# Singularity Structure and Stability Analysis of the Dirac Equation on the Boundary of the Nutku Helicoid Solution

T.Birkandan* and M. Hortaçsu* [1]

* Istanbul Technical University, Department of Physics, Istanbul, Turkey.

November 1, 2021

**Abstract**

Dirac equation written on the boundary of the Nutku helicoid space consists of a system of ordinary differential equations. We tried to analyze this system and we found that it has a higher singularity than those of the Heun's equations which give the solutions of the Dirac equation in the bulk. We also lose an independent integral of motion on the boundary. This facts explain why we could not find the solution of the system on the boundary in terms of known functions. We make the stability analysis of the helicoid and catenoid cases and end up with an appendix which gives a new example where one encounters a form of the Heun equation.

PACS: 04.62.+v, 02.30.Hq

[1] E-mail addresses: hortacsu@itu.edu.tr, birkandant@itu.edu.tr

# 1 Introduction

Although one usually needs only different forms of the hypergeometric equation or its confluent forms to describe many different phenomena in theoretical physics, functions with higher singularity structure are seen more and more in the literature [1-12]. A common form is the Heun function [13], which is studied extensively in the books by Ronveaux and Slavyanov et al [14][15], the seminal book by Ince [16] as well as in several articles [17-20]. Although for linear equations both Ince and Slavyanov et al end their singularity analysis with Heun type functions, sometimes equations with even more singularities are needed for relatively simple situations, which lack some symmetries.

As an example of such a case, here we study the singularity structure of the Dirac equations, written in the background of the Nutku helicoid solution[21], restricted to the boundary of the helicoid. This metric was formerly studied by Lorenz-Petzold [22]. One can study the scalar field in this background and obtain the propagator in a closed form [23], thanks to four integrals of motion allowed by the metric. One can also write the Dirac equation and obtain the solutions in terms of Mathieu functions [24-26]. One needs to study the eigenvalue problem on the boundary to impose the boundary conditions in this problem. The similar problem in the bulk, using partial differential equations, can be solved in terms of known functions. On the boundary, we get a system of ordinary differential equations. At first glance, this system seems to be easier to analyze. When we investigate the system further, we find that this system has higher form of singularities and fewer integrals of motion. These turn out to be the reasons why we cannot express the solution in terms of the functions cited in Ince's book[16].

In this note, we comment on the symmetries of the new problem and study the singularity structure of the new system. We show that one gets an equation of higher form of singularity. Since we do not have closed solutions, we then use stability analysis to see if this system describes a stable system. To our surprise we find that although the answer is not affirmative for the helicoid case, we get a limit cycle for the related catenoid solution.

Below we summarize our results. In an appendix, we show a new example where one encounters a form of the Heun equation. This is the solution of the laplacian in the background of the Eguchi-Hanson solution [27], trivially extended to five dimensions.

# 2 Our Analysis

The Nutku helicoid metric is given as

$$
\begin{aligned}
ds^2 &= \frac{1}{\sqrt{1+\frac{a^2}{r^2}}}[dr^2 + (r^2+a^2)d\theta^2 + \left(1+\frac{a^2}{r^2}\sin^2\theta\right)dy^2 \\
&\quad -\frac{a^2}{r^2}\sin 2\theta dydz + \left(1+\frac{a^2}{r^2}\cos^2\theta\right)dz^2]. \qquad (1)
\end{aligned}
$$

where $0 < r < \infty$, $0 \leq \theta \leq 2\pi$, $y$ and $z$ are along the Killing directions and will be taken to be periodic coordinates on a 2-torus [23]. This is an example of a multi-center metric. This metric reduces to the flat metric if we take $a = 0$.

$$
ds^2 = dr^2 + r^2 d\theta^2 + dy^2 + dz^2. \qquad (2)
$$

If we make the following transformation

$$
r = a \sinh x, \qquad (3)
$$

the metric is written as

$$
\begin{aligned}
ds^2 &= \frac{a^2}{2}\sinh 2x(dx^2 + d\theta^2) \\
&\quad + \frac{2}{\sinh 2x}[(\sinh^2 x + \sin^2\theta)dy^2 \qquad (4) \\
&\quad - \sin 2\theta dydz + (\sinh^2 x + \cos^2\theta)dz^2].
\end{aligned}
$$

The solutions of the Dirac equation, written in the background of the Nutku helicoid metric, can be expressed as a special form of Heun functions [25][26]. We could reduce the double confluent Heun function obtained for the radial equation to the Mathieu function with coordinate transformations. Mathieu function is a related but much more studied function with similar singularity structure. In the work cited above, [25][26], we tried to get the solution of the little Dirac equation, the name used for the equation restricted to a boundary of the helicoid. We needed this solution also to be able to calculate the index of the differential operator. For a fixed value of the radial coordinate we had only a coupled system of ordinary differential equations, which, in general, should be much simpler to solve than the coupled system of partial differential equations obtained for the full Dirac equation, written for the bulk. We were successful to obtain the solution in this latter case in terms of Mathieu functions. We were not able to identify the solutions for the little Dirac equation , though.

The first thing we check is whether we lose any of the three Killing vectors and one Killing tensor. For the metric in question, one has three integrals of motion, [23] namely $p_y, p_z$ and $g^{\mu\nu}p_\mu p_\nu = \mu^2$, namely

$$
\left(\frac{dS_x}{dx}\right)^2 + \left(\frac{dS_\theta}{d\theta}\right)^2 + a^2\left(p_y^2 + p_z^2\right)\sinh^2 x - \frac{\mu^2 a^2}{2}\sinh 2x + a^2\left(\cos\theta\, p_y + \sin\theta p_z\right)^2 = 0, \qquad (5)
$$

and an extra integral of motion, the Killing tensor [23] ,

$$K = -p_\theta^2 - a^2(cos\theta p_y + sin\theta p_z)^2 \tag{6}$$

which gives us the angular equation for a fixed value of the constant $\lambda$ . See eq.s (46) and (47) of [23]. When one restricts the solution to a fixed value of the radial coordinate, the value of $g^{\mu\nu}p_\mu p_\nu$ is not an independent constant of motion from the other two Killing vectors and the Killing tensor.
Trying to see the effect of this result on our problem, we investigate the singularity structure of the equation we get for the little Dirac operator. Here we study the simplest case, where the eigenvalue $\lambda$ is equal to zero, since the answer to our problem is already apparent here. For this case, instead of getting a system of four coupled equations we get coupling only between two of them at a time. When we analyze the system by reducing them to a second order equation for a single dependent variable, we find an operator with two irregular and one regular singularities, which is one more than allowed for the equations considered among the Heun functions. The double confluent Heun function, the solution one obtains for the full Dirac equation, has two irregular singularities, missing the extra regular singularity of the case studied here.
To make our discussion concrete we explicitly perform the calculation in the next section.

### 2.1 Singularities

The Dirac equation written in the background of the Nutku helicoids metric is written as

$$(\partial_x + i\partial_\theta)\Psi_3 \ + iak[cos(\theta - \phi + ix)]\Psi_4 = 0, \tag{7}$$

$$(\partial_x - i\partial_\theta)\Psi_4 \ - iak[cos(\theta - \phi - ix)]\Psi_3 = 0, \tag{8}$$

$$(-\partial_x + i\partial_\theta)f_1 \ + iak[cos(\theta - \phi + ix)]f_2 = 0, \tag{9}$$

$$(-\partial_x - i\partial_\theta)f_2 \ - iak[cos(\theta - \phi - ix)]f_1 = 0. \tag{10}$$

These equations have simple solutions [24] which can also be expanded in terms of products of radial and angular Mathieu functions [28][25]. Problem arises when these solutions are restricted to boundary [26].
To impose these boundary conditions we need to write the little Dirac equation, the Dirac equation restricted to the boundary, where the variable $x$ takes a fixed value $x_0$. We choose to write the equations in the form,

$$\frac{\sqrt{2}}{a}\{i\frac{d}{d\theta}\Psi_3 \ + ikacos(\theta - \phi + ix_0)\Psi_4\} = \lambda f_1, \tag{11}$$

$$\frac{\sqrt{2}}{a}\{-i\frac{d}{d\theta}\Psi_4 \ - iakcos(\theta - \phi - ix_0)\Psi_3\} = \lambda f_2, \tag{12}$$

$$\frac{\sqrt{2}}{a}\{-i\frac{d}{d\theta}f_1 \ - iakcos(\theta - \phi + ix_0)f_2\} = \lambda\Psi_3, \tag{13}$$

$$\frac{\sqrt{2}}{a}\{i\frac{d}{d\theta}f_2 \ + iakcos(\theta-\phi-ix_0)f_1\} = \lambda\Psi_4. \tag{14}$$

Here $\lambda$ is the eigenvalue of the little Dirac equation. We take $\lambda = 0$ as the simplest case. The transformation

$$\Theta = \theta - \phi - ix_0 \tag{15}$$

can be used. Then we solve $f_1$ in the latter two equations in terms of $f_2$:

$$-\frac{d^2}{d\Theta^2}f_2 - \tan\Theta\frac{d}{d\Theta}f_2 + \frac{(ak)^2}{2}[\cos(2\Theta)\cosh(2x_0) - i\sin(2\Theta)\sinh(2x_0) + \cosh(2x_0)]f_2 = 0 \tag{16}$$

When we make the transformation

$$u = e^{2i\Theta}, \tag{17}$$

the equation reads,

$$\{4(u+1)u[u\frac{d^2}{du^2} + \frac{d}{du}] - 2iu(u-1)\frac{d}{du} + \frac{(ak)^2}{2}(u+1)[ue^{-2x_0} + \frac{1}{u}e^{2x_0} + \cosh(2x_0)]\}f_2 = 0. \tag{18}$$

This equation has irregular singularities at $u = 0$ and $\infty$ and a regular singularity at $u = -1$. If we try a solution in the form $\sum\limits_{n=-\infty}^{\infty} a_n u^n$ around the irregular singularity $u = 0$ we end up with a four-term recursion relation as

$$\begin{aligned} &a_{n-1}[4(n^2-2n+1) - 2i(n-1) + \frac{(ak)^2}{2}(\frac{3}{2}e^{-2x_0} + \frac{1}{2}e^{2x_0})] \\ &+a_n[4n^2 + 2in + \frac{(ak)^2}{2}(\frac{3}{2}e^{-2x_0} + \frac{1}{2}e^{2x_0})] + \\ &a_{n-2}[\frac{(ak)^2}{2}e^{-2x_0}] + a_{n+1}[\frac{(ak)^2}{2}e^{2x_0}] \end{aligned} \tag{19}$$

As it is known, in the Heun equation case, this kind of series solution gives a three-term relation [29].
If we search for a solution of the Thomé type we may try a solution of the form $f_2 = e^{\frac{A}{\sqrt{u}}}g(u)$. This form does not allow us to get a Taylor series expansion around the irregular point $u = 0$ [5] [30].
If we try a series solution around the regular singularity at $u = -1$ as $\sum\limits_{n=0}^{\infty} a_n(u+1)^{n+\alpha}$ we find a relation between five consecutive coefficients for the solution. Therefore, we may conclude that the solution of this equation cannot be written in terms of Heun functions or simplier special functions.
To check this further, we first set the coefficient of $\frac{1}{u}$ term in equation 18 equal to zero to change our irregular singularity at zero to a regular one. Then we keep this term and discard the $ue^{-2x_0}$ term to reduce the singularity structure of infinity. In both cases one can check that the solution can be expressed in terms

of confluent Heun functions. This shows that reducing one of the singularities yields a Heun function. Thus, we conclude that the full equation 18 is not one of the better known equations in the literature, which are included in the computer packages like Maple, cited in the seminal book by Ince [16].
To investigate the type of our equation we try to get a confluent form of a new equation,

$$y''(z)+(\frac{1-\mu_0}{z}+\frac{1-\mu_1}{z+1}+\frac{1-\mu_2}{z-a})y'(z)+\frac{\beta_0+\beta_1 z+\beta_2 z^2}{z^2(z-a)}y(z)=0 \quad (20)$$

with regular singularities at 0, $-1$ and $a$ and an irregular singularity at infinity. This equation differs from the generalized Heun equation [7][17]:

$$y''(z)+(\frac{1-\mu_0}{z}+\frac{1-\mu_1}{z+1}+\frac{1-\mu_2}{z-a}-\alpha)y'(z)+\frac{\beta_0+\beta_1 z+\beta_2 z^2}{z(z+1)(z-a)}y(z)=0 \quad (21)$$

which also has regular singularities at 0, $-1$ and $a$ and an irregular singularity at infinity. These two equations both have four-term recursion relations. They, however, have different singularity ranks according to the classification given in [31]. When we put $a=0$ in equation 20, we get

$$y''(z)+(\frac{2-\mu_0-\mu_2}{z}+\frac{1-\mu_1}{z+1})y'(z)+\frac{\beta_0+\beta_1 z+\beta_2 z^2}{z^3}y(z)=0. \quad (22)$$

We get a singularity structure as a regular singularity at $-1$ and two irregular singularities at zero and infinity like the equation 18.
Both equations 18 and 22 have four-term recursion relations when a Laurent power series solution is attempted. We may name this equation as the confluent form of the equation 20. It is in the same form as our original equation rewritten as

$$\{\frac{d^2}{du^2}+[\frac{1}{u}(1+\frac{i}{2})+\frac{i}{u+1}]\frac{d}{du}+\frac{(ak)^2}{2}[\frac{e^{-2x_0}}{u}+\frac{e^{2x_0}}{u^3}+\frac{\cosh(2x_0)}{u^2}]\}f_2=0. \quad (23)$$

Both of these equations have s-rank multisymbols $\{1,\frac{3}{2},\frac{3}{2}\}$ referring to the singularities at $\{-1,0,\infty\}$ [31].
We could not obtain a confluent equation similar to the equation 18 from the generalized Heun equation 21. If we simply put $a=0$ in this equation we get the confluent Heun solution. We can obtain an equation with the same singularity structure as our equation only if we write the equation,

$$y''(z)+(\frac{1-\mu_0}{z}+\frac{1-\mu_1}{z+1}+\frac{1-\mu_2}{z-a})y'(z)+\frac{\frac{\beta_{-1}}{z}+\beta_0+\beta_1 z+\beta_2 z^2}{z(z+1)(z-a)}y(z)=0, \quad (24)$$

and make $a$ approach zero. Then we end up with an equation having the same singularity structure as in equation 18.
If we want to compare equation 18 with equation 21 we have to form a confluent form of the latter equation. To coalesce the singularities at zero, we make a

detour and then use standart techiques [32]. We first translate the singularity at zero to a singularity at $b$,

$$y''(z)+(\frac{1-\mu_0}{z-b}+\frac{1-\mu_1}{z+1}+\frac{1-\mu_2}{z-a}-\alpha)y'(z)+\frac{\beta_0+\beta_1 z+\beta_2 z^2}{(z-b)(z+1)(z-a)}y(z)=0 \quad (25)$$

This equation has regular singularities at $z=-1,a,b$ and an irregular singularity at infinity. We make the transformation $z=\frac{1}{v}$. Then, the equation 25 becomes,

$$\begin{aligned} & y''(v)+(\frac{2-(1-\mu_0)-(1-\mu_1)-(1-\mu_2)}{v}+\frac{\alpha}{v^2} \\ & -\frac{1-\mu_0}{\frac{1}{b}-v}-\frac{1-\mu_1}{v+1}-\frac{1-\mu_2}{\frac{1}{a}-v})y'(v) \\ & +\frac{\frac{\beta_0}{v}+\frac{\beta_1}{v^2}+\frac{\beta_2}{v^3}}{(1-bv)(1+v)(1-av)}y(v)=0 \end{aligned} \quad (26)$$

We set $\mu_0=-\mu_1=1/b=2/a=\epsilon$ and take the limit $\epsilon\to\infty$. Then we transform back to the original variables using $v=\frac{1}{z}$ to obtain

$$y''(z)+(\frac{3-\mu_1}{z}-\alpha-\frac{\mu_1-1}{z(z+1)}+\frac{1}{z^2})y'(z)+\frac{\beta_0+\beta_1 z+\beta_2 z^2}{z^2(z+1)}y(z)=0 \quad (27)$$

This equation is a confluent form of the equation 25. It has a regular singularity at $-1$ and two irregular singularities at zero and infinity and a four-way recursion relation when expanded around zero using a Laurent expansion. This equation has s-rank multisymbols $\{1,2,2\}$ referring to singularities at $\{-1,0,\infty\}$ [31]. Here we get the rank-2 irregular singularities at zero and infinity only from the coefficient of the first derivative whereas in equation 18 and in equation 22 the coefficient of the term without derivatives gives us these singularities. They also have different ranks. Even if we set $\alpha=0$ in equation 27 then we get rank=$\{1,2,\frac{3}{2}\}$ which is different from the rank of our original equation. Hence, our equation 18 may be a confluent form only of a variation of the equation 21, like equation 24.

## 2.2 Stability analysis for the little Dirac Equation

The little Dirac equation is a system of linear differential equations with periodic coefficients. Then we can write the system as [33][34]:

$$\partial_\theta\psi=P(\theta)\psi \quad (28)$$

Here $P(\theta+\tau)=P(\theta)$ and $\tau$ is the period of the coefficients. ($\tau\neq 0$). According to Bellman, if $\psi(0)=I$ , we can write

$$\psi(\theta)=Q(\theta)e^{B\theta.} \quad (29)$$

The matrix $Q(\theta)$ is also periodic with the period $\tau$. Then we have

$$\begin{aligned}\psi(\theta+\tau) &= Q(\theta+\tau)e^{B\theta}e^{B\tau} \\ &= Q(\theta)e^{B\theta}e^{B\tau} \\ &= \psi(\theta)e^{B\tau}.\end{aligned} \tag{30}$$

We can define $e^{B\tau} \equiv C$ and use

$$C = \psi(\theta)^{-1}\psi(\theta+\tau) \tag{31}$$

to obtain $C$. The Jordan normal form of $C$, $T$ being the transformation matrix,

$$C = T \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_r \end{pmatrix} T^{-1} \tag{32}$$

gives us the $B$ matrix:

$$B = \frac{1}{\tau} \ln L \tag{33}$$

The eigenvalues of $B$ give us the characteristic roots. We will use these characteristic roots with different parameters in our stability analysis. The characteristic roots are given by $\alpha_i$, $(i = 1..4)$.

Table 1. The change in the characteristic roots with respect to parameters (helicoid case)

| $a$ | $k$ | $x_0$ | $\lambda$ | real parts of the characteristic roots ($\times 2\pi$) | signs of the characteristic roots |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | $\alpha_{1,2,3,4} = 9.25029$ | $++--$ |
| 0.5 | ” | ” | ” | $\alpha_{1,2,3,4} = 4.03926$ | ” |
| 0.8 | ” | ” | ” | $\alpha_{1,2,3,4} = 7.2163$ | ” |
| 1.1 | ” | ” | ” | $\alpha_{1,2,3,4} = 10.2542$ | ” |
| 1.5 | ” | ” | ” | $\alpha_{1,2,3,4} = 14.2215$ | ” |
| 1 | 0.5 | 1 | 1 | $\alpha_{1,2,3,4} = 5.8327$ | ” |
| ” | 0.8 | ” | ” | $\alpha_{1,2,3,4} = 7.75421$ | ” |
| ” | 1.1 | ” | ” | $\alpha_{1,2,3,4} = 10.0319$ | ” |
| ” | 1.5 | ” | ” | $\alpha_{1,2,3,4} = 13.2745$ | ” |
| 1 | 1 | 0.5 | 1 | $\alpha_{1,2,3,4} = 6.56206$ | ” |
| ” | ” | 0.8 | ” | $\alpha_{1,2,3,4} = 7.91586$ | ” |
| ” | ” | 1.1 | ” | $\alpha_{1,2,3,4} = 10.0632$ | ” |
| ” | ” | 1.5 | ” | $\alpha_{1,2,3,4} = 14.4648$ | ” |
| 1 | 1 | 1 | 0.5 | $\alpha_{1,2,3,4} = 8.30164$ | ” |
| ” | ” | ” | 0.8 | $\alpha_{1,2,3,4} = 8.81342$ | ” |
| ” | ” | ” | 1.1 | $\alpha_{1,2,3,4} = 9.49249$ | ” |
| ” | ” | ” | 1.5 | $\alpha_{1,2,3,4} = 10.5888$ | ” |

Our calculations indicate that $f_1$ and $f_2$ solutions are not stable (positive characteristic root) while, $\Psi_3$ and $\Psi_4$ solutions are stable (negative characteristic root). As it is seen in Table 1, when we keep all the other parameters constant and vary only $a$, the value of the roots are influenced most, whereas the effect of the variation in the value of $\lambda$ changes the value of the roots least. We also find that when these parameters exceed unity in absolute value, we encounter inconsistencies in the numerical values. The separation between consecutive roots increase and some negative roots go to positive values for large values of the parameters. If we keep the values of the parameters in the range $[-1, 1]$, we seem to have no such problems.

The periodicity of the defined $Q$ can be checked using numerical means.

We use

$$\psi(\theta + \tau) = Q(\theta + \tau)e^{B\theta}C \tag{34}$$

and for $\theta = 0$, $\psi(\theta) = Q(\theta)e^{B\theta}$ to give,

$$\psi(\tau)C^{-1} = Q(\tau) = Q(0) = \psi(0) = I. \tag{35}$$

We check numerically that this equation is satisfied; hence, $Q$ is periodic.

For the catenoid case, one replaces $a$ with $ia$ in the metric. The same stability procedure is performed for this case and we find the characteristic roots given in the Table 2.

Table 2. The change in the characteristic roots with respect to parameters (catenoid case)

| $a$ ($\times$ $i$) | $k$ | $x_0$ | $\lambda$ | characteristic roots ($\times 2\pi$) | signs of the characteristic roots |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | $\alpha_{1,2} = -5.84543 \times 10^{-8} + 2.08426i$<br>$\alpha_{3,4} = -6.40967 \times 10^{-8} + 0.72266i$ | $+-+-$ |
| 0.5 | " | " | " | $\alpha_{1,2} = -2.40553 \times 10^{-7} - 1.19724i$<br>$\alpha_{3,4} = -2.49225 \times 10^{-7} - 2.3721i$ | $-+-+$ |
| 0.8 | " | " | " | $\alpha_{1,2} = -5.63046 \times 10^{-7} - 0.233835i$<br>$\alpha_{3,4} = -5.78178 \times 10^{-7} - 2.42696i$ | $-+-+$ |
| 1.1 | " | " | " | $\alpha_{1,2} = -7.82856 \times 10^{-7} + 2.9982i$<br>$\alpha_{3,4} = -7.88358 \times 10^{-7} - 0.186969i$ | $+-+-$ |
| 1.5 | " | " | " | $\alpha_{1,2} = -7.53539 \times 10^{-7} - 0.526163i$<br>$\alpha_{3,4} = -7.7255 \times 10^{-7} - 2.35978i$ | $-+-+$ |
| 1 | 0.5 | 1 | 1 | $\alpha_{1,2} = 3.49081 \times 10^{-8} + 0.661356i$<br>$\alpha_{3,4} = 2.02794 \times 10^{-8} - 1.31726i$ | $+--+$ |
| " | 0.8 | " | " | $\alpha_{1,2} = -4.04957 \times 10^{-7} + 0.864554i$<br>$\alpha_{3,4} = -4.22823 \times 10^{-7} + 2.3638i$ | $+-+-$ |
| " | 1.1 | " | " | $\alpha_{1,2} = -2.2577 \times 10^{-7} + 2.7328i$<br>$\alpha_{3,4} = -2.30929 \times 10^{-7} + 0.120058i$ | $+-+-$ |
| " | 1.5 | " | " | $\alpha_{1,2} = -1.33916 \times 10^{-7} + 0.715427i$<br>$\alpha_{3,4} = -1.38904 \times 10^{-7} - 2.75007i$ | $+--+$ |
| 1 | 1 | 0.5 | 1 | $\alpha_{1,2} = -4.68625 \times 10^{-8} - 0.232525i$<br>$\alpha_{3,4} = -5.34771 \times 10^{-8} - 2.02431i$ | $-+-+$ |
| " | " | 0.8 | " | $\alpha_{1,2} = -4.69901 \times 10^{-7} - 1.06977i$<br>$\alpha_{3,4} = -4.72986 \times 10^{-7} - 2.11709i$ | $-+-+$ |
| " | " | 1.1 | " | $\alpha_{1,2} = -6.06425 \times 10^{-7} - 2.74153i$<br>$\alpha_{3,4} = -6.14188 \times 10^{-7} - 0.143986i$ | $-+-+$ |
| " | " | 1.5 | " | $\alpha_{1,2} = -4.26375 \times 10^{-7} + 0.299098i$<br>$\alpha_{3,4} = -4.4214 \times 10^{-7} + 1.55822i$ | $+-+-$ |
| 1 | 1 | 1 | 0.5 | $\alpha_{1,2} = -4.67773 \times 10^{-7} + 0.761149i$<br>$\alpha_{3,4} = -4.69627 \times 10^{-7} + 0.741915i$ | $+-+-$ |
| " | " | " | 0.8 | $\alpha_{1,2} = -2.82256 \times 10^{-7} - 1.5208i$<br>$\alpha_{3,4} = -2.86571 \times 10^{-7} - 0.79892i$ | $-+-+$ |
| " | " | " | 1.1 | $\alpha_{1,2} = -4.14479 \times 10^{-7} + 2.38047i$<br>$\alpha_{3,4} = -4.1985 \times 10^{-7} + 0.650337i$ | $+-+-$ |
| " | " | " | 1.5 | $\alpha_{1,2} = -3.33914 \times 10^{-7} - 2.62503i$<br>$\alpha_{3,4} = -3.42067 \times 10^{-7} - 0.145628i$ | $-+-+$ |

We see that the real parts of these roots are compatible with assigning to value zero within numerical errors. This corresponds to a limit cycle[35][36].

## 3 Conclusion

Here we performed a systematic analysis of the Dirac equation restricted to the boundary when it is written in the background of the Nutku helicoid solution

[23] We find that the resulting system of ordinary differential equations has a singularity which is higher than those of the Heun functions which are solutions for the bulk. We also lose an independent integral of motion. This fact explains why we could not obtain the solution of the system on the boundary in terms of well known functions.
The stability analysis we performed shows that although this system is not stable, a related system, the catenoid solution is. We can, thus, give a meaning to its solutions, although we can not get explicit solutions for the little Dirac equation obtained from it too.

# 4 Appendix: Scalar field in the background of the extended Eguchi-Hanson solution

To go to five dimensions, we can add a time component to the Eguchi-Hanson metric [27] so that we have

$$ds^2 = -dt^2 + \frac{1}{1-\frac{a^4}{r^4}}dr^2 + r^2(\sigma_x^2+\sigma_y^2) + r^2(1-\frac{a^4}{r^4})\sigma_z^2 \tag{36}$$

where

$$\sigma_x = \frac{1}{2}(-\cos\xi d\theta - \sin\theta\sin\xi d\phi) \tag{37}$$

$$\sigma_y = \frac{1}{2}(\sin\xi d\theta - \sin\theta\cos\xi d\phi) \tag{38}$$

$$\sigma_z = \frac{1}{2}(-d\xi - \cos\theta d\phi). \tag{39}$$

This is a vacuum solution.
If we take

$$\Phi = e^{ikt}e^{in\phi}e^{i(m+\frac{1}{2})\xi}\varphi(r,\theta), \tag{40}$$

we find the scalar equation as

$$\begin{aligned} H\varphi(r,\theta) &= (\frac{r^4-a^4}{r^2}\partial_{rr} + \frac{3r^4+a^4}{r^3}\partial_r + k^2r^2 + \frac{4a^4m^2}{a^4-r^4} + \\ &\quad 4\partial_{\theta\theta} + 4\cot\theta\partial_\theta + \frac{8mn\cos\theta - 4(m^2+n^2)}{\sin^2\theta})\varphi(r,\theta). \end{aligned} \tag{41}$$

If we take $\varphi(r,\theta) = f(r)g(\theta)$, the solution of the radial part is expressed in terms of confluent Heun ($H_C$) functions.

$$f\left(r\right) = \left(-a^4+r^4\right)^{\frac{1}{2}m}\ H_C\left(0,m,m,\frac{1}{2}\,k^2a^2,\frac{1}{2}\,m^2-\frac{1}{4}\,\lambda-\frac{1}{4}\,k^2a^2,\,\frac{a^2+r^2}{2a^2}\right)$$

$$+\left(a^2+r^2\right)^{-\frac{1}{2}m}\left(r^2-a^2\right)^{\frac{1}{2}m}H_C\left(0,-m,m,\frac{1}{2}k^2a^2,\frac{1}{2}m^2-\frac{1}{4}\lambda-\frac{1}{4}k^2a^2,\frac{a^2+r^2}{2a^2}\right) \tag{42}$$

The angular solution is in terms of hypergeometric solutions.

$$g(\theta)=\frac{1}{\sin\theta}\{\sqrt{2-2\cos(\theta)}\left(\frac{1}{2}\cos(\theta)-\frac{1}{2}\right)^{\frac{1}{2}m}\left(\frac{1}{2}\cos(\theta)-\frac{1}{2}\right)^{-\frac{1}{2}n}$$

$$[(2\cos(\theta)+2)^{\frac{1}{2}-\frac{1}{2}n-\frac{1}{2}m}$$

$$\times {}_2F_1([-n+\frac{1}{2}\sqrt{\lambda+1}+\frac{1}{2},-n-\frac{1}{2}\sqrt{\lambda+1}+\frac{1}{2}],[1-n-m],\frac{1}{2}\cos(\theta)+\frac{1}{2}) \tag{43}$$

$$+(2\cos(\theta)+2)^{\frac{1}{2}+\frac{1}{2}n+\frac{1}{2}m}$$

$$\times {}_2F_1([m+\frac{1}{2}\sqrt{\lambda+1}+\frac{1}{2},m-\frac{1}{2}\sqrt{\lambda+1}+\frac{1}{2}],[1+n+m],\frac{1}{2}\cos(\theta)+\frac{1}{2})]\}$$

If the variable transformation $r=a\sqrt{\cosh x}$ is made, the solution can be expressed as

$$f(x)=\frac{1}{\sinh x}\{(\sinh(x))^{m+1}H_C\left(0,m,m,\frac{1}{2}k^2a^2,\frac{1}{2}m^2-\frac{1}{4}\lambda-\frac{1}{4}k^2a^2,\frac{1}{2}\cosh(x)+\frac{1}{2}\right)$$

$$+(2\cosh(x)+2)^{-\frac{1}{2}m+\frac{1}{2}}(2\cosh(x)-2)^{\frac{1}{2}m+\frac{1}{2}} \tag{44}$$

$$\times H_C\left(0,-m,m,\frac{1}{2}k^2a^2,\frac{1}{2}m^2-\frac{1}{4}\lambda-\frac{1}{4}k^2a^2,\frac{1}{2}\cosh(x)+\frac{1}{2}\right)\}.$$

We tried to express the equation for the radial part in terms of $u=\frac{a^2+r^2}{2a^2}$ to see the singularity structure more clearly. Then the radial differential operator reads

$$4\frac{d^2}{du^2}+4\left(\frac{1}{u-1}+\frac{1}{u}\right)\frac{d}{du}+k^2a^2\left(\frac{1}{u-1}+\frac{1}{u}\right)+\frac{m^2}{u^2(1-u)^2}. \tag{45}$$

This operator has two regular singularities at zero and one, and an irregular singularity at infinity, the singularity structure of the confluent Heun equation. This is different from the hypergeometric equation, which has regular singularities at zero, one and infinity.

**Acknowledgement**: We would like to thank Prof. Ayşe Bilge for correspondence and discussions. This work is supported by TÜBİTAK, the Scientific and Technological Council of Turkey. The work of M.H. is also supported by TÜBA, the Academy of Sciences of Turkey.

## References


[1] S.A. Teukolsky, Physical Review Letters **29**, 1114 (1972)

[2] E.W. Leaver, Proceedings of Royal Society London A **402**, 285 (1985)

[3] A.H. Wilson, Proceedings of Royal Society **A118**, 617-647 (1928)

[4] B.D.B. Figueiredo, math-phys/0509013, also Journal of Mathematical Physics **46** (11), 113503 (2005)

[5] S.Y. Slavyanov, *Asymptotic Solutions of the One-dimensional Schrodinger Equation* (Leningrad University Press) (in Russian) (1991), Translation into English: S.Y. Slavyanov, *Asymptotic Solutions of the One-dimensional Schrodinger Equation* (Amer. Math. Soc. Trans. of Math. Monographs) 151 (1996)

[6] G. Siopsis, hep-th/0407157, Nuclear Physics **B715**, 483-498 (2005)

[7] D. Batic, H. Schmid, M. Winklmeier, gr-qc/0607017, Journal of Physics A: Math. General **39**, 12559 (2006)

[8] P.P. Fiziev, Classical and Quantum Gravity **23**, 2447 (2006), gr-qc/0509123

[9] P.P. Fiziev, gr-qc/0603003

[10] D. Batic, H. Schmid, Journal of Mathematical Physics **48**, 042502 (2007)

[11] R. Manvelyan et al., hep-th/0001179, Nuclear Physics **B579**, 177-208 (2000)

[12] T. Oota, Y. Yasui, Nuclear Physics **B742**, 275 (2006)

[13] K. Heun, Mathematische Annalen **33**, 161 (1889)

[14] A. Ronveaux (ed.), *Heun's Differential Equations* (Oxford University Press) (1995)

[15] S.Y. Slavyanov, W. Lay, *Special Functions, A Unified Theory Based on Singularities*(Oxford University Press) (2000)

[16] E.L. Ince, *Ordinary Differential Equations* (Dover Publications) (1926,1956)

[17] R. Schafke, D. Schmidt, SIAM Journal of Mathematical Analysis **11**, 848 (1980)

[18] R.S. Maier, math.CA/0408317

[19] R.S. Maier, math.CA/0203264, also Journal of Differential Equations **213**, 171-203 (2005)

[20] N. Gurappa, P.K. Panigrahi, Journal of Physics A: Math. General **37** L605-L608 (2004)

[21] Y. Nutku, Physical Review Letters **77**, 4702 (1996)

[22] D. Lorenz-Petzold, Journal of Mathematical Physics **24**, 2632 (1983)

[23] A.N. Aliev, M. Hortaçsu, J. Kalaycı , Y. Nutku, Classical and Quantum Gravity **16**, 631 (1999)

[24] Y. Sucu, N. Ünal, Classical and Quantum Gravity **21**, 1443 (2004)

[25] T. Birkandan, M. Hortaçsu, Journal of Physics A: Math. Theor. **40**, 1 (2007)

[26] T. Birkandan, M. Hortaçsu, Journal of Mathematical Physics **48**, 092301 (2007)

[27] T. Eguchi, A.J. Hanson, Physics Letters **74B**, 249 (1978)

[28] L. Chaos-Cador, E. Ley-Koo, Revista Mexicana de Fisica **48**, 67 (2002)

[29] A. Ronveaux (ed.), *Heun's Differential Equations* (Oxford University Press) p.155 (1995)

[30] F.W.J. Olver, *Asyptotics and Special Functions* (Academic Press) (1974)

[31] A. Ronveaux (ed.), *Heun's Differential Equations* (Oxford University Press) p.293 (1995)

[32] P. Dennery, A. Krzywicki, *Mathematics for Physicists* (Dover Publications) (1967)

[33] R. Bellman, *Stability Theory of Differential Equations* (McGraw-Hill) (1953)

[34] E.A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations* (Krieger Pub Co.) (1984)

[35] W. Hurewicz, *Lectures on Ordinary Differential Equations* (Dover Publications) (2002)

[36] S. Lefschetz, *Differential Equations: Geometric Theory* (Dover Publications) (2005)