frac{13}{16} + \left(-\frac{21817}{480} + \frac{147}{64} \pi^2 \right) \nu + \frac{169}{8} \nu^2 \right) + \frac{1}{j^2} \left(\frac{3}{2} + \left(\frac{621787}{13440} - \frac{273}{128} \pi^2 \right) \nu - \frac{1789}{128} \nu^2 \right) \right), \tag{7.11d}
\end{aligned}$$

$$\begin{aligned}
f_t^{\text{MH}} - f_t^{\text{ADM}} = & \varepsilon^2 \frac{19}{8} \frac{\sqrt{(1-j)}}{\sqrt{j}} \nu + \frac{\varepsilon^3}{\sqrt{j(1-j)}} \left(-1 + \left(-\frac{296083}{6720} + \frac{21}{32} \pi^2 \right) \nu + \frac{989}{64} \nu^2 \right. \\
& \left. + j \left(\frac{361}{64} \nu - \frac{171}{64} \nu^2 \right) + \frac{1}{j} \left(1 + \left(\frac{276133}{6720} - \frac{21}{32} \pi^2 \right) \nu - \frac{799}{64} \nu^2 \right) \right), \tag{7.11e}
\end{aligned}$$

$$g_t^{\text{MH}} - g_t^{\text{ADM}} = 0, \tag{7.11f}$$

$$h_t^{\text{MH}} - h_t^{\text{ADM}} = -\frac{\varepsilon^3}{192} (1-j)^{\frac{3}{2}} j^{-\frac{3}{2}} \nu (-23 + 73\nu), \tag{7.11g}$$

$$i_t^{\text{MH}} - i_t^{\text{ADM}} = -\frac{11}{32} \varepsilon^3 (1-j) j^{-\frac{3}{2}} \nu (-19 + 10\nu), \tag{7.11h}$$

$$f_\phi^{\text{MH}} - f_\phi^{\text{ADM}} = -\varepsilon^2 \left(\frac{1}{j} - \frac{1}{j^2} \right) \left(\frac{1}{8} + \frac{9}{4} \nu \right) + \frac{\varepsilon^3}{j} \left(\frac{1}{32} + \frac{1045}{192} \nu - \frac{99}{32} \nu^2 \right)$$

$$\begin{aligned}
& + \frac{1}{j} \left(-\frac{5}{4} + \left(-\frac{139633}{3360} + \frac{21}{16}\pi^2 \right) \nu + \frac{117}{8} \nu^2 \right) \\
& + \frac{1}{j^2} \left(\frac{3}{2} + \left(\frac{92307}{2240} - \frac{21}{16}\pi^2 \right) \nu - \frac{351}{32} \nu^2 \right) \Big), \tag{7.11i}
\end{aligned}$$

$$\begin{aligned}
g_\phi^{\text{MH}} - g_\phi^{\text{ADM}} = & \varepsilon^2 \frac{1}{32} \frac{(1-j)^{3/2}}{j^2} \nu + \varepsilon^3 \sqrt{1-j} \left(\frac{1}{j} \left(\frac{7}{128} \nu - \frac{5}{32} \nu^2 \right) + \right. \\
& \left. \frac{1}{j^2} \left(\left(-\frac{49709}{13440} + \frac{21}{128}\pi^2 \right) \nu + \frac{445}{128} \nu^2 \right) + \frac{1}{j^3} \left(\left(\frac{100783}{26880} - \frac{21}{128}\pi^2 \right) \nu - \frac{847}{256} \nu^2 \right) \right), \tag{7.11j}
\end{aligned}$$

$$h_\phi^{\text{MH}} - h_\phi^{\text{ADM}} = -\frac{\varepsilon^3}{256} (1-j)^{\frac{5}{2}} j^{-3} \nu (-1+5\nu), \tag{7.11k}$$

$$i_\phi^{\text{MH}} - i_\phi^{\text{ADM}} = -\frac{\varepsilon^3}{384} (-1+j)^2 j^{-3} \nu (-149+198\nu). \tag{7.11l}$$

Finally we note that in the case of a circular orbit the angular momentum variable, say j_\circ , is related to the constant of energy ε by the 3PN gauge-invariant expansion

$$j_\circ = 1 + \frac{\varepsilon}{4} (9 + \nu) + \frac{\varepsilon^2}{16} (81 - 32\nu + \nu^2) + \frac{\varepsilon^3}{192} (2835 - 7699\nu + 246\nu\pi^2 + 96\nu^2 + 3\nu^3), \tag{7.12}$$

which is easily deduced using either MH or ADM coordinates. This expression can be used to compute all the orbital elements for circular orbits and we can check that all of the eccentricities e_r , e_t or e_ϕ are zero.

VIII. ORBITAL AVERAGE OF THE 3PN ENERGY FLUX

To average the energy flux over an orbit we will require the use of the previous 3PN quasi-Keplerian representation of the motion. Consequently, the averaging is only possible in MH or ADM coordinates without the logarithms as discussed before. The average of the (instantaneous part of the) energy flux is defined by

$$\langle \mathcal{F}_{\text{inst}} \rangle = \frac{1}{P} \int_0^P dt \mathcal{F}_{\text{inst}} = \frac{1}{2\pi} \int_0^{2\pi} du \frac{d\ell}{du} \mathcal{F}_{\text{inst}}. \tag{8.1}$$

As we have seen the energy flux (2.6) is made of instantaneous terms and hereditary (tail) terms. The hereditary terms have already been computed and averaged in Paper I.

Using the QK representation of the orbit discussed in Sect. VII, we can re-express the energy flux $\mathcal{F}_{\text{inst}}$ [or, more exactly, $(d\ell/du) \mathcal{F}_{\text{inst}}$], which is a function of its natural variables r , \dot{r} and v^2 , as a function of the frequency-related parameter x defined by Eqs. (7.9)–(7.10), the “time” eccentricity e_t and the eccentric anomaly u .¹⁰ We note that in the expression of the energy flux at the 3PN order there are some logarithmic terms of the type $\ln(r/r_0)$ even in MH coordinates. Indeed, we

¹⁰ Ref. [19] uses Gm/a_r and e_r while [25] employs Gmn/c^3 and e_t . We propose the use of $x = (Gm\omega/c^3)^{2/3}$ for reasons outlined in the previous Section. The choice of e_t rather than say e_r is a matter of convenience since it appears in the Kepler equation which is directly dealt with when averaging over an orbit.

recall that the MH coordinates permit the removal of the log-terms $\ln(r/r'_0)$, where r'_0 is the scale associated with Hadamard's self-field regularization, but there are still the terms $\ln(r/r_0)$ which involve the constant r_0 entering the definition of the multipole moments for general sources. As a result we find that the general structure of $\mathcal{F}_{\text{inst}}$ (in MH or ADM coordinates) is

$$\frac{d\ell}{du} \mathcal{F}_{\text{inst}} = \sum_{N=3}^{11} \left\{ \alpha_N(e_t) \frac{1}{(1 - e_t \cos u)^N} + \beta_N(e_t) \frac{\sin u}{(1 - e_t \cos u)^N} + \gamma_N(e_t) \frac{\ln(1 - e_t \cos u)}{(1 - e_t \cos u)^N} \right\}, \quad (8.2)$$

where the coefficients α_N , β_N , γ_N so defined are straightforwardly computed using the QK parametrization (they are too long to be listed here). It is worth noting that the β_N 's correspond to all the 2.5PN terms while the γ_N 's represent the logarithmic terms at order 3PN. The dependence on the constant $\ln r_0$ has been included into the coefficients α_N 's. To compute the average we have at our disposal some integration formulas. First of all,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\sin u}{(1 - e \cos u)^N} du = 0, \quad (8.3)$$

which shows that in the final result there will be no terms (of the ‘‘instantaneous’’ type) at 2.5PN order. The 2.5PN instantaneous contribution is proportional to \dot{r} and vanishes after averaging since it includes only odd functions of u . Next, we have

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{(1 - e \cos u)^N} = \frac{(-)^{N-1}}{(N-1)!} \left(\frac{d^{N-1}}{dy^{N-1}} \left[\frac{1}{\sqrt{y^2 - e^2}} \right] \right)_{y=1}, \quad (8.4)$$

which can also be formulated with the help of the standard Legendre polynomial P_{N-1} as

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{(1 - e \cos u)^N} = \frac{1}{(1 - e^2)^{N/2}} P_{N-1} \left(\frac{1}{\sqrt{1 - e^2}} \right). \quad (8.5)$$

Finally for the log-terms we have a less trivial formula but which takes a structure similar as in Eq. (8.4), namely

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\ln(1 - e \cos u)}{(1 - e \cos u)^N} du = \frac{(-)^{N-1}}{(N-1)!} \left(\frac{d^{N-1} Y(y, e)}{dy^{N-1}} \right)_{y=1}, \quad (8.6)$$

in which

$$Y(y, e) = \frac{1}{\sqrt{y^2 - e^2}} \left\{ \ln \left[\frac{\sqrt{1 - e^2} + 1}{2} \right] + 2 \ln \left[1 + \frac{\sqrt{1 - e^2} - 1}{y + \sqrt{y^2 - e^2}} \right] \right\}. \quad (8.7)$$

A. Orbital average in MH coordinates

The expression for the instantaneous energy flux in MH coordinates is given by Eqs. (5.1)–(5.2) together with the modification (6.8) for transforming to MH coordinates. Implementing all the above integrations, the flux can be averaged over an orbit to order 3PN extending the results

of [19] at 2PN¹¹. The result is presented in the form

$$\langle \mathcal{F}_{\text{inst}} \rangle = \frac{32c^5}{5G} v^2 x^5 \left(\mathcal{I}_{\text{N}}^{\text{MH}} + x \mathcal{I}_{\text{1PN}}^{\text{MH}} + x^2 \mathcal{I}_{\text{2PN}}^{\text{MH}} + x^3 \mathcal{I}_{\text{3PN}}^{\text{MH}} \right), \quad (8.8)$$

where the ‘‘instantaneous’’ post-Newtonian pieces $\mathcal{I}_{n\text{PN}}^{\text{MH}}$ depend on v and the time eccentricity e_t in MH coordinates (note that $e_t \equiv e_t^{\text{MH}}$ here), and read¹²

$$\mathcal{I}_{\text{N}}^{\text{MH}} = \frac{1}{(1 - e_t^2)^{7/2}} \left\{ 1 + \frac{73}{24} e_t^2 + \frac{37}{96} e_t^4 \right\}, \quad (8.9a)$$

$$\begin{aligned} \mathcal{I}_{\text{1PN}}^{\text{MH}} = & \frac{1}{(1 - e_t^2)^{9/2}} \left\{ -\frac{1247}{336} - \frac{35}{12} v + e_t^2 \left(\frac{10475}{672} - \frac{1081}{36} v \right) \right. \\ & \left. + e_t^4 \left(\frac{10043}{384} - \frac{311}{12} v \right) + e_t^6 \left(\frac{2179}{1792} - \frac{851}{576} v \right) \right\}, \end{aligned} \quad (8.9b)$$

$$\begin{aligned} \mathcal{I}_{\text{2PN}}^{\text{MH}} = & \frac{1}{(1 - e_t^2)^{11/2}} \left\{ -\frac{203471}{9072} + \frac{12799}{504} v + \frac{65}{18} v^2 \right. \\ & + e_t^2 \left(-\frac{3807197}{18144} + \frac{116789}{2016} v + \frac{5935}{54} v^2 \right) \\ & + e_t^4 \left(-\frac{268447}{24192} - \frac{2465027}{8064} v + \frac{247805}{864} v^2 \right) \\ & + e_t^6 \left(\frac{1307105}{16128} - \frac{416945}{2688} v + \frac{185305}{1728} v^2 \right) \\ & + e_t^8 \left(\frac{86567}{64512} - \frac{9769}{4608} v + \frac{21275}{6912} v^2 \right) \\ & + \sqrt{1 - e_t^2} \left[\frac{35}{2} - 7v + e_t^2 \left(\frac{6425}{48} - \frac{1285}{24} v \right) \right. \\ & \left. + e_t^4 \left(\frac{5065}{64} - \frac{1013}{32} v \right) + e_t^6 \left(\frac{185}{96} - \frac{37}{48} v \right) \right] \right\}, \end{aligned} \quad (8.9c)$$

$$\begin{aligned} \mathcal{I}_{\text{3PN}}^{\text{MH}} = & \frac{1}{(1 - e_t^2)^{13/2}} \left\{ \frac{2193295679}{9979200} + \left[\frac{8009293}{54432} - \frac{41\pi^2}{64} \right] v - \frac{209063}{3024} v^2 - \frac{775}{324} v^3 \right. \\ & + e_t^2 \left(\frac{20506331429}{19958400} + \left[\frac{649801883}{272160} + \frac{4879\pi^2}{1536} \right] v - \frac{3008759}{3024} v^2 - \frac{53696}{243} v^3 \right) \\ & + e_t^4 \left(-\frac{3611354071}{13305600} + \left[\frac{755536297}{136080} - \frac{29971\pi^2}{1024} \right] v - \frac{179375}{576} v^2 - \frac{10816087}{7776} v^3 \right) \\ & \left. + e_t^6 \left(\frac{4786812253}{26611200} + \left[\frac{1108811471}{1451520} - \frac{84501\pi^2}{4096} \right] v + \frac{87787969}{48384} v^2 - \frac{983251}{648} v^3 \right) \right\} \end{aligned}$$

¹¹ Results of [19] are given in ADM coordinates.

¹² The Newtonian coefficient $\mathcal{I}_{\text{N}}^{\text{MH}}$ is nothing but the Peters & Mathews [1] ‘‘enhancement’’ function of eccentricity $f(e_t) \equiv (1 + \frac{73}{24}e_t^2 + \frac{37}{96}e_t^4)/(1 - e_t^2)^{7/2}$, called that way because it enhances the numerical value of the orbital decay of the binary pulsar by gravitational radiation (*viz* the orbital \dot{P}).

$$\begin{aligned}
& + e_i^8 \left(\frac{21505140101}{141926400} + \left[-\frac{32467919}{129024} - \frac{4059\pi^2}{4096} \right] v + \frac{79938097}{193536} v^2 - \frac{4586539}{15552} v^3 \right) \\
& + e_i^{10} \left(-\frac{8977637}{11354112} + \frac{9287}{48384} v + \frac{8977}{55296} v^2 - \frac{567617}{124416} v^3 \right) \\
& + \sqrt{1 - e_i^2} \left[-\frac{14047483}{151200} + \left[-\frac{165761}{1008} + \frac{287\pi^2}{192} \right] v + \frac{455}{12} v^2 \right. \\
& + e_i^2 \left(\frac{36863231}{100800} + \left[-\frac{14935421}{6048} + \frac{52685\pi^2}{4608} \right] v + \frac{43559}{72} v^2 \right) \\
& + e_i^4 \left(\frac{759524951}{403200} + \left[-\frac{31082483}{8064} + \frac{41533\pi^2}{6144} \right] v + \frac{303985}{288} v^2 \right) \\
& + e_i^6 \left(\frac{1399661203}{2419200} + \left[-\frac{40922933}{48384} + \frac{1517\pi^2}{9216} \right] v + \frac{73357}{288} v^2 \right) \\
& + e_i^8 \left(\frac{185}{48} - \frac{1073}{288} v + \frac{407}{288} v^2 \right) \left. \right] \\
& + \left(\frac{1712}{105} + \frac{14552}{63} e_i^2 + \frac{553297}{1260} e_i^4 + \frac{187357}{1260} e_i^6 + \frac{10593}{2240} e_i^8 \right) \ln \left[\frac{x}{x_0} \frac{1 + \sqrt{1 - e_i^2}}{2(1 - e_i^2)} \right] \Bigg\}. \quad (8.9d)
\end{aligned}$$

For ease of presentation we have not put a label on e_i to indicate that it is the time eccentricity in MH coordinates. Of course, since x is gauge invariant, no such label is required on it. It is important to keep track of this fact when comparing formulas in different gauges, as we will eventually do.

The last term in the 3PN coefficient $\mathcal{I}_{3\text{PN}}^{\text{MH}}$ given by Eq. (8.9d) is proportional to some logarithm which directly arises from the integration formula (8.6)–(8.7). Inside the logarithm we posed

$$x_0 \equiv \frac{Gm}{c^2 r_0}, \quad (8.10)$$

exhibiting the dependence of the instantaneous part of the 3PN energy flux upon the arbitrary constant length scale r_0 . Only after computing the complete energy flux can one discuss the structure of the logarithmic term in the energy flux and the required cancellation of the $\ln r_0$. Therefore we now add the hereditary contribution to the 3PN flux which has been computed in Paper I. From Eq. (6.2) in Paper I we write the result as

$$\langle \mathcal{F}_{\text{hered}} \rangle = \frac{32c^5}{5G} v^2 x^5 \left(x^{3/2} \mathcal{H}_{1.5\text{PN}}^{\text{MH}} + x^{5/2} \mathcal{H}_{2.5\text{PN}}^{\text{MH}} + x^3 \mathcal{H}_{3\text{PN}}^{\text{MH}} \right), \quad (8.11)$$

where the ‘‘hereditary’’ post-Newtonian coefficients (starting at 1.5PN order) read

$$\mathcal{H}_{1.5\text{PN}}^{\text{MH}} = 4\pi \varphi(e_i), \quad (8.12a)$$

$$\mathcal{H}_{2.5\text{PN}}^{\text{MH}} = -\frac{8191}{672} \pi \psi(e_i) - \frac{583}{24} v \pi \zeta(e_i), \quad (8.12b)$$

$$\mathcal{H}_{3\text{PN}}^{\text{MH}} = -\frac{116761}{3675} \kappa(e_i) + \left[\frac{16}{3} \pi^2 - \frac{1712}{105} C - \frac{1712}{105} \ln \left(\frac{4x^{3/2}}{x_0} \right) \right] F(e_i). \quad (8.12c)$$

The function $F(e_i)$ in factor of the logarithm in the 3PN coefficient does admit a closed analytic

form which was determined in paper I as

$$F(e_t) = \frac{1}{(1 - e_t^2)^{13/2}} \left[1 + \frac{85}{6}e_t^2 + \frac{5171}{192}e_t^4 + \frac{1751}{192}e_t^6 + \frac{297}{1024}e_t^8 \right]. \quad (8.13)$$

On the other hand paper I found that the four ‘‘enhancement’’ functions of eccentricity $\varphi(e_t)$, $\psi(e_t)$, $\zeta(e_t)$ and $\kappa(e_t)$ very likely do not admit any analytic closed-form expressions. Numerical plots of the four enhancement factors $\varphi(e_t)$, $\psi(e_t)$, $\theta(e_t)$ and $\kappa(e_t)$ as functions of eccentricity e_t have been presented in Paper I. The coefficients in Eqs. (8.12) have been introduced in such a way that the circular orbit limit of all the functions $F(e_t)$ and $\varphi(e_t), \dots, \kappa(e_t)$ is one.

Finally, the PN coefficients in the total averaged energy flux \mathcal{F} in MH coordinates are given by the sum of the instantaneous and hereditary contributions, say

$$\mathcal{K}_{n\text{PN}}^{\text{MH}} = \mathcal{I}_{n\text{PN}}^{\text{MH}} + \mathcal{H}_{n\text{PN}}^{\text{MH}}. \quad (8.14)$$

We notice that up to 2.5PN order there is a clean separation between the instantaneous terms which are at even PN orders (recall that there is no 2.5PN term in the averaged flux), and the hereditary terms which appear at odd PN orders and are specifically due to tails (*i.e.* $\mathcal{H}_{1.5\text{PN}}^{\text{MH}}$ and $\mathcal{H}_{2.5\text{PN}}^{\text{MH}}$). On the contrary, at 3PN order – and, indeed, at any higher PN order – there is a mixture of instantaneous and hereditary terms. The 3PN hereditary term $\mathcal{H}_{3\text{PN}}^{\text{MH}}$ is due to the so-called GW tails of tails (see Paper I).

The analytical result (8.13) is crucial for checking that the arbitrary constant x_0 disappears from the final result, namely from the 3PN coefficient $\mathcal{K}_{3\text{PN}}^{\text{MH}}$. Indeed we immediately verify from comparing the last term in Eq. (8.9d) with Eq. (8.12c) and the explicit expression (8.13) of $F(e_t)$ that x_0 cancels out from the sum of the instantaneous and hereditary contributions, extending to non circular orbits this fact which was already observed for the circular case in Ref. [30]. Finally the complete 3PN coefficient (independent of x_0) reads

$$\begin{aligned} \mathcal{K}_{3\text{PN}}^{\text{MH}} = & \frac{1}{(1 - e_t^2)^{13/2}} \left\{ \frac{2193295679}{9979200} + \left[\frac{8009293}{54432} - \frac{41\pi^2}{64} \right] \nu - \frac{209063}{3024} \nu^2 - \frac{775}{324} \nu^3 \right. \\ & + e_t^2 \left(\frac{20506331429}{19958400} + \left[\frac{649801883}{272160} + \frac{4879\pi^2}{1536} \right] \nu - \frac{3008759}{3024} \nu^2 - \frac{53696}{243} \nu^3 \right) \\ & + e_t^4 \left(-\frac{3611354071}{13305600} + \left[\frac{755536297}{136080} - \frac{29971\pi^2}{1024} \right] \nu - \frac{179375}{576} \nu^2 - \frac{10816087}{7776} \nu^3 \right) \\ & + e_t^6 \left(\frac{4786812253}{26611200} + \left[\frac{1108811471}{1451520} - \frac{84501\pi^2}{4096} \right] \nu + \frac{87787969}{48384} \nu^2 - \frac{983251}{648} \nu^3 \right) \\ & + e_t^8 \left(\frac{21505140101}{141926400} + \left[-\frac{32467919}{129024} - \frac{4059\pi^2}{4096} \right] \nu + \frac{79938097}{193536} \nu^2 - \frac{4586539}{15552} \nu^3 \right) \\ & + e_t^{10} \left(-\frac{8977637}{11354112} + \frac{9287}{48384} \nu + \frac{8977}{55296} \nu^2 - \frac{567617}{124416} \nu^3 \right) \\ & + \sqrt{1 - e_t^2} \left[-\frac{14047483}{151200} + \left[-\frac{165761}{1008} + \frac{287\pi^2}{192} \right] \nu + \frac{455}{12} \nu^2 \right. \\ & \left. + e_t^2 \left(\frac{36863231}{100800} + \left[-\frac{14935421}{6048} + \frac{52685\pi^2}{4608} \right] \nu + \frac{43559}{72} \nu^2 \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + e_t^4 \left(\frac{759524951}{403200} + \left[-\frac{31082483}{8064} + \frac{41533\pi^2}{6144} \right] v + \frac{303985}{288} v^2 \right) \\
& + e_t^6 \left(\frac{1399661203}{2419200} + \left[-\frac{40922933}{48384} + \frac{1517\pi^2}{9216} \right] v + \frac{73357}{288} v^2 \right) \\
& + e_t^8 \left(\frac{185}{48} - \frac{1073}{288} v + \frac{407}{288} v^2 \right) \\
& + \left(\frac{1712}{105} + \frac{14552}{63} e_t^2 + \frac{553297}{1260} e_t^4 + \frac{187357}{1260} e_t^6 + \frac{10593}{2240} e_t^8 \right) \\
& \times \left[-C + \frac{35}{107} \pi^2 - \frac{1}{2} \ln(16x) + \ln \left(\frac{1 + \sqrt{1 - e_t^2}}{2(1 - e_t^2)} \right) \right] \Bigg\} - \frac{116761}{3675} \kappa(e_t). \quad (8.15)
\end{aligned}$$

The 1.5PN and 2.5PN coefficients are only due to tails, thus

$$\mathcal{K}_{1.5N}^{\text{MH}} = 4\pi \varphi(e_t), \quad (8.16a)$$

$$\mathcal{K}_{2.5PN}^{\text{MH}} = -\frac{8191}{672} \pi \psi(e_t) - \frac{583}{24} v \pi \zeta(e_t). \quad (8.16b)$$

The Newtonian, 1PN and 2PN coefficients reduce to their instantaneous contributions $\mathcal{I}_N^{\text{MH}}$, $\mathcal{I}_{1PN}^{\text{MH}}$ and $\mathcal{I}_{2PN}^{\text{MH}}$ already given in Eqs. (8.9).

Since the enhancement functions $\varphi(e_t)$, $\psi(e_t)$, $\zeta(e_t)$ and $\kappa(e_t)$ reduce to one in the circular case, when $e_t = 0$, the circular-orbit limit of the energy flux is immediately deduced from inspection of Eqs. (8.9) and (8.16) as

$$\begin{aligned}
\langle \mathcal{F} \rangle_{\odot} &= \frac{32c^5}{5G} x^5 v^2 \left\{ 1 + x \left(-\frac{1247}{336} - \frac{35}{12} v \right) + 4\pi x^{3/2} \right. \\
& + x^2 \left(-\frac{44711}{9072} + \frac{9271}{504} v + \frac{65}{18} v^2 \right) + \pi x^{5/2} \left(-\frac{8191}{672} - \frac{583}{24} v \right) \\
& + x^3 \left(\frac{6643739519}{69854400} - \frac{1712}{105} C + \frac{16}{3} \pi^2 - \frac{856}{105} \ln(16x) \right. \\
& \left. \left. + \left[-\frac{134543}{7776} + \frac{41}{48} \pi^2 \right] v - \frac{94403}{3024} v^2 - \frac{775}{324} v^3 \right) \right\}. \quad (8.17)
\end{aligned}$$

This limiting case is in exact agreement with Eq. (12.9) of [30] (after taking into account the values of the ambiguity parameters $\lambda = -\frac{1987}{3080}$ and $\theta = -\frac{11831}{9240}$ computed in Refs. [32, 33, 34]). Notice that the flux in the circular-orbit limit (8.17) depends only on the parameter x and hence its expression becomes gauge invariant.

B. Orbital average in ADM coordinates

We start from the expression for the instantaneous energy flux in ADM coordinates as given by (6.11), employ the appropriate 3PN QK representation and follow the procedure for performing the average as outlined in the previous Section. We find that the β_N 's and γ_N 's in ADM coordinates [cf. Eq. (8.2)] are exactly the same as in MH coordinates; the α_N 's, however, are different in general

(except for α_{11}). The result for the average energy flux in ADM coordinates is of the form

$$\langle \mathcal{F}_{\text{inst}} \rangle = \frac{32c^5}{5G} \nu^2 x^5 \left(\mathcal{I}_{\text{N}}^{\text{ADM}} + x \mathcal{I}_{\text{1PN}}^{\text{ADM}} + x^2 \mathcal{I}_{\text{2PN}}^{\text{ADM}} + x^3 \mathcal{I}_{\text{3PN}}^{\text{ADM}} \right), \quad (8.18)$$

where the coefficients depend on the time eccentricity in ADM coordinates (hence $e_t \equiv e_t^{\text{ADM}}$ here) and on ν , and read

$$\mathcal{I}_{\text{N}}^{\text{ADM}} = \frac{1}{(1 - e_t^2)^{7/2}} \left\{ 1 + \frac{73}{24} e_t^2 + \frac{37}{96} e_t^4 \right\}, \quad (8.19a)$$

$$\begin{aligned} \mathcal{I}_{\text{1PN}}^{\text{ADM}} = & \frac{1}{(1 - e_t^2)^{9/2}} \left\{ -\frac{1247}{336} - \frac{35}{12} \nu + e_t^2 \left(\frac{10475}{672} - \frac{1081}{36} \nu \right) \right. \\ & \left. + e_t^4 \left(\frac{10043}{384} - \frac{311}{12} \nu \right) + e_t^6 \left(\frac{2179}{1792} - \frac{851}{576} \nu \right) \right\}, \end{aligned} \quad (8.19b)$$

$$\begin{aligned} \mathcal{I}_{\text{2PN}}^{\text{ADM}} = & \frac{1}{(1 - e_t^2)^{11/2}} \left\{ -\frac{203471}{9072} + \frac{12799}{504} \nu + \frac{65}{18} \nu^2 \right. \\ & + e_t^2 \left(-\frac{3866543}{18144} + \frac{4691}{2016} \nu + \frac{5935}{54} \nu^2 \right) \\ & + e_t^4 \left(-\frac{369751}{24192} - \frac{3039083}{8064} \nu + \frac{247805}{864} \nu^2 \right) \\ & + e_t^6 \left(\frac{1302443}{16128} - \frac{215077}{1344} \nu + \frac{185305}{1728} \nu^2 \right) \\ & + e_t^8 \left(\frac{86567}{64512} - \frac{9769}{4608} \nu + \frac{21275}{6912} \nu^2 \right) \\ & + \sqrt{1 - e_t^2} \left[\frac{35}{2} - 7\nu + e_t^2 \left(\frac{6425}{48} - \frac{1285}{24} \nu \right) \right. \\ & \left. + e_t^4 \left(\frac{5065}{64} - \frac{1013}{32} \nu \right) + e_t^6 \left(\frac{185}{96} - \frac{37}{48} \nu \right) \right] \right\}, \end{aligned} \quad (8.19c)$$

$$\begin{aligned} \mathcal{I}_{\text{3PN}}^{\text{ADM}} = & \frac{1}{(1 - e_t^2)^{13/2}} \left\{ \frac{2193295679}{9979200} + \left[\frac{8009293}{54432} - \frac{41\pi^2}{64} \right] \nu - \frac{209063}{3024} \nu^2 - \frac{775}{324} \nu^3 \right. \\ & + e_t^2 \left(\frac{2912411147}{2851200} + \left[\frac{249108317}{108864} + \frac{31255}{1536} \pi^2 \right] \nu - \frac{3525469}{6048} \nu^2 - \frac{53696}{243} \nu^3 \right) \\ & + e_t^4 \left(-\frac{4520777971}{13305600} + \left[\frac{473750339}{108864} - \frac{7459\pi^2}{1024} \right] \nu + \frac{697997}{576} \nu^2 - \frac{10816087}{7776} \nu^3 \right) \\ & + e_t^6 \left(\frac{3630046753}{26611200} + \left[-\frac{8775247}{145152} - \frac{78285\pi^2}{4096} \right] \nu + \frac{31147213}{12096} \nu^2 - \frac{983251}{648} \nu^3 \right) \\ & + e_t^8 \left(\frac{21293656301}{141926400} + \left[-\frac{36646949}{129024} - \frac{4059\pi^2}{4096} \right] \nu + \frac{85830865}{193536} \nu^2 - \frac{4586539}{15552} \nu^3 \right) \\ & + e_t^{10} \left(-\frac{8977637}{11354112} + \frac{9287}{48384} \nu + \frac{8977}{55296} \nu^2 - \frac{567617}{124416} \nu^3 \right) \\ & + \sqrt{1 - e_t^2} \left[-\frac{14047483}{151200} + \left[-\frac{165761}{1008} + \frac{287\pi^2}{192} \right] \nu + \frac{455}{12} \nu^2 \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + e_t^2 \left(\frac{36863231}{100800} + \left[-\frac{14935421}{6048} + \frac{52685\pi^2}{4608} \right] \nu + \frac{43559}{72} \nu^2 \right) \\
& + e_t^4 \left(\frac{759524951}{403200} + \left[-\frac{31082483}{8064} + \frac{41533\pi^2}{6144} \right] \nu + \frac{303985}{288} \nu^2 \right) \\
& + e_t^6 \left(\frac{1399661203}{2419200} + \left[-\frac{40922933}{48384} + \frac{1517\pi^2}{9216} \right] \nu + \frac{73357}{288} \nu^2 \right) \\
& + e_t^8 \left(\frac{185}{48} - \frac{1073}{288} \nu + \frac{407}{288} \nu^2 \right) \\
& + \left(\frac{1712}{105} + \frac{14552}{63} e_t^2 + \frac{553297}{1260} e_t^4 + \frac{187357}{1260} e_t^6 + \frac{10593}{2240} e_t^8 \right) \ln \left[\frac{x}{x_0} \frac{1 + \sqrt{1 - e_t^2}}{2(1 - e_t^2)} \right] \Bigg\} .
\end{aligned} \tag{8.19d}$$

We recall that the Newtonian and 1PN orders are the same in MH and ADM coordinates (the coefficients $\mathcal{I}_N^{\text{ADM}}$ and $\mathcal{I}_{\text{1PN}}^{\text{ADM}}$ agree with their MH counterparts). On the other hand, adding up the hereditary contribution (8.11)–(8.12) [which is the same in MH and ADM coordinates], we obtain the total 3PN coefficient $\mathcal{K}_{\text{3PN}}^{\text{ADM}}$, analogous to Eq. (8.15) but in ADM coordinates,

$$\begin{aligned}
\mathcal{K}_{\text{3PN}}^{\text{ADM}} = & \frac{1}{(1 - e_t^2)^{13/2}} \left\{ \frac{2193295679}{9979200} + \left[\frac{8009293}{54432} - \frac{41\pi^2}{64} \right] \nu - \frac{209063}{3024} \nu^2 - \frac{775}{324} \nu^3 \right. \\
& + e_t^2 \left(\frac{2912411147}{2851200} + \left[\frac{249108317}{108864} + \frac{31255}{1536} \pi^2 \right] \nu - \frac{3525469}{6048} \nu^2 - \frac{53696}{243} \nu^3 \right) \\
& + e_t^4 \left(-\frac{4520777971}{13305600} + \left[\frac{473750339}{108864} - \frac{7459\pi^2}{1024} \right] \nu + \frac{697997}{576} \nu^2 - \frac{10816087}{7776} \nu^3 \right) \\
& + e_t^6 \left(\frac{3630046753}{26611200} + \left[-\frac{8775247}{145152} - \frac{78285\pi^2}{4096} \right] \nu + \frac{31147213}{12096} \nu^2 - \frac{983251}{648} \nu^3 \right) \\
& + e_t^8 \left(\frac{21293656301}{141926400} + \left[-\frac{36646949}{129024} - \frac{4059\pi^2}{4096} \right] \nu + \frac{85830865}{193536} \nu^2 - \frac{4586539}{15552} \nu^3 \right) \\
& + e_t^{10} \left(-\frac{8977637}{11354112} + \frac{9287}{48384} \nu + \frac{8977}{55296} \nu^2 - \frac{567617}{124416} \nu^3 \right) \\
& + \sqrt{1 - e_t^2} \left[-\frac{14047483}{151200} + \left[-\frac{165761}{1008} + \frac{287\pi^2}{192} \right] \nu + \frac{455}{12} \nu^2 \right. \\
& + e_t^2 \left(\frac{36863231}{100800} + \left[-\frac{14935421}{6048} + \frac{52685\pi^2}{4608} \right] \nu + \frac{43559}{72} \nu^2 \right) \\
& + e_t^4 \left(\frac{759524951}{403200} + \left[-\frac{31082483}{8064} + \frac{41533\pi^2}{6144} \right] \nu + \frac{303985}{288} \nu^2 \right) \\
& + e_t^6 \left(\frac{1399661203}{2419200} + \left[-\frac{40922933}{48384} + \frac{1517\pi^2}{9216} \right] \nu + \frac{73357}{288} \nu^2 \right) \\
& + e_t^8 \left(\frac{185}{48} - \frac{1073}{288} \nu + \frac{407}{288} \nu^2 \right) \\
& \left. + \left(\frac{1712}{105} + \frac{14552}{63} e_t^2 + \frac{553297}{1260} e_t^4 + \frac{187357}{1260} e_t^6 + \frac{10593}{2240} e_t^8 \right) \right\}
\end{aligned}$$

$$\times \left[-C + \frac{35}{107}\pi^2 - \frac{1}{2} \ln(16x) + \ln \left(\frac{1 + \sqrt{1 - e_t^2}}{2(1 - e_t^2)} \right) \right] - \frac{116761}{3675} \kappa(e_t), \quad (8.20)$$

in which again $e_t = e_t^{\text{ADM}}$. A useful internal consistency check of the algebraic correctness of different coordinate representations of the energy flux, is the verification that the equality of Eqs. (8.9) and (8.19) holds if and only if we have the transformation between the time eccentricities e_t^{MH} and e_t^{ADM} given by

$$\begin{aligned} \frac{e_t^{\text{MH}}}{e_t^{\text{ADM}}} = & 1 + \frac{x^2}{1 - e_t^2} \left(-\frac{1}{4} - \frac{17}{4} \nu \right) \\ & + \frac{x^3}{(1 - e_t^2)^2} \left(-\frac{1}{2} + \left[-\frac{16739}{1680} + \frac{21}{16}\pi^2 \right] \nu + \frac{83}{24} \nu^2 + e_t^2 \left(-\frac{1}{2} - \frac{249}{16} \nu + \frac{241}{24} \nu^2 \right) \right). \end{aligned} \quad (8.21)$$

(There is no ambiguity in not having a label on the e_t in the 2PN and 3PN terms above.) We find that the relation (8.21) is perfectly equivalent to what is predicted from using different QK representations of the motion, namely Eq. (7.11c) together with (7.10).

C. Gauge invariant formulation

In the previous section, the averaged energy flux was represented using x – a gauge invariant variable defined by (7.9) – and the eccentricity e_t which however is coordinate dependent (but is useful in extracting the circular limit of the result). In the present Section we provide a gauge invariant formulation of the energy flux.

Perhaps the most natural choice is to express the result in terms of the conserved energy E and angular momentum J (per unit of reduced mass), or, rather, in terms of the pair of rescaled variables (ε, j) defined by Eqs. (7.5) and (7.6). However there are other possible choices for a couple of gauge invariant quantities. As we have seen in Eqs. (7.8) the mean motion n and the periastron precession K are gauge invariant so we may define as our first choice the pair of variables (x, ι) , where we recall that x is related to the orbital frequency $\omega = K n$ by Eq. (7.9), and where we define

$$\iota \equiv \frac{3x}{k}, \quad (8.22)$$

with $k \equiv K - 1$. Here we have introduced a factor 3 so that ι reduces to j in first approximation (*i.e.* when $\varepsilon \rightarrow 0$). To 3PN order this parameter is related to the energy and angular momentum variables ε and j by

$$\begin{aligned} \iota = & j + \varepsilon \left\{ -\frac{27}{4} + \frac{5}{2} \nu - j \frac{5}{12} \nu \right\} \\ & + \varepsilon^2 \left\{ \frac{205}{16} + \left[-\frac{1201}{48} + \frac{41}{128} \pi^2 \right] \nu + \frac{35}{24} \nu^2 + j^{\frac{1}{2}} (-5 + 2\nu) + j \left(\frac{35}{16} + \frac{1}{72} \nu^2 \right) \right. \\ & \left. + \frac{1}{j} \left(-\frac{331}{16} + \left[\frac{725}{12} - \frac{205}{128} \pi^2 \right] \nu + \frac{15}{8} \nu^2 \right) \right\} \\ & + \varepsilon^3 \left\{ \frac{495}{64} + \left[-\frac{1145}{24} + \frac{205}{512} \pi^2 \right] \nu + \left(\frac{2341}{72} - \frac{451}{1536} \pi^2 \right) \nu^2 - \frac{415}{144} \nu^3 \right\} \end{aligned}$$

$$\begin{aligned}
& + j^{\frac{1}{2}} \left(\frac{95}{8} - \frac{115}{24} \nu + \frac{17}{12} \nu^2 \right) + j \left(-\frac{415}{192} - \frac{385}{192} \nu - \frac{5}{32} \nu^2 + \frac{161}{1296} \nu^3 \right) \\
& + \frac{1}{j^{\frac{1}{2}}} \left(-\frac{5}{4} + \left[\frac{202}{9} - \frac{41}{96} \pi^2 \right] \nu \right) + \frac{1}{j} \left(-\frac{12345}{32} + \left[\frac{147283}{192} - \frac{3895}{512} \pi^2 \right] \nu \right. \\
& \left. + \left[-\frac{77945}{288} + \frac{4715}{1536} \pi^2 \right] \nu^2 + \frac{445}{32} \nu^3 \right) \\
& \left. + \frac{1}{j^2} \left(\frac{193351}{192} + \left[-\frac{165835}{96} + \frac{7175}{256} \pi^2 \right] \nu + \left[\frac{1300}{3} - \frac{1025}{128} \pi^2 \right] \nu^2 - \frac{25}{4} \nu^3 \right) \right\}. \quad (8.23)
\end{aligned}$$

We have performed two calculations of the gauge-invariant result, in terms of the variables (x, ι) , starting from the expression of the averaged flux in either MH and ADM coordinates. The instantaneous part of the flux takes the form

$$\langle \mathcal{F}_{\text{inst}} \rangle = \frac{32c^5}{5G} \nu^2 x^5 \iota^{-13/2} \left(\mathcal{I}_{\text{N}} + x \mathcal{I}_{\text{1PN}} + x^2 \mathcal{I}_{\text{2PN}} + x^3 \mathcal{I}_{\text{3PN}} \right), \quad (8.24)$$

in which the PN coefficients are polynomials of ι and the mass ratio ν and given by

$$\mathcal{I}_{\text{N}} = \frac{425}{96} \iota^3 - \frac{61}{16} \iota^4 + \frac{37}{96} \iota^5, \quad (8.25a)$$

$$\begin{aligned}
\mathcal{I}_{\text{1PN}} = & \left(-\frac{289}{3} + \frac{3605}{384} \nu \right) \iota^2 + \left(\frac{1865}{24} + \frac{3775}{384} \nu \right) \iota^3 \\
& + \left(-\frac{5297}{336} - \frac{2725}{384} \nu \right) \iota^4 + \left(\frac{139}{112} + \frac{259}{1152} \nu \right) \iota^5, \quad (8.25b)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{\text{2PN}} = & \left(\frac{267725837}{258048} + \left[\frac{1440583}{2304} - \frac{609875}{24576} \pi^2 \right] \nu + \frac{24395}{1024} \nu^2 \right) \iota \\
& + \left(-\frac{51894953}{82944} + \left[-\frac{583921}{512} + \frac{497125}{24576} \pi^2 \right] \nu + \frac{1625}{48} \nu^2 \right) \iota^2 \\
& + \left(\frac{49183667}{387072} + \left[\frac{14718145}{32256} - \frac{32595}{8192} \pi^2 \right] \nu + \frac{37145}{4608} \nu^2 \right) \iota^3 \\
& + \left(-\frac{305}{16} + \frac{61}{8} \nu \right) \iota^{7/2} + \left(-\frac{2145781}{64512} \right. \\
& \left. + \left[-\frac{505639}{10752} + \frac{1517}{8192} \pi^2 \right] \nu - \frac{105}{16} \nu^2 \right) \iota^4 \\
& + \left(\frac{185}{48} - \frac{37}{24} \nu \right) \iota^{9/2} + \left(\frac{744545}{258048} + \frac{19073}{32256} \nu + \frac{2849}{27648} \nu^2 \right) \iota^5, \quad (8.25c)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{\text{3PN}} = & \frac{149899221067}{7741440} + \left[-\frac{186950547065}{3096576} + \frac{46739713}{32768} \pi^2 \right] \nu \\
& + \left[\frac{66297815}{6144} - \frac{8315825}{32768} \pi^2 \right] \nu^2 - \frac{415625}{12288} \nu^3 - \frac{161249}{192} \iota^{1/2} \\
& + \left(-\frac{66998702987}{2073600} + \left[\frac{71728525525}{1032192} - \frac{117241181}{98304} \pi^2 \right] \nu \right.
\end{aligned}$$

$$\begin{aligned}
& + \left[-\frac{24611099}{2304} + \frac{6633185}{49152}\pi^2 \right] v^2 + \frac{4346075}{12288}v^3 \Big) \iota + \frac{3727559}{2880}\iota^{3/2} \\
& + \left(\frac{4774135897}{322560} + \left[-\frac{332003303819}{13934592} + \frac{29862965}{114688}\pi^2 \right] v \right. \\
& + \left. \left[\frac{15103071}{7168} - \frac{3258475}{294912}\pi^2 \right] v^2 - \frac{2249695}{18432}v^3 \right) \iota^2 \\
& + \left(-\frac{928043}{5760} + \left[-\frac{1879}{1152} - \frac{2501}{1536}\pi^2 \right] v - \frac{5605}{192}v^2 \right) \iota^{5/2} \\
& + \left(-\frac{2740721737}{1290240} + \left[\frac{225135517}{73728} - \frac{8351167}{688128}\pi^2 \right] v \right. \\
& + \left. \left[\frac{6154165}{64512} - \frac{615}{1024}\pi^2 \right] v^2 + \frac{298895}{6144}v^3 \right) \iota^3 \\
& + \left(-\frac{3913177}{37800} + \left[-\frac{351499}{12096} + \frac{1517}{4608}\pi^2 \right] v + \frac{1153}{32}v^2 \right) \iota^{7/2} \\
& + \left(\frac{1758850201}{141926400} + \left[-\frac{186455099}{1032192} + \frac{68757}{229376}\pi^2 \right] v \right. \\
& + \left. \left[-\frac{5900711}{387072} - \frac{1517}{16384}\pi^2 \right] v^2 - \frac{2568655}{331776}v^3 \right) \iota^4 \\
& + \left(\frac{51335}{2688} - \frac{10951}{2688}v - \frac{481}{192}v^2 \right) \iota^{9/2} \\
& + \left(\frac{2635805}{405504} + \frac{891535}{3096576}v + \frac{4537}{27648}v^2 + \frac{106375}{995328}v^3 \right) \iota^5 \\
& + \left(\frac{161249}{192} - \frac{125939}{80}\iota + \frac{263113}{288}\iota^2 - \frac{168953}{1008}\iota^3 + \frac{10593}{2240}\iota^4 \right) \ln \left[\frac{x}{x_0} \frac{1 + \sqrt{\iota}}{2\iota} \right]. \quad (8.25d)
\end{aligned}$$

Similarly, we can also obtain the equivalent expression of the flux in terms of the rescaled variables (ε, j) defined by Eqs. (7.5) and (7.6).

The hereditary part of the flux given by (8.11)–(8.12) is straightforwardly added. In this part we have simply to replace e_t by its expression in terms of x and ι at the 1PN order, namely

$$e_t = \left[1 - \iota + x \left\{ -\frac{35}{4} + \frac{9}{2}v + \iota \left(\frac{17}{4} - \frac{13}{6}v \right) \right\} \right]^{1/2}. \quad (8.26)$$

(At this order there is no difference between MH and ADM coordinates.) Note also that with the latter choice of gauge-invariant variables the circular-orbit limit is not directly readable from the expressions. However, it can be easily obtained by using the expression for the variable j_\odot as reduced to circular orbits in terms of ε , Eq. (7.12).

IX. THE TEST PARTICLE LIMIT OF THE 3PN ENERGY FLUX

An important check on our result is the test particle limit for which the energy flux in the eccentric orbit case is available (to second order in the eccentricity) from computations based on perturbation theory around a Schwarzschild background. We compare the end result of our

computation – composed of the instantaneous terms and the hereditary terms computed in paper I – with the result obtained in Ref. [47]. Thus, we take the test particle limit of our result (*i.e.* $\nu \equiv \mu/m \rightarrow 0$), say in the form given by Eqs. (8.8)–(8.9) in which $e_t \equiv e_t^{\text{MH}}$, and expand it in powers of e_t retaining only terms up to e_t^2 . The instantaneous contribution to the energy flux in the test mass limit is then given by

$$\begin{aligned} \langle \mathcal{F}_{\text{inst}} \rangle = & \frac{32c^5}{5G} v^2 x^5 \left\{ 1 - \frac{1247}{336} x - \frac{44711}{9072} x^2 + \left[\frac{1266161801}{9979200} + \frac{1712}{105} \ln\left(\frac{x}{x_0}\right) \right] x^3 \right. \\ & \left. + e_t^2 \left(\frac{157}{24} - \frac{187}{168} x - \frac{84547}{756} x^2 + \left[\frac{22718275589}{9979200} + \frac{106144}{315} \ln\left(\frac{x}{x_0}\right) \right] x^3 \right) + \mathcal{O}(\nu) \right\} + \mathcal{O}(e_t^4). \end{aligned} \quad (9.1)$$

On the other hand, the hereditary contribution has been reported in Eqs. (8.11)–(8.12) and admits the test-mass limit

$$\begin{aligned} \langle \mathcal{F}_{\text{hered}} \rangle = & \frac{32c^5}{5G} v^2 x^5 \left\{ 4\pi x^{3/2} \varphi(e_t) - \frac{8191}{672} \pi x^{5/2} \psi(e_t) \right. \\ & \left. + x^3 \left[-\frac{116761}{3675} \kappa(e_t) + \left[\frac{16}{3} \pi^2 - \frac{1712}{105} C - \frac{1712}{105} \ln\left(\frac{4x^{3/2}}{x_0}\right) \right] F(e_t) \right] + \mathcal{O}(\nu) \right\}. \end{aligned} \quad (9.2)$$

To proceed further, all the enhancement functions should be expanded up to power e_t^2 . This is easy for $F(e_t)$ which is known analytically from Eq. (8.13) and we have

$$F(e_t) = 1 + \frac{62}{3} e_t^2 + \mathcal{O}(e_t^4). \quad (9.3)$$

The other enhancement functions are only known numerically for general eccentricity. We have however succeeded in obtaining analytically their leading correction term e_t^2 by implementing our calculation of the tails in paper I at order e_t^2 from the start. The results we thereby obtained [Eqs. (6.8) of paper I] are

$$\varphi(e_t) = 1 + \frac{2335}{192} e_t^2 + \mathcal{O}(e_t^4), \quad (9.4a)$$

$$\psi(e_t) = 1 - \frac{22988}{8191} e_t^2 + \mathcal{O}(e_t^4), \quad (9.4b)$$

$$\kappa(e_t) = 1 + \left(\frac{62}{3} - \frac{4613840}{350283} \ln 2 + \frac{24570945}{1868176} \ln 3 \right) e_t^2 + \mathcal{O}(e_t^4). \quad (9.4c)$$

[We do not need $\zeta(e_t)$ here since it is in factor of a ν -dependent term.] Our final result to $\mathcal{O}(\nu)$ and $\mathcal{O}(e_t^4)$ is therefore

$$\begin{aligned} \langle \mathcal{F} \rangle = & \frac{32c^5}{5G} v^2 x^5 \left\{ 1 - \frac{1247}{336} x + 4\pi x^{3/2} - \frac{44711}{9072} x^2 - \frac{8191}{672} \pi x^{5/2} \right. \\ & \left. + \left[\frac{6643739519}{69854400} + \frac{16}{3} \pi^2 - \frac{1712}{105} C - \frac{856}{105} \ln(16x) \right] x^3 \right. \\ & \left. + e_t^2 \left(\frac{157}{24} - \frac{187}{168} x + \frac{2335}{48} \pi x^{3/2} - \frac{84547}{756} x^2 + \frac{821}{24} \pi x^{5/2} \right) \right\} \end{aligned}$$

$$+ \left[\frac{113160471971}{69854400} + \frac{18832}{45} \ln 2 - \frac{234009}{560} \ln 3 + \frac{992}{9} \pi^2 - \frac{106144}{315} C - \frac{53072}{315} \ln(16x) \right] x^3 + \mathcal{O}(e_t^4) + \mathcal{O}(\nu) \Big\}. \quad (9.5)$$

The above expression is in terms of our chosen eccentricity e_t . One should note that the ‘‘Schwarzschild’’ eccentricity e appearing in the black-hole perturbation theory [47] is *a priori* different from e_t ; therefore the above result can only be compared modulo a transformation of these eccentricities. We find that indeed Eq. (9.5) is equivalent to the black-hole perturbation result given by Eq. (180) of [47], if and only if the two eccentricities are linked together by

$$e_t^2 = e^2 (1 - 6x + 4x^2 - 8x^3). \quad (9.6)$$

(Recall that $e_t = e_t^{\text{MH}}$ here.)

X. CONCLUDING REMARKS

The instantaneous contributions to the 3PN gravitational wave luminosity from the inspiral phase of a binary system of compact objects moving in an elliptical orbit is computed using the Multipolar post-Minkowskian wave generation formalism¹³. The non-trivial inputs for this calculation include the mass octupole and current quadrupole at 2PN order for general orbits and the 3PN accurate mass quadrupole. Using the 3PN quasi-Keplerian representation of elliptical orbits obtained recently the flux is averaged over the binary’s orbit. The instantaneous part of the energy flux is computed in the standard harmonic coordinate system (with logarithms). For technical reasons the average over an orbit of the instantaneous contributions is presented in other coordinate systems: Modified harmonic coordinates (without logarithms) and ADM coordinates. Alternative *gauge invariant* expressions are also provided. Supplementing the instantaneous contributions of this paper by the important hereditary contributions arising from tails, tails-of-tails and tails squared terms calculated in paper I [36], the complete energy flux has been obtained.

For binaries moving on circular orbits the 3PN energy flux agrees with that computed in [30]. However the circular-orbit results are known to the higher 3.5PN order [30]. The extension of the 3.5PN term to eccentric orbits would be interesting, but some uncomputed modules remain in the general formalism to compute the multipole moments for general sources required for the 3.5PN generation in the eccentric orbit case. We leave this to a future investigation.

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¹³ The instantaneous part of the 3PN gravitational wave flux of angular momentum and linear momentum from inspiralling compact binaries moving on elliptical orbits has been computed [48, 49].

to the results of this paper are done with the software MATHEMATICA.

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