

Canonical formulation of the embedded theory of gravity equivalent to Einstein's General Relativity

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Abstract

We study the approach in which independent variables describing gravity are functions of the space-time embedding into a flat space of higher dimension. We formulate a canonical formalism for such a theory in a form, which requires imposing additional constraints, which are a part of Einstein's equations. As a result, we obtain a theory with an eight-parameter gauge symmetry. This theory becomes equivalent to Einstein's general relativity either after partial gauge fixing or after rewriting the metric in the form that is invariant under the additional gauge transformations. We write the action for such a theory.

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1. Introduction

In the usual description of gravity in the framework of Einstein's general relativity (GR), the four-dimensional space-time is a Riemannian (more precisely, pseudo-Riemannian) space. An example of the Riemannian space is a d -dimensional surface in a flat space of a higher dimension if we consider the metric on this surface to be induced by the trivial metric of the ambient space. But a surface in a flat ambient space turns out to be not just a particular example of a Riemannian space; considering an arbitrary Riemannian space can be presumably replaced by considering such a surface. According to the Janet-Cartan theorem [1, 2] (see, e. g., Remark 18 in [3]), an arbitrary Riemannian space W^d of dimension d can be locally embedded isometrically in any Riemannian space of dimension greater than or equal to

$$N = \frac{d(d+1)}{2}, \tag{1}$$

and therefore, in particular, in a flat Riemannian space of such dimension. Friedman [4] generalized this theorem to the case of a nonpositive definite space signature.

This theorem ensures only a local existence of an isometric embedding, i. e., for a finite part of a manifold. We note that if we address the problem of a global manifold embedding, then the necessary dimension N may increase dramatically depending on the manifold topology (see Remark 18 in [3]).

The Janet-Cartan theorem ensures only the existence, not the uniqueness, of the embedding. This means that different surfaces with the same metric may exist. In this case, we speak of a possible isometric bending of a surface. For instance, it is clear that the uniqueness of the embedding is certainly lost for $N > d(d+1)/2$ because we can first embed the surface into a space of dimension $d(d+1)/2$ and then isometrically embed this space in different ways as a part of a cylinder in an N -dimensional space. Comparing the numbers of variables and equations shows that in the general case, the uniqueness of embedding presumably holds (up to trivial shifts and rotations in the ambient space) when condition (1) is satisfied exactly. Indeed, the metric $g_{\mu\nu}$ has $d(d+1)/2$ independent components, and we have the same number of equations for N functions describing the embedding. But in some cases, the uniqueness may be absent even if condition (1) holds. As a nontrivial example of a surface admitting isometric bending, we can take a sufficiently small but finite part of a sphere.

In accordance with condition (1), we can take the ten-dimensional space as an ambient space for the four-dimensional space-time. The signature of the former can be conveniently taken to be $(+, -, -, \dots, -)$, i. e., we take the Minkowski space $R^{1,9}$ as an ambient space. Because instead of considering space-time as a Riemannian space, we can, with the above precaution, consider the four-dimensional surface W^4 in the ten-dimensional space $R^{1,9}$, the idea arises to use not the metric field $g_{\mu\nu}(x)$ but variables describing the surface W^4 as independent variables in the gravity description. As such variables, it is convenient to take the embedding function $y^a(x^\mu)$ describing the map

$$y^a(x^\mu) : R^4 \longrightarrow R^{1,9}. \tag{2}$$

Here and hereafter, the indices a, b, \dots range the values $0, 1, 2, \dots, 9$ and $\mu, \nu, \dots = 0, 1, 2, 3$. We assume that y^a are the Lorentzian coordinates in $R^{1,9}$ and we can raise and lower the indices a, b, \dots using the constant pseudo-Euclidean metric of the ambient space

$$\eta_{ab} = \text{diag}(1, -1, -1, \dots, -1).$$

This idea was first advanced in 1975 in a talk by Regge and Teitelboim and was then published in [5]. In this way, we can try to study another approach to quantizing gravity and, in particular, obtain a new insight into the causality problem in the theory because we have fixed light cones in the ambient flat space. Moreover, we can relate this theory to other theories, such as superstring theory, using the extra dimensions.

In [5], the theory action was chosen to be the standard GR action

$$S = \int d^4x \sqrt{-g} R, \quad (3)$$

where R is the scalar curvature and $g = \det g_{\mu\nu}$, where the metric is induced and expressed in terms of the embedding function,

$$g_{\mu\nu} = \partial_\mu y^a \partial_\nu y_a. \quad (4)$$

Varying this action with respect to $y^a(x)$ produces the equations of the theory of embedding:

$$\nabla_\mu (G^{\mu\nu} \partial_\nu y^a) = 0, \quad (5)$$

where $G^{\mu\nu}$ is Einstein's tensor constructed from metric (4) and ∇_μ is the covariant derivative induced by this metric. In the case of matter with the energy-momentum tensor $T^{\mu\nu}$, Eqs. (5) become

$$\nabla_\mu ((G^{\mu\nu} - \varkappa T^{\mu\nu}) \partial_\nu y^a) = 0. \quad (6)$$

Because adding matter does not play a principal role in describing the theory, for simplicity in what follows, we consider the gravitational field with matter absent.

Equations (5) are more general than Einstein's equations, i. e., any solution of Einstein's equations is a solution of the theory of embedding, but not vice versa. The theory of embedding is therefore not equivalent to GR. There are extra solutions in the former. The question therefore arises whether it is possible to introduce additional restrictions into the theory of embedding such that it becomes equivalent to GR.

To exclude extra solutions, the following idea was advanced in [5]: it was proposed to complete the set of equations of motion arising from the action by imposing additional constraints $G_{\mu\perp} = 0$ on some of Einstein's equations, where the symbol \perp denotes the direction orthogonal to the constant-time surface. We discuss this possibility in Sec. 3 below, where we show that at least in the general case, imposing the constraints $G_{\mu\perp} = 0$ only at the initial instant suffices to ensure the equivalence to Einstein's equations, i. e., these constraints are analogous to the first-class constraints in the canonical formalism. This result was derived in more detail in [6], where a detailed exposition of the formalism convenient for describing the theory of embedding can also be found. The basic equations of this formalism are given in Sec. 2.

We mention that an artificial, *ad hoc*, introduction of additional equations into the theory seems not quite satisfactory, as was noted in [7], where it was also argued that a proper way out might be to find a modification of the action that generates the necessary additional equations, but it was also mentioned that how to do this is unknown. We propose such a modification of the action in Sec. 5 below.

There were attempts to construct a canonical formalism for the theory of embedding. It is known that disregarding surface terms allows reducing the gravitational action to a form

in which the Lagrangian contains only first derivatives of the metric $g_{\mu\nu}$ with respect to time. Because formula (4) contains the differentiation, the Lagrangian turns out to contain second-order time derivatives of the embedding function y^a . This considerably hinders constructing the canonical formalism. Nevertheless, a possibility of constructing a canonical formulation of this theory was studied in [8] based on a special technique developed for this case. The progress along this route was achieved only after a complete gauge fixing.

A detailed consideration shows that the embedding theory equations do not contain time derivatives of y^a of order higher than two. This would indicate that the action can be rewritten in a form in which the Lagrangian contains only first derivatives in time. It was already noted in [5] that this can be achieved if the action is written in the Arnowitt-Deser-Misner form [9]. If we represent the action in this form, then we can develop the canonical formalism standardly, but extremely complex constraints appear in the theory. Their form was studied in [10], where it was found that some of the constraints cannot be written explicitly and can be represented only in the form of the existence of coincident roots of a certain pair of polynomials.

A possibility of adding the conditions $G_{\mu\perp} = 0$ (which can also be considered constraints) to the set of arising constraints was considered in [5], but the problem of closing such a unified system of constraints remained open. We devote Sec. 4 of this paper to constructing such a canonical formalism with the additional constraints. The found system of constraints, which is found explicitly in this case, differs from that in [5] because the restrictions on the generalized momentums that appeared there were not taken into account correctly. We demonstrate that the obtained system of constraints is closed.

In Sec. 5, we construct and analyze the action corresponding to the found canonical formulation of the theory.

An extended bibliography related to the theory of embedding and related questions can be found in [11].

2. A concise exposition of the embedding theory formalism

The description of a surface W^4 in the flat space $R^{1,9}$ using the embedding function $y^a(x)$ is invariant under transformations of the coordinates x^μ on the surface. The components of the functions $y^a(x)$ act like scalars with respect to these transformations. We can therefore regard the function $y^a(x)$ as a ten-component field defined in the four-dimensional Riemann space and carrying the superscript of the global internal symmetry group $SO(1,9)$, corresponding to the Lorentz transformations of the ambient space $R^{1,9}$.

Because $y^a(x)$ is a scalar, its covariant derivative coincides with the standard derivative,

$$\nabla_\mu y^a = \partial_\mu y^a \equiv e_\mu^a. \quad (7)$$

Analogously, the covariant derivatives of quantities carrying both Greek and Latin indices are constructed standardly as if there were no Latin indices. The quantity e_μ^a resembles a tetrad used in the tetrad description of gravity, but it differs from the standard tetrad in that its index a ranges more values than the index μ . This quantity can be treated as the union of four vectors (if we take $\mu = 0, 1, 2, 3$) of the ambient space. These vectors constitute a basis (nonorthogonal in general) in the subspace tangent to the surface W^4 at a given point. At the same time, the quantity e_μ^a is a vector with respect to its index μ at a fixed a .

The induced metric is

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} = e_\mu^a e_{\nu,a} = \partial_\mu y^a \partial_\nu y_a. \quad (8)$$

It is convenient to introduce the quantity

$$e_a^\mu = g^{\mu\nu} e_{\nu,a}, \quad g^{\mu\nu} g_{\nu\alpha} = \delta_\alpha^\mu. \quad (9)$$

It is then easy to note that the equalities

$$e_a^\gamma e_\beta^a = \delta_\beta^\gamma, \quad g^{\mu\nu} = e^{\mu,a} e_a^\nu \quad (10)$$

are satisfied. From the condition of covariant constancy of the metric and the absence of torsion (because of which $\nabla_\mu e_\nu^a = \nabla_\nu e_\mu^a$) and from form (8) of the metric, we can establish (see [6]) that

$$e_{\alpha,a} \nabla_\mu e_\nu^a = 0, \quad (11)$$

whence we expression the connection

$$\Gamma_{\mu\nu}^\beta = e_a^\beta \partial_\mu e_\nu^a = e_a^\beta \partial_\mu \partial_\nu y^a. \quad (12)$$

But in calculations within this formalism, we can disregard such a noncovariant quantity as the connection, which is a definite advantage of the formalism. Usually, we cannot write a formula for the covariant differentiation without a connection, but it is possible here. We write the covariant derivative of a vector using property (11):

$$\nabla_\alpha a^\mu = e_a^\mu e_\nu^a \nabla_\alpha a^\nu = e_a^\mu \nabla_\alpha (e_\nu^a a^\nu) - e_a^\mu (\nabla_\alpha e_\nu^a) a^\nu = e_a^\mu \nabla_\alpha (e_\nu^a a^\nu) = e_a^\mu \partial_\alpha (e_\nu^a a^\nu). \quad (13)$$

We can write analogous formulas for the covariant differentiation of tensors of arbitrary rank. We obtain a simple rule for the covariant differentiation: contracting with respect to every index with the quantity e_ν^a , we must "transfer" the tensor from the Riemannian space to the ambient space, take the standard derivative there, and then "transfer" it back performing the contraction with e_ν^a . In cases where it cannot lead to confusion, we merely write a^a instead of the contraction $e_\nu^a a^\nu$ (and analogously for quantities with several indices).

We now introduce the quantity $\Pi_b^a(x)$, which is extremely useful for calculations and is the projection on the plane tangent to the surface W^4 at a given point. It is easy to verify that such a projection operator can be written as

$$\Pi_b^a = e_\mu^a e_b^\mu. \quad (14)$$

It is also convenient to introduce the operator of projection to the space dual to the tangent plane:

$$\Pi_\perp^a = \delta_b^a - \Pi_b^a. \quad (15)$$

We write several properties of the introduced objects useful for calculations (see the proof in [6]):

$$\begin{aligned} \Pi_{ab} &= \Pi_{ba}, & \delta \Pi_{ab} &= -\delta \Pi_{\perp ab}, & \Pi_b^a (\delta \Pi_c^b) \Pi_d^c &= 0, \\ \Pi_\perp^a (\delta \Pi_c^b) \Pi_d^c &= 0, & \delta \Pi_{ab} &= \Pi_a^c (\delta \Pi_{cd}) \Pi_\perp^d + \Pi_\perp^c (\delta \Pi_{cd}) \Pi_b^d. \end{aligned} \quad (16)$$

The second fundamental form of the surface $b_{\mu\nu}^a$ plays an important role in describing the geometry of embedded surfaces. By definition (see, e. g., Sec. 3 in Chap. 7 in [3]), for any tangent vector field f^b ,

$$(\partial_\mu f^b) \Pi_\perp^a = b_{\mu\nu}^a f^\nu, \quad (17)$$

whence, noting that $f^b \Pi_{\perp b}^a = 0$, we obtain the equality

$$b_{\mu\nu}^a = e_\nu^b \partial_\mu \Pi_b^a. \quad (18)$$

We note that the condition

$$\Pi_a^b b_{\mu\nu}^a = 0 \quad (19)$$

is satisfied identically for $b_{\mu\nu}^a$, i. e., $b_{\mu\nu}^a$ can be regarded as the set of vectors indexed by μ and ν with components indexed by a that are orthogonal to the tangent plane. We have one more convenient representation for the second fundamental form of the surface. Replacing Π_b^a with $-\Pi_{\perp b}^a$ in (18) and pushing the derivative to e_ν^b , we obtain the equality

$$b_{\mu\nu}^a = \Pi_{\perp b}^a \partial_\mu e_\nu^b = \Pi_{\perp b}^a \partial_\mu \partial_\nu y^b, \quad (20)$$

whence we immediately see that $b_{\mu\nu}^a$ is symmetric in the lower indices. We can also easily note that formula (20) can be rewritten in the explicitly covariant form

$$b_{\mu\nu}^a = \nabla_\mu e_\nu^a = \nabla_\mu \nabla_\nu y^a. \quad (21)$$

In the case where the codimension of the surface is one and hence $\Pi_{\perp ab} = n_a n_b$, because of condition (19), instead of $b_{\mu\nu}^a$, it suffices to consider the quantity

$$K_{\mu\nu} = n_a b_{\mu\nu}^a, \quad (22)$$

which is also called the second fundamental (or second quadratic) form of the surface in this case.

The second fundamental form of the surface plays an important role in describing gravity in terms of the embedding function because the Riemann-Christoffel curvature tensor in the case of a flat ambient space is expressed in terms of precisely this quantity:

$$R_{\alpha\beta\mu\nu} = b_{\alpha\mu}^a b_{a,\beta\nu} - b_{\alpha\nu}^a b_{a,\beta\mu}. \quad (23)$$

We note that this equation is the Gauss equation for a surface embedded in a flat ambient space. The scalar curvature can be written in the form

$$R = g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\mu\nu} = (g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}) b_{\alpha\mu}^a b_{a,\beta\nu}. \quad (24)$$

Substituting expression (23) in the known representation of the Einstein tensor,

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{1}{4} g_{\xi\zeta} E^{\mu\xi\alpha\beta} E^{\nu\zeta\gamma\delta} R_{\alpha\beta\gamma\delta}, \quad (25)$$

where $E^{\mu\xi\alpha\beta} = \frac{1}{\sqrt{-g}} \varepsilon^{\mu\xi\alpha\beta}$ and $\varepsilon^{\mu\xi\alpha\beta}$ is the unit totally antisymmetric tensor, we can write it in the form

$$G^{\mu\nu} = \frac{1}{2} g_{\xi\zeta} E^{\mu\xi\alpha\beta} E^{\nu\zeta\gamma\delta} b_{\alpha\gamma}^a b_{a,\beta\delta}. \quad (26)$$

We use this expression for the Einstein tensor below.

3. Comparing equations of the embedding theory with Einstein's equations

Embedding theory equations (5) are ten equations obtained by varying with respect to the ten components of the embedding function y^a . But among these equations, four are satisfied identically. Indeed, Eqs. (5) can be rewritten as the set of equations

$$\Pi_a^b \nabla_\mu (G^{\mu\nu} e_\nu^a) = 0, \quad (27)$$

$$\Pi_{\perp a}^b \nabla_\mu (G^{\mu\nu} e_\nu^a) = 0. \quad (28)$$

By virtue of formulas (21),(19) the first of these equations can be written equivalently in the form

$$\nabla_\mu G^{\mu\nu} = 0 \quad (29)$$

and is satisfied identically because of the Bianci identity. Therefore, only six independent embedding equations (28) remain. Again using (21) and taking identity (29) into account, we can rewrite them in the form

$$G^{\mu\nu} b_{\mu\nu}^a = 0. \quad (30)$$

We recall that the quantity $b_{\mu\nu}^a$ is regarded as the set of vectors with components indexed by a directed normal to the tangent plane, and Eqs. (30) therefore contain only six independent equations.

We now rewrite Eq. (30) using formula (26):

$$g_{\xi\zeta} E^{\mu\xi\alpha\beta} E^{\nu\zeta\gamma\delta} b_{\alpha\gamma}^b b_{b,\beta\delta} b_{\mu\nu}^a = 0. \quad (31)$$

It is clear in this form of writing that the embedding theory equations do not contain derivatives of an order higher than two. It is interesting that Einstein's equations have the same property if they are written in terms of y^a :

$$g_{\xi\zeta} E^{\mu\xi\alpha\beta} E^{\nu\zeta\gamma\delta} b_{\alpha\gamma}^b b_{b,\beta\delta} = 0. \quad (32)$$

We now compare the embedding theory equations and Einstein's equations. For this, we fix the choice of time on the surface W^4 , i. e., we define a system of three-dimensional spacelike surfaces W^3 corresponding to constant time. These surfaces are described by the embedding functions

$$y^a(x^i) \equiv y^a(x^\mu)|_{x^0=t}. \quad (33)$$

Here and hereafter, the indices i, k, \dots range the values 1, 2, 3. For each such surface, we can introduce all the quantities described in Sec. 2, and we label such quantities with the digit 3 over the letter: ${}^3e_i^a, {}^3g_{ik}, {}^3\Pi_{ab}, {}^3\Pi_{\perp ab}, {}^3E_{abc}, {}^3b_{ik}^a, {}^3R_{iklm}, \dots$. We note that tensors with upper Latin indices marked by the digit 3 can be always obtained by raising these indices using the matrix ${}^3g^{ik}$, which is inverse to the matrix ${}^3g_{ik}$. We also introduce the unit vector n_a that is tangent to the surface W^4 at a given point and is normal to W^3 . From the first equality in (10), we have

$$e_a^0 e_i^a = 0 \quad \Rightarrow \quad e_a^0 {}^3e_i^a = 0, \quad (34)$$

whence we obtain

$$n_a = \frac{e_a^0}{\sqrt{e_b^0 e^{0,b}}} = \frac{e_a^0}{\sqrt{g^{00}}}. \quad (35)$$

For definiteness, we set $n_0 > 0$. It is clear that

$$\Pi_{ab} = \overset{3}{\Pi}_{ab} + n_a n_b, \quad \Pi_{\perp ab} = \overset{3}{\Pi}_{\perp ab} - n_a n_b. \quad (36)$$

We obtain one more useful representation for the vector n^a . Using formulas (36),(35), we have

$$\overset{3}{\Pi}_{\perp b}^a e_0^b = n^a n_b e_0^b = \frac{n^a e_b^0 e_0^b}{\sqrt{g^{00}}} = \frac{n^a}{\sqrt{g^{00}}}, \quad (37)$$

whence

$$n^a = \sqrt{g^{00}} \overset{3}{\Pi}_{\perp b}^a \partial_0 y^b = \frac{\overset{3}{\Pi}_{\perp b}^a \partial_0 y^b}{\sqrt{\partial_0 y^c \overset{3}{\Pi}_{\perp cd} \partial_0 y^d}}. \quad (38)$$

As is known, the second fundamental form of the surface W^3 as a submanifold in W^4 is given by

$$K_{ik} = -\nabla_i n_k, \quad (39)$$

where the covariant derivative is determined by the Riemannian connection in W^4 . Using covariant differentiation rule (13), we find that

$$K_{ik} = -e_k^a \partial_i (e_a^\mu n_\mu) = -\overset{3}{e}_k^a \partial_i n_a = n_a \partial_i \overset{3}{e}_k^a = n_a \overset{3}{b}_{ik}^a = n_a \partial_i \partial_k y^a. \quad (40)$$

Applying formula (20) for the second fundamental forms of the surfaces W^4 and W^3 and using the second relation in (36), we can easily obtain the simple law for adding second fundamental forms:

$$b_{ik}^a = \overset{3}{b}_{ik}^a - n^a K_{ik}. \quad (41)$$

Using the above relations, among all Einstein's equations taken in form (32), those four equation that can be written as

$$n_a G^{ac} = 0, \quad (42)$$

can be represented as the equation

$$\left(\overset{3}{g}^{ik} \overset{3}{g}^{lm} - \overset{3}{g}^{il} \overset{3}{g}^{km} \right) \overset{3}{b}_{ik}^a \overset{3}{b}_{lm}^b n_a n_b - \overset{3}{R} = 0, \quad (43)$$

taken together with the three equations

$$\left(\overset{3}{g}^{ik} \overset{3}{g}^{lm} - \overset{3}{g}^{im} \overset{3}{g}^{lk} \right) \overset{3}{b}_{im}^a \partial_l n_a = 0 \quad (44)$$

(see [6]). Applying the identity

$$\begin{aligned} \prod_{\perp a}^3 b^3 \nabla_l \left(\left(g^{ik} g^{lm} - g^{im} g^{lk} \right) b_{a,im} \right) &= \prod_{\perp a}^3 b^3 g^{ik} g^{lm} \left(\nabla_l \nabla_i e_m^a - \nabla_i \nabla_l e_m^a \right) = \\ &= \prod_{\perp a}^3 b^3 g^{ik} g^{lm} R_{mli}^a e_n^a = 0, \end{aligned} \quad (45)$$

where we use formula (21), we can conveniently transform Eqs. (44) to the form

$$\nabla_l \left(\left(g^{ik} g^{lm} - g^{im} g^{lk} \right) b_{im}^a n_a \right) = 0. \quad (46)$$

We can easily see that Eqs. (43) and (46) do not contain time derivatives of $y^a(x)$ of an order higher than one, and such derivatives enter the equations only through the quantity n^a . Four equations (42) can be therefore considered constraints, i. e., conditions, which impose restrictions on the initial data $y^a(x^i)$, $\partial_0 y^a(x^i)$ defined on the surface $x^0 = const$. It is interesting that Eqs. (42), which were constraints in GR, remain constraints in the embedding theory although the GR independent variables (the metric $g_{\mu\nu}$) are now expressed in terms of the independent variables of the embedding theory (the functions $y^a(x)$) by means of differentiation. We call Eqs. (42) Einstein's constraints.

We now consider the previously obtained embedding theory equations (30), writing them in the form

$$G^{ab} b_{ab}^c = 0. \quad (47)$$

We assume that they are satisfied and treat them as the equations describing the time evolution of the three-dimensional spacelike surface W^3 . We find what additional restrictions must be introduced in the theory for it to be equivalent to GR, i. e., for Einstein's equations to hold. The analysis just performed demonstrates that for this, we must at least choose the initial data, i. e., the values of $y^a(x^i)$ and $\partial_0 y^a(x^i)$ at the initial instant, that satisfy Einstein's constraints.

We assume that this was done, i. e., Eqs. (42) are satisfied at the initial instant. Using formula (36) at the initial instant, we can then write the expression

$$G^{ab} e_a^i e_b^k b_{ik}^h = 0. \quad (48)$$

instead of (47). The quantity b_{ik}^h in this equation can be interpreted as a matrix with the multi-indices h and $\{ik\}$. We can then assume that h ranges not ten but six values because four identities (19) are satisfied, and the multi-index $\{ik\}$ also ranges six values because the quantity b_{ik}^h is symmetric. The quantity b_{ik}^h can therefore be considered a square matrix of the size 6×6 . We additionally assume that this matrix is nondegenerate at all points of the initial surface, which can be conditionally written as

$$\det (b_{ik}^h) \neq 0. \quad (49)$$

This assumption is purely technical; it just excludes a certain initial data subset of measure zero. Condition (49) was discussed in [6]. In particular, it was shown there that breaking this condition results in a special situation for Einstein's equations written in terms of the embedding function.

If the matrix b_{ik}^h is nondegenerate, then Eq. (48) is equivalent to the equation

$$G^{ab} \overset{3}{e}_a^i \overset{3}{e}_b^k = 0, \quad (50)$$

which, together with imposed constraints (42), can be written as Einstein's equations

$$G^{\mu\nu} = 0. \quad (51)$$

We have thus obtained the first result: the embedding theory equations together with constraints (42) and condition (49), imposed at some instant, result in Einstein's equations being satisfied at this instant.

It was shown in [6] that using the Bianci identity, we can obtain the relation for the time derivative of constraints (42):

$$\partial_0 (n_a G^{ac}) = -\frac{1}{\sqrt{g^{00}}} \Pi_b^c e_a^i \partial_i G^{ab} - n_a G^{ab} \partial_0 \Pi_b^c + G^{ac} \partial_0 n_a. \quad (52)$$

This relation easily implies the second result: if Einstein's equations (51) are satisfied at some instant, then the time derivative of constraints (42) vanishes at this instant.

Together with the first result above, we conclude that if the embedding theory equations are always satisfied, condition (49) is satisfied during some time interval, and Einstein's constraints (42) are imposed at the beginning of this time interval, then Einstein's equations are satisfied during the whole time interval. Because, in particular, Einstein's constraints, being imposed at the initial instant, are satisfied automatically at subsequent times, we can expect that these constraints become first-class constraints in the framework of a canonical formalism.

If the initial data are chosen as corresponding to the general case and condition (49) is therefore satisfied, then the embedding theory with Einstein's constraints imposed at the initial instant is equivalent to GR.

4. Canonical formalism with additionally imposed Einstein's constraints

We now develop the canonical formalism for the embedding theory. Dropping the total divergence term in the integrand in action (3), we write it in the Arnowitt-Deser-Misner form

$$S = \int d^4x \sqrt{-g} \left((K_i^i)^2 - K_{ik} K^{ik} + \overset{3}{R} \right). \quad (53)$$

If we rewrite this expression in terms of the embedding function $y^a(x)$ and use one of the forms of representing formula (40), then it becomes

$$S = \int d^4x \sqrt{-g} \left(n_a n_b \overset{3}{b}_{ik}^a \overset{3}{b}_{lm}^b L^{ik,lm} + \overset{3}{R} \right), \quad (54)$$

where

$$L^{ik,lm} = \overset{3}{g}^{ik} \overset{3}{g}^{lm} - \overset{3}{g}^{il} \overset{3}{g}^{km}. \quad (55)$$

We note that formula (38) implies the equality

$$g^{00} = \frac{1}{\overset{3}{y}^a \overset{3}{\Pi}_{ab} \overset{3}{y}^b}, \quad (56)$$

where $\dot{y}^a \equiv \partial_0 y^a$. Using the relation $g = \overset{3}{g}/g^{00}$ and formulas (38),(56),(23), we can rewrite action (54) in the form in which the derivatives of the variables $y^a(x)$ with respect to the time x^0 are written explicitly:

$$S = \int dx^0 L(y^a, \dot{y}^a), \quad L = \int d^3x \frac{1}{2} \left(\frac{\dot{y}^a B_{ab} \dot{y}^b}{\sqrt{\dot{y}^a \overset{3}{\Pi}_{ab} \dot{y}^b}} + \sqrt{\dot{y}^a \overset{3}{\Pi}_{ab} \dot{y}^b} B_c^c \right), \quad (57)$$

where the quantity

$$B^{ab} = 2\sqrt{-\overset{3}{g}} \overset{3}{b}_{ik}^a \overset{3}{b}_{lm}^b L^{ik,lm}, \quad (58)$$

and also the projection operator $\overset{3}{\Pi}_{ab}$ do not contain time derivatives.

We can treat the quantity $\overset{3}{b}_{ik}^a$ as the set of six vectors (at the fixed values of the indices i and k with respect to which it is symmetric). On the other hand, this quantity satisfies three identities $\overset{3}{b}_{ik}^a \overset{3}{e}_{a,l} = 0$. In the general case, we therefore have a unique normalized vector w_a determined by the conditions

$$w_a \overset{3}{e}_i^a = 0, \quad w_a \overset{3}{b}_{ik}^a = 0. \quad (59)$$

We note that action of the matrix B^{ab} on this vector gives zero, and this matrix is therefore not invertible even in the seven-dimensional space orthogonal to the vectors $\overset{3}{e}_i^a$. This was not mentioned in [5], which resulted in an incorrect form of one of the constraints.

We find the generalized momentum π_a for the variable y^a from action (57) (we use formulas (38),(56)):

$$\pi_a = \frac{\delta L}{\delta \dot{y}^a} = B_{ab} n^b - \frac{1}{2} n_a (n_c B^{cd} n_d - B_c^c). \quad (60)$$

Taking the properties of the quantity $\overset{3}{b}_{ik}^a$ into account, we obtain the constraints

$$\Phi_i = \pi_a \overset{3}{e}_i^a = 0. \quad (61)$$

In the general case, relation (60) must generate one more constraint arising as a restriction on the momentum π_a following from the identity $n^a n_a = 1$. We note that it is extremely difficult to write this constraint as an algebraic expression (this problem was studied in [10]). But instead of studying the general case, we assume that Einstein's constraints (42) are additionally imposed when constructing the canonical formalism. Using formulas (43),(46),(58),(23), we write these constraints in the form

$$\mathcal{H}^i = -2\sqrt{-\overset{3}{g}} \overset{3}{\nabla}_k \left(L^{ik,lm} \overset{3}{b}_{lm}^a n_a \right) = 0, \quad (62)$$

$$\mathcal{H}^0 = n_c B^{cd} n_d - B_c^c = 0, \quad (63)$$

where we choose the common factors for convenience. Accounting for constraint (63) in expression (60) for the momentum, we obtain the momentum in the form

$$\pi_a = B_{ab} n^b. \quad (64)$$

As a result, the identity $n^a n_a = 1$ does not restrict the momentum π_a (because the matrix B_{ab} , as stated above, is not invertible even in the subspace orthogonal to the vectors $\overset{3}{e}_i^a$); instead, the constraint

$$\Phi_0 = \pi_a w^a = 0 \quad (65)$$

arises in addition to constraint (61).

Using formulas (60),(57),(38),(56), we can easily find that the theory Hamiltonian

$$H = \int d^3x \pi_a \dot{y}^a - L \quad (66)$$

vanishes. The generalized Hamiltonian then reduces to a linear combination of constraints (61)-(63),(65), and we must verify whether these constraints are governed by a constraint algebra of the first kind.

In the canonical formalism, constraints must be expressed via generalized coordinates and momenta, i. e., via y^a and π_a but not \dot{y}^a in our case. Constraints (61) and (65) satisfy this requirement (we note that the vector w_a determined by conditions (59) depends on y^a but not on \dot{y}^a), while constraints (62) and (63) do not satisfy it. They must therefore be transformed to the necessary form. For this, we introduce the quantity α_a^{ik} unambiguously determined by the conditions

$$\alpha_a^{ik} = \alpha_a^{ki}, \quad \alpha_a^{ik} \overset{3}{e}_l^a = 0, \quad \alpha_a^{ik} w^a = 0, \quad \alpha_a^{ik} \overset{3}{b}_{lm}^a = \frac{1}{2} (\delta_l^i \delta_m^k + \delta_m^i \delta_l^k). \quad (67)$$

It is clear that this quantity as well as w_a depends on y^a but not on \dot{y}^a . Relation (64) implies that

$$\overset{3}{b}_{ik}^b n_b = \frac{1}{2\sqrt{-g}} \hat{L}_{ik,lm} \alpha_a^{lm} \pi^a, \quad (68)$$

where

$$\hat{L}_{pr,lm} = \frac{1}{2} g_{pr} g_{lm} - g_{pl} g_{rm}, \quad \hat{L}_{pr,lm} L^{ik,lm} = \delta_p^i \delta_r^k. \quad (69)$$

Using formula (68), we can write constraints (62),(63) as

$$\mathcal{H}^i = -\sqrt{-g} \overset{3}{\nabla}_k \left(\frac{1}{\sqrt{-g}} \pi^a \alpha_a^{ik} \right), \quad (70)$$

$$\mathcal{H}^0 = \frac{1}{2\sqrt{-g}} \pi^a \alpha_a^{ik} \hat{L}_{ik,lm} \alpha_b^{lm} \pi^b - 2\sqrt{-g} \overset{3}{R}. \quad (71)$$

Constraints represented in this form cease to contain \dot{y}^a . It is clear that constraint (71) is an expression quadratic in the momentum π^a and constraint (70) and also constraints (61),(65) are linear in this momentum.

We note that the set of constraints (61),(65),(70),(71) found here differs from that in [5] in that constraint (65) looks different (see the note after formula (59)).

We must now calculate the Poisson brackets between obtained constraints (61),(65),(70), (71). Because Einstein's constraints, as shown in Sec. 3, must be preserved on the equations of motion, it can be hoped that these Poisson brackets are expressed by linear combinations of the constraints, i. e., we have a first-class constraint algebra. But this must be verified explicitly.

It is useful to find the result of the action of transformations generated by constraints (61),(65),(70) on some combinations of variables. Calculation shows that

$$\left\{ \int d^3 \tilde{x} \Phi_i \xi^i, y^a \right\} = \xi^i \partial_i y^a, \quad \left\{ \int d^3 \tilde{x} \Phi_i \xi^i, \frac{\pi_a}{\sqrt{-g}} \right\} = \xi^i \partial_i \frac{\pi_a}{\sqrt{-g}}, \quad (72)$$

$$\left\{ \int d^3 \tilde{x} \Phi_0 \xi, g_{lm} \right\} = 0, \quad \left\{ \int d^3 \tilde{x} \Phi_0 \xi, \pi^a \alpha_a^{lm} \right\} \approx 0, \quad (73)$$

$$\left\{ \int d^3 \tilde{x} (\mathcal{H}_i + \Phi_i) \xi^i, g_{lm} \right\} = 0, \quad \left\{ \int d^3 \tilde{x} (\mathcal{H}_i + \Phi_i) \xi^i, \pi^a \alpha_a^{lm} \right\} \approx 0, \quad (74)$$

where $\mathcal{H}_i = g_{ik} \mathcal{H}^k$ and ξ, ξ^i are arbitrary infinitesimal quantities depending on x^k , $\{\dots\}$ is the Poisson bracket, the sign \approx denotes the equality up to adding a linear combination of constraints, and we use the notation $f \equiv f(x)$, $\tilde{f} \equiv f(\tilde{x})$.

Formulas (72) imply that the constraint Φ_k generates transformations of three-dimensional coordinates on the constant-time surface W^3 (we note that the generalized momentum π^a is a three-dimensional scalar density). Because the constraints Φ_0 and \mathcal{H}^0 are scalar densities and Φ_i and \mathcal{H}^i are vector densities, this means that the Poisson brackets between the constraint Φ_i and all other constraints are linear combinations of the constraints.

The first formula in (73) states that the constraint Φ_0 generates a transformation that is an isometric bending of the surface W^3 . Because the constraints \mathcal{H}^i and \mathcal{H}^0 are expressed via the quantities g_{lm} and $\pi^a \alpha_a^{lm}$, formulas (73) imply that the Poisson brackets between the constraint Φ_0 and the constraints $\mathcal{H}^i, \mathcal{H}^0$ are linear combinations of the constraints.

Analogously, formulas (74) demonstrate that the constraint combination $\mathcal{H}_i + \Phi_i$ also generates isometric bendings of the surface W^3 and that its Poisson brackets with the constraints $\mathcal{H}^i, \mathcal{H}^0$ reduce to linear combinations of the constraints. We note that the total number (four) of the found generators of three-dimensional isometric bendings corresponds to comparing the number of independent components of the three-dimensional metric (six) and the dimension (ten) of the space into which the three-dimensional surface is embedded.

Taking all the above into account, to prove the closedness of the constraint algebra (61),(65), (70),(71), it remains to verify that the two Poisson brackets $\left\{ \Phi_0, \Phi_0 \right\}$ and $\left\{ \mathcal{H}^0, \mathcal{H}^0 \right\}$ are zero. This is a direct calculation, which is very cumbersome, especially in the case of the Poisson bracket $\left\{ \mathcal{H}^0, \mathcal{H}^0 \right\}$. We have thus proved that constraints (61),(65),(70),(71) are first-class constraints.

In accordance with everything said after formula (65), we can write the Hamiltonian of the embedding theory with the additionally imposed Einstein's constraints in the form of a linear combination of all eight constraints with the Lagrange multipliers,

$$H = \int d^3 x (\lambda^i \Phi_i + \lambda^0 \Phi_0 + N_i \mathcal{H}^i + N_0 \mathcal{H}^0). \quad (75)$$

But because four of the eight constraints appear not from the canonical formalism construction but are merely imposed artificially and added to the Hamiltonian with their Lagrange multipliers, the equations of motion generated by this Hamiltonian may not exactly reproduce the initial equations of the embedding theory and may contain some of the Lagrange multipliers as additional variables.

The presence of the eight first-class constraints suggests the presence of an eight-parameter gauge symmetry in the theory, while only a four-parameter gauge group corresponding to changing the coordinates on the surface W^4 is present in the initial theory. This means that the additional symmetry transformations must act on the additional variables (the Lagrange multipliers).

It can be conjectured that cleverly fixing the arising additional gauge freedom, i. e., imposing some special conditions on the Lagrange multipliers arising in the equations of motion, results in these equations becoming the embedding theory equations and correspondingly, by the satisfaction of the constraint equations, Einstein's equations. In the next section, we show that this is true.

It is interesting that a simple calculation shows that the quantity

$$\pi^{lm} = -\frac{1}{2}\pi^a\alpha_a^{lm} \quad (76)$$

is canonically conjugate to the three-dimensional metric ${}^3g_{ik}$. If we confined ourself to considering only the dynamics of quantities composed from ${}^3g_{ik}$ and π^{lm} , then by virtue of formulas (73),(74), we would drop the term in Hamiltonian (75) that is proportional to Φ_0 and replace Φ_i with $-\mathcal{H}_i$. As a result, the Hamiltonian would become

$$\begin{aligned} H &= \int d^3x \left((N_i - \lambda_i)\mathcal{H}^i + N_0\mathcal{H}^0 \right) = \\ &= \int d^3x \left(2(N_i - \lambda_i)\sqrt{-g}{}^3\nabla_k \left(\frac{\pi^{ik}}{\sqrt{-g}} \right) + 2N_0 \left(\frac{\pi^{ik}\hat{L}_{ik,lm}\pi^{lm}}{\sqrt{-g}} - \sqrt{-g}{}^3R \right) \right), \end{aligned} \quad (77)$$

in which form it reduces to a combination of only four constraints rather than eight and coincides exactly with the known expression for the Hamiltonian in the Arnowitt-Deser-Misner formalism.

5. The action for the embedding theory with additional Einstein's constraints

We construct the action corresponding to Hamiltonian (75). For this, we calculate the quantity \dot{y}^a :

$$\dot{y}^a = \frac{\delta H}{\delta \pi_a} = \lambda^i c_i^a + \lambda^0 w^a + \alpha^{a,ik} {}^3\nabla_i N_k + \frac{N_0}{\sqrt{-g}} \alpha^{a,ik} \hat{L}_{ik,lm} \alpha_b^{lm} \pi^b. \quad (78)$$

Contracting this equality with the quantity ${}^3b_{a,pr}$ and using its properties and formulas (67), (69), we find the relation

$$\pi^a \alpha_a^{ik} = \frac{\sqrt{-g}}{N_0} L^{ik,lm} \left(\dot{y}_a {}^3b_{ik}^a - \frac{1}{2} \left({}^3\nabla_i N_k + {}^3\nabla_k N_i \right) \right). \quad (79)$$

Substituting relations (78) and (79) in the formula relating the Lagrangian and Hamiltonian of the theory, we can easily obtain the expression for the desired action,

$$\begin{aligned}
S &= \int dx^0 \left(\int d^3x \pi_a \dot{y}^a - H \right) = \int d^4x \left(\frac{N_0}{2\sqrt{-g^3}} \pi^a \alpha_a^{ik} \hat{L}_{ik,lm} \alpha_b^{lm} \pi^b + 2N_0 \sqrt{-g^3} R \right) = \\
&= \int d^4x \sqrt{-g^3} \left[\frac{1}{2N_0} \left(\dot{y}_a b_{ik}^a - \frac{\nabla_i N_k + \nabla_k N_i}{2} \right) L^{ik,lm} \left(\dot{y}_b b_{lm}^b - \frac{\nabla_l N_m + \nabla_m N_l}{2} \right) + \right. \\
&\qquad \qquad \qquad \left. + 2N_0 R \right]. \quad (80)
\end{aligned}$$

We see that together with the initial theory variables y^a , the Lagrange multipliers N_i and N_0 enter this action as additional independent variables. Hence, these variables also enter the equations of motion obtained from action (80). The result is that such equations of motion do not exactly reproduce initial equations (5) of the embedding theory.

Equations of motion corresponding to action (80) impose no restrictions on the time evolution of the variables N_i, N_0 because they are Lagrange multipliers. These variables can be assigned arbitrarily chosen values using the additional gauge transformations discussed above.

Comparing expression (80) with initial action (57), we can easily see that they coincide under the conditions

$$N_i = 0, \quad N_0 = \frac{1}{2} \sqrt{\dot{y}^a \Pi_{ab}^3 \dot{y}^b} = \frac{1}{2\sqrt{g^{00}}} \quad (81)$$

(equality (56) is used here) on the Lagrange multipliers N_i and N_0 . This means that if we impose conditions (81), thus partially fixing the gauge freedom in the equations of motion obtained from action (80), then these equations become embedding theory equations (5) supplied with Einstein's constraints (the latter appear after varying the action over the variables N_i, N_0). As shown above, if the initial data are in the general position, then this set of equations is equivalent to Einstein's equations.

We can therefore conclude that a theory equivalent to Einstein's GR can be obtained by partially fixing the gauge freedom in the generalized embedding theory with action (80), which has the eight-parameter gauge symmetry.

It is interesting that action (80) up to nonintegral terms can be written in the form of the initial GR action

$$S = \int d^4x \sqrt{-g'} R(g'), \quad (82)$$

if we here substitute for the metric $g'_{\mu\nu}$ not its induced expression (4) but the modification of it

$$g'_{ik} = \overset{3}{g}_{ik} = \partial_i y^a \partial_k y_a, \quad g'_{0k} = \partial_0 y^a \partial_k y_a - N_k, \quad g'_{00} = 4N_0^2 + g'_{0i} \overset{3}{g}^{ik} g'_{0k}, \quad (83)$$

whence we obtain $g'^{00} = \frac{1}{4N_0^2}$. To see this, we must rewrite action (82) in form (53) replacing $g_{\mu\nu}$ with $g'_{\mu\nu}$ and using the formula $g' = g'/g'^{00}$:

$$S = \int d^4x \sqrt{-g'} \left[\frac{1}{\sqrt{g'^{00}}} K_{ik}(g') L^{ik,lm}(g') K_{lm}(g') + \frac{1}{\sqrt{g'^{00}}} \overset{3}{R}(g') \right]. \quad (84)$$

Using the known relation

$$K_{ik} = \frac{\sqrt{g^{00}}}{2} \left(-\partial_0 g_{ik} + \overset{3}{\nabla}_i g_{0k} + \overset{3}{\nabla}_k g_{0i} \right), \quad (85)$$

formulas (83), and the equality

$$K_{ik} = \sqrt{g^{00}} \overset{3}{y}_a \overset{3}{b}_{ik}^a, \quad (86)$$

which follows from (38),(40), we can see that

$$K_{ik}(g') = \frac{1}{2N_0} \left(\overset{3}{y}_a \overset{3}{b}_{ik}^a - \frac{\overset{3}{\nabla}_i N_k + \overset{3}{\nabla}_k N_i}{2} \right). \quad (87)$$

Substituting this expression in (84) and taking formulas (83) and the fact that $L^{ik,lm}(g') = L^{ik,lm}(g)$ and $\overset{3}{R}(g') = \overset{3}{R}(g)$ (where g is the induced metric) into account, we can easily see that the result coincides with formula (80).

As stated above, the considered theory with the independent variables y^a , N_i , N_0 whose action can be written in form (82), has an eight-parameter gauge symmetry. We see that the quantity $g'_{\mu\nu}$ is invariant under four of these eight transformations; the generators of the former in the canonical formalism are the constraint Φ_0 and the combinations of the constraints $\mathcal{H}_i + \Phi_i$. Those are the transformations that are isometric bendings of surfaces of the constant time W^3 (cf. formulas (73),(74) and the reasoning after them) supplied with the corresponding transformations of the variables N_i, N_0 . We can also show that the quantity $g'_{\mu\nu}$ behaves as a tensor under the remaining four transformations, which just results in action (82) being invariant under these transformations.

After gauge conditions (81) are imposed, the quantity $g'_{\mu\nu}$ coincides with the induced metric. Therefore, it satisfies Einstein's equations in this gauge (this is true if the initial data are in the general position; see above). But because this quantity is invariant under the transformations that we use to reduce arbitrary values of variables to those restricted by gauge conditions (81), it satisfies Einstein's equations even if we do not impose gauge conditions (81). Therefore, we can in principal consider the quantity $g'_{\mu\nu}$ to be the metric, which is invariant under additional symmetry transformations and coincides with the induced metric only in gauge (81).

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References

- [1] *M. Janet.* Ann. Soc. Math. Pol. 1926. V. 5. P. 38.
- [2] *E. Kartan.* Ann. Soc. Pol. Math. 1927. V. 6. P. 1.
- [3] *Sh. Kobayashi, K. Nomizu.* Foundations of Differential Geometry. Vols. 1, 2. Wiley, New York, 1963, 1969.

- [4] *A. Friedman*. J. Math. Mech. 1961. V. 10. P. 625.
- [5] *T. Regge, C. Teitelboim*. General relativity a la string: a progress report. In Proceedings of the First Marcel Grossmann Meeting, Trieste, Italy, 1975. Ed. R. Ruffini, North Holland, Amsterdam, 1977. P. 77.
- [6] *S.A. Paston, V.A. Franke*. The gravity as a theory of embedding of space-time into the flat space of higher dimensions. In: "Proceedings of the 15 International V.A. Fock school for advances of physics 2005". Ed. by V. Novozhilov. Publishing house of St.Petersburg State University, St.Petersburg. 2006. P. 34.
- [7] *S. Deser, F.A.E. Pirani, D.C. Robinson*. Phys. Rev. D. 1976. V. 14. P. 3301.
- [8] *V. Tapia*. Clas. Quantum Grav. 1989. V. 6. P. L49.
- [9] *R. Arnowitt, S. Deser, C.W. Misner*. "Gravitation: an introduction to current research", Louis Witten ed., Wiley, 1962. Chapter 7. P. 227-265. gr-qc/0405109.
- [10] *V.A. Franke, V. Tapia*. Nuovo Cimento B. 1992. V. 107. N 6, P. 611.
- [11] *M. Pavsic, V. Tapia*. gr-qc/0010045.