Gravitational Lensing Analyzed by Graded Refractive Index of Vacuum

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We found strong similarities between the gravitational lensing and the conventional optical lensing. The similarities imply a graded refractive index description of the light deflection in gravitational field. We got a general approach to this refractive index in a static spherically symmetric gravitational field and obtained its exterior and interior solutions exactly through the general relativity. In weak field case, the two solutions come to a simple unified exponential function of the gravitational potential. With these results, the gravitational lensing can be analyzed in a convenient optical way. Especially, the long puzzling problem of the central image missing can be solved easily. We also pointed out that the graded refraction property of the gravitational spacetime is related to the vacuum influenced by the gravitational matter.

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I. INTRODUCTION

Gravitational lensing is an effect predicted by general relativity [1, 2, 3]. It has been now a powerful tool for the study of the so intractable problems in astrophysics and cosmology such as the value of Hubble constant [4], the physics of quasars [5], the mass and mass distribution of a galaxy or a galaxy cluster[2], the large scale structure of the universe [6, 7], the existence of dark matter [8, 9, 10], the nature of dark energy [11, 12] and so on.

It is told that, the light deflection of a gravitational lens is caused by the gravity or in Einstein's words the curved spacetime, while the light deflection of an optical lens is caused by the variance of refractive index of the medium. But is there any deeper similarity between the two lenses besides the deflection of light? Can we analyze the gravitational lensing as the conventional optical one?

In fact, people have long tried an optical description of the lensing effect. In 1920, Eddington [13] suggested that the light deflection in solar gravitational field can be conceived as a refraction effect of the space in a flat spacetime. The idea was further studied by Wilson [14], Dicke [15], Felice [16], and Nandi *et al*. [17, 18, 19]. Recently, this thought of light deflection has been investigated further by Puthoff [20, 21], Vlokh [22], Ye and Lin [23] etc. In Puthoff's paper, the light deflection is related to the vacuum polarization in gravitational field. Vlokh discussed further this light refraction effect. Ye and Lin used the refractive index to simulate the gravitational lensing.

The purpose of this paper is, on the basis of the general relativity, to find the similarities between the gravitational lensing and the optical lensing, and to analyze the former in a simple optical way. The paper is organized as follows: in Sec. II, we point out the strong similarities between the two lenses and suggest a refractive index analysis of the gravitational lensing; in Sec. III, we find the exact solutions of this refractive index for outside and inside the lens matter system respectively; in Sec. IV, we make a weak field approximation; in Sec. V, we apply the obtained result to the problem of gravitational lensing, especially to the central image missing problem puzzled physicists for a long time [24]; in Sec. VI, we make a discussion on the refraction property of the gravitational spacetime; finally in Sec. VII, we draw our conclusions.

II. SIMILARITIES BETWEEN THE TWO LENSES

A. Similarity in Fermat's principle

Landau and Lifshitz have derived from the general relativity the Fermat's principle for the propagation of light in a static gravitational field as follows [25]:

$$\delta \int {g_{00}}^{-1/2} dl = 0, \tag{1}$$

where dl is the length element of the passing light measured by the local observer, g_{00} is a component of the metric tensor $g_{\mu\nu}$, $g_{00}^{-1/2}dl$ corresponds to an element of optical path length. ${g_{00}}^{-1/2} = dt/d\tau$, where $d\tau$ represents the time interval measured by the local observer for a light ray passing through the length dl, while dt is the corresponding time measured by the observer at infinity. Eq. (1) could then be rewritten as

$$\delta \int \frac{dt}{d\tau}\frac{dl}{ds} ds = 0, \tag{2}$$

where ds is the length element measured by the observer at infinity, corresponding to the local length dl.

For an optical lens, the light propagation satisfies the conventional Fermat's principle

$$\delta \int n ds = 0, \tag{3}$$

where n is the refractive index of the medium.

The similarity between Eq. (2) and Eq. (3) indicates that if we set the scale of length and time at infinity as a standard scale for the whole gravitational space and time, the light

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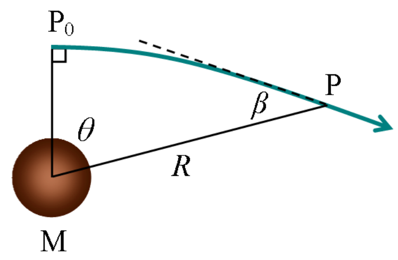

FIG. 1: Light deflection in a gravitational lens.

propagating in a gravitational lens could then be regarded as that in an optical medium with the refractive index being

$$n = \frac{dt}{d\tau}\frac{dl}{ds}, \tag{4}$$

where $dt/d\tau$ relates to the curved time and dl/ds relates to the curved space.

B. Similarity in light deflection formula

First we consider the light deflection in a gravitational lens. For a static and spherically symmetric lens matter system, the metric has the standard form

$$d\mathcal{T}^2 = B(R)c^2dt^2 - A(R)dR^2 - R^2(d\theta^2 + sin^2\theta d\phi^2), \tag{5}$$

where R is the radial coordinate of the metric.

The light deflection is shown in Fig. 1, where curve P_0P represents the light ray, M is the mass of the lens matter, θ is the angular displacement of the coordinate radius R, β is the angle between the coordinate radius R and the tangent of the ray at point P. The general relativity gives the angular displacement as follows [26, 27, 28]:

$$d\theta = \frac{dR}{R/\sqrt{A(R)}\sqrt{\left[\frac{R/\sqrt{B(R)}}{R_0/\sqrt{B(R_0)}}\right]^2 - 1}}, \tag{6}$$

where R_0 represents the radial coordinate at the nearest point P_0 .

Now we consider the light deflection in a medium of a spherically symmetric refractive index n. According to the Fermat's principle, the light propagation in such a lens satisfies the following relation [29]:

$$nr\sin\beta = \text{constant}, \tag{7}$$

where r, differing from the above said coordinate radius R, is the distance from the light to the center of the lens. The relation can be rewritten as

$$nr\sin\beta = n_0r_0, \tag{8}$$

where r_0 and n_0 represent the radial distance and the refractive index at the point closest to the center respectively.

Since

$$\tan\beta = \frac{rd\theta}{dr}, \tag{9}$$

associating with Eq. (8) reaches

$$d\theta = \frac{dr}{r\sqrt{\left(\frac{nr}{n_0r_0}\right)^2 - 1}}. \tag{10}$$

Eq. (10) and Eq. (6) show another strong similarity between the two lenses, which once more indicates that there could be a refractive index analysis of the gravitational lensing. The similarity tells us that this refractive index can be figured out through the following two equations:

$$\frac{dR}{R/\sqrt{A(R)}} = \frac{dr}{r}, \tag{11}$$

$$n = \frac{R}{r\sqrt{B(R)}}. \tag{12}$$

For the detailed derivation of the above two equations, see the APPENDIX.

III. EXACT SOLUTIONS OF THE REFRACTIVE INDEX

A. Exterior solution

The coefficients $A(R)$ and $B(R)$ in Eqs. (11) and (12) can be obtained from the Schwarzschild solutions. The Schwarzschild exterior solution ($R \geqslant R_L$, R_L is the radial coordinate at the surface of the lens matter system) gives [28]:

$$A(R) = \left(1 - \frac{2GM}{Rc^2}\right)^{-1}, \tag{13}$$

$$B(R) = 1 - \frac{2GM}{Rc^2}, \tag{14}$$

where G is the gravitational constant, c is the velocity of light in vacuum without the influence of gravitational field.

Substituting Eq. (13) into Eq. (11) gives

$$\frac{r}{k} = \frac{\sqrt{1 - \frac{2GM}{Rc^2}} + 1}{\frac{2GM}{Rc^2}} - \frac{1}{2}, \tag{15}$$

where k is a constant. Since in the Schwarzschild metric, the ratio $r/R \to 1$ at infinity, we have

$$k = \frac{GM}{c^2}. \tag{16}$$

Thus Eq. (15) gives

$$r = \frac{R}{2}\left(\sqrt{1 - \frac{2GM}{Rc^2}} + 1 - \frac{GM}{Rc^2}\right), \tag{17}$$

or

$$R = r\left(1 + \frac{GM}{2rc^2}\right)^2. \tag{18}$$

Then through Eqs. (12), (14) and (17), we get the exterior refractive index

$$n = \frac{1}{\frac{1}{2}\left(\sqrt{1-\frac{2GM}{Rc^2}} + 1 - \frac{GM}{Rc^2}\right)\sqrt{1-\frac{2GM}{Rc^2}}}, \tag{19}$$

or through Eqs. (12), (14) and (18), we get

$$n = \left(1+\frac{GM}{2rc^2}\right)^3\left(1-\frac{GM}{2rc^2}\right)^{-1}, \tag{20}$$

which is exactly in agreement with that given by Felice in a different way [16].

B. Interior solution

The Schwarzschild interior solution ($R \leqslant R_L$) gives [28]

$$A(R) = \left(1-\frac{2GM(R)}{Rc^2}\right)^{-1}, \tag{21}$$

$$B(R) = \exp\left\{-\int_R^\infty \frac{2G}{R^2c^2}\left[M(R)+\frac{4\pi R^3 p(R)}{c^2}\right]\left[1-\frac{2GM(R)}{Rc^2}\right]^{-1} dR\right\}, \tag{22}$$

where $M(R) = \int_0^R 4\pi R^2 \rho(R) dR$, $\rho(R)$ is the mass density, and for an ordinary lens matter system, the pressure $p(R) = 0$. So

$$B(R) = \exp\left\{-\int_R^{R_L} \frac{2GM(R)}{R^2c^2}\left[1-\frac{2GM(R)}{Rc^2}\right]^{-1} dR\right\} \exp\left\{-\int_{R_L}^\infty \frac{2GM(R_L)}{R^2c^2}\left[1-\frac{2GM(R_L)}{Rc^2}\right]^{-1} dR\right\}. \tag{23}$$

Substituting Eq. (21) into Eq. (11), and considering the relation between the radial coordinates r_L and R_L at the surface determined by Eq. (17), we obtain

$$r = \frac{R_L}{2}\left(\sqrt{1-\frac{2GM(R_L)}{R_Lc^2}} + 1 - \frac{GM(R_L)}{R_Lc^2}\right) \exp\left(-\int_R^{R_L} \frac{dR}{R\sqrt{1-\frac{2GM(R)}{Rc^2}}}\right). \tag{24}$$

Then combining Eqs. (12), (23) and (24) gives the interior refractive index

$$n = n_L \frac{R}{R_L} \exp\left(\int_R^{R_L} \frac{dR}{R\sqrt{1-\frac{2GM(R)}{Rc^2}}}\right) \exp\left(\int_R^{R_L} \frac{GM(R)}{R^2c^2}\left[1-\frac{2GM(R)}{Rc^2}\right]^{-1} dR\right), \tag{25}$$

where

$$n_L = \frac{1}{\frac{1}{2}\left(\sqrt{1-\frac{2GM(R_L)}{R_Lc^2}} + 1 - \frac{GM(R_L)}{R_Lc^2}\right)\sqrt{1-\frac{2GM(R_L)}{R_Lc^2}}}, \tag{26}$$

is the refractive index at the surface.

IV. WEAK FIELD APPROXIMATION

For an ordinary lens matter system, the gravitational field is not extremely strong, i.e., GM/Rc^2 or $GM/rc^2 << 1$, then we have:

The exterior refractive index

$$n \cong \exp\left(\frac{2GM}{rc^2}\right) = \exp\left(\frac{-2P_r}{c^2}\right), \tag{27}$$

where $P_r = -GM/r$ is the gravitational potential at position r outside the lens matter system.

The above expression of graded refractive index has been verified in the problem of light deflection in the solar gravitational field [23], with the result being in agreement with that given by the general relativity [3, 28] and the actual measurements [30].

And the interior refractive index

$$n \cong \exp\left[\frac{2GM(r_L)}{r_Lc^2} + \int_r^{r_L} \frac{2GM(r)}{r^2c^2} dr\right] = \exp\left(\frac{-2P_r}{c^2}\right), \tag{28}$$

where $P_r = -GM(r_L)/r_L - \int_r^{r_L} [GM(r)/r^2] dr$ is the gravitational potential at position r inside the lens matter system.

In general, we have the refractive index profile both outside and inside the lens matter system in weak field case as follows:

$$n = \exp\left(\frac{-2P_r}{c^2}\right). \tag{29}$$

The above result is derived from a single static spherically symmetric lens matter system. For a multi-body system, the total gravitational potential will be the superposition of each potential; therefore, the refractive index can be expressed as

$$\begin{aligned} n &= \exp\left(\frac{-2P_r}{c^2}\right) \\ &= \exp\left[\frac{-2(P_{r1}+P_{r2}+P_{r3}+\cdots)}{c^2}\right] \\ &= n_1 n_2 n_3 \cdots, \end{aligned} \tag{30}$$

where $P_{r1}, P_{r2}, P_{r3}, \cdots$ and $n_1, n_2, n_3, \cdots$ are the gravitational potential and the corresponding refractive index caused by each gravitational body respectively. This expression may be extended to arbitrary distributed matter systems. Fig. 2 shows such an example, where the brighter cyanine represents the higher value of refractive index, and the closed curves are the isolines — the denser the lines, the quicker the change of refractive index.

V. APPLICATION TO GRAVITATIONAL LENSING

Eq. (29) and Eq. (30) are the main results of this paper, which provide a convenient optical way to describe the effect of gravitational lensing. Considering a source S and a lens L of mass M, the light emitted from S is bent due to the gravitational field of the lens. The bent light could be figured out

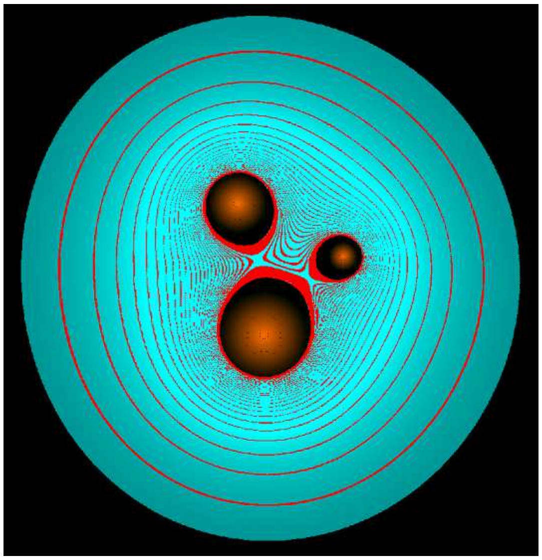

FIG. 2: Refractive index profile of a gravitational lens composed of three celestial bodies of different mass.

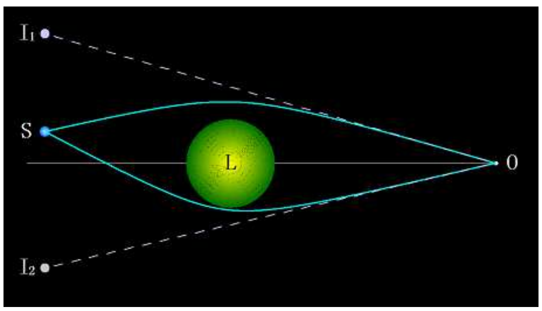

FIG. 3: A computer simulation of gravitational lensing.

through Eq. (7) and Eq. (29). Drawing the extension line of the light from the observer O, the apparent (observed) position of the source image could then be found out. The result is shown in Fig. 3, where I_1, I_2 represent the upper and lower images respectively.

The refractive index given by Eq. (29) could also be applied in studying the formation of the central image, which is predicted by the general relativity but not observed in almost all known cases of gravitational lensing. This problem has puzzled people for many years [24].

For the central imaging, the refractive index profile inside the lens matter system should also be considered. As a model for discussion, we suppose a matter system (for example, a galaxy or a cluster of galaxies) of radius r_L with a density distribution given by

$$\rho = \rho_c \left[1 - \left(\frac{r}{r_L} \right)^k \right], \tag{31}$$

where ρ_c is the central density of the system, $0 \leqslant r \leqslant r_L$, $k > 0$. The density ρ decreases with the distance r from the center of mass; the decreasing rate depends on the parameter k. This model gives the distribution of gravitational potential

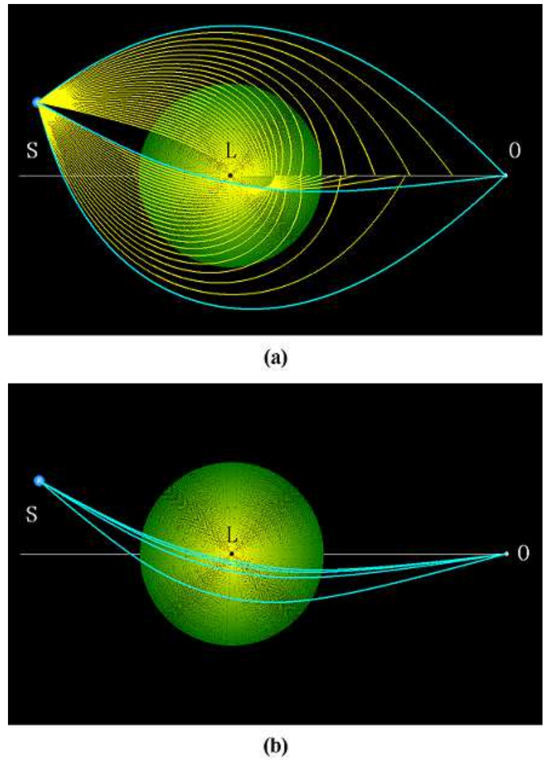

FIG. 4: Ray tracing results for the central imaging. (a) Tracing the central imaging ray for a given lensing system; (b) Tracing the central imaging rays for lenses of different mass.

as

$$P_{ro} = -4\pi\rho_c G \frac{k}{3(3+k)} \frac{r_L^3}{r}, \tag{32}$$

$$P_{ri} = -4\pi\rho_c G \left\{ \frac{k}{2(2+k)} r_L^2 - \left[\frac{1}{6} - \frac{1}{(2+k)(3+k)} \left(\frac{r}{r_L} \right)^k \right] r^2 \right\}, \tag{33}$$

for outside ($r \geqslant r_L$) and inside ($r \leqslant r_L$) the matter system respectively.

The refractive index profile outside and inside the lens matter system then reads

$$n_o = \exp \left[\frac{8\pi\rho_c G}{c^2} \frac{k}{3(3+k)} \frac{r_L^3}{r} \right], \tag{34}$$

$$n_i = \exp \left\{ \frac{8\pi\rho_c G}{c^2} \left\{ \frac{k}{2(2+k)} r_L^2 - \left[\frac{1}{6} - \frac{1}{(2+k)(3+k)} \left(\frac{r}{r_L} \right)^k \right] r^2 \right\} \right\}. \tag{35}$$

Fig. 4 (a) shows a ray tracing result for the imaging of a gravitational lens with the above described refractive index profile. In the figure, only three paths of ray (indicated by the three thick lines) could pass through the observer O, forming the upper, lower and central images respectively. From the figure, we find that, the larger the distance OL from the observer to the lens body, the closer the central imaging ray to the lens center. In the same way, the larger the distance SL from the source to the lens body, the closer the central imaging ray to the lens center. This can be easily seen by interchanging the source and observer in the figure.

Fig. 4 (b) shows the dependence of the ray position of the central image on the lens mass. The lens mass is expressed by $M = \frac{4}{3}\pi r_L^3 \rho_c k/(3+k)$, which can be derived from Eq. (31). The four curved lines in the figure represent respectively the four central imaging rays in four lenses of different mass. The mass ratio of the corresponding lenses is $2:3:4:5$ from bottom to top. We find that, when the mass M increases, the central imaging ray will be closer to the lens center.

This imaging characteristic of Fig. 4 can be used to explain the missing of the central image of gravitational lensing. In practical observations, the distances OL, SL and the mass M are all in astronomical scale; therefore, the light of central imaging is extremely close to the lens center. For a lens matter system denser in the center, the possibility of the central imaging light to be blocked increases. The relatively longer inner path of the central imaging ray adds the possibility of scattering and absorption on the way. Besides, the central image, if still exists, is relatively faint compared with the bright lens core. All these factors lead to little chance of finding the central image. This accounts for the central image missing in almost all observed cases of gravitational lensing.

VI. DISCUSSION ON THE REFRACTION PROPERTY OF THE GRAVITATIONAL SPACETIME

Though we have successfully given a refractive index description of the gravitational lensing and solved a puzzling problem of it, there is still a doubt about the refraction property of the gravitational spacetime as shown in Eq. (4). The doubt can be eliminated if we consider that, around the gravitational matter, there exists a special medium — vacuum, which may have a graded refractive index in gravitational field.

A graded vacuum refractive index needs at least an influence of matter or field on the vacuum. It is exhilarating to see that the recent theoretical and experimental progresses in quantum vacuum have provided a strong support to such an influence. Ahmadi and Nouri-Zonoz pointed out that under the influence of electromagnetic field, vacuum can be polarized, which has led to astonishingly precise agreement between predicted and observed values of the electron magnetic moment and Lamb shift, and may have effects on the photon propagation [31]. Rikken and Rizzo considered the anisotropy of the optical properties of the vacuum when a static magnetic field $\mathbf{B}_0$ and a static electric field $\mathbf{E}_0$ are simultaneously applied perpendicular to the direction of light propagation [32]. They predicted that magnetoelectric birefringence will occur in vacuum under such conditions. They also demonstrated that the propagation of light in vacuum becomes anisotropic with the anisotropy in the refractive index being proportional to $\mathbf{B}_0 \times \mathbf{E}_0$. Dupays *et al.* studied the propagation of light in the neighborhood of magnetized neutron stars. They pointed out that the light emitted by background astronomical objects will be deviated due to the optical properties of quantum vacuum in the presence of a magnetic field [33].

The fact that the light propagation in vacuum can be modified by applying electromagnetic fields to the vacuum indicates that vacuum is actually a special kind of optical medium [31, 33]. The similarity between the vacuum and the dielectric medium implies that vacuum must also have its inner structure, which could be influenced by matter or fields. Actually, the structure of quantum vacuum has been investigated in quite a number of papers recently [34, 35, 36].

Besides the electromagnetic field, the existence of matter can also influence the vacuum. For example, the vacuum inside a microcavity is modified due to the existence of the cavity mirrors, which will alter the zero-point energy inside the cavity and cause an attractive force between the two mirrors known as Casimir effect [37, 38], which has been verified experimentally [39, 40].

The above said makes it natural to suppose that the refractive index of vacuum can be influenced by the gravitational matter. This thought is also supported by the theory of quantum fields, which tells us that particles are actually the excited vacuum. And it is unthinkable that the excited vacuum will exert no influence on the vacuum around.

In brief, we could relate the graded refraction property of the gravitational spacetime to the vacuum influenced by the gravitational matter.

VII. CONCLUSIONS

We have shown two strong similarities between the gravitational lensing and the conventional optical lensing: one is in the Fermat's principle, the other is in light deflection formula. The similarities indicate a graded refractive index analysis of the gravitational lensing. We derived through the general relativity and the Fermat's principle the general expression of this refractive index for a static spherically symmetric gravitational field. From this expression and the Schwarzschild exterior and interior solutions, the exact refractive index profile is obtained. In weak field case, we got a simple unified exponential function of the gravitational potential for the calculation of the refractive index profile outside and inside the lens matter system. Since the derivation is based on the general relativity, there is no inconsonance between the general relativity and our result. By using the obtained result, we investigated the gravitational lensing in a conventional optical way. Some results from computer simulations interpreted the long puzzling problem of the central image missing in a clear way. We also suggested the graded refractive index of the gravitational spacetime be interpreted as that of the gravitational vacuum. We hope our work will be a useful means of gravitational lensing and be a positive stimulus to the vacuum-based investigation of gravitation.

Acknowledgments

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APPENDIX: DERIVATION OF EQS. (11) AND (12)

Suppose the relation between R and r is

$$\frac{dR}{f(R)} = \frac{dr}{r}, \tag{A.1}$$

where $f(R)$ is a function of R.

Now we consider the light path $P_\infty P_0$ in Fig. 5, where P_0 is the closest point to the gravitational center. Combining Eqs. (6), (10) and (A.1), we get

$$\frac{R/\sqrt{A(R)}}{f(R)}\sqrt{\left[\frac{R/\sqrt{B(R)}}{R_0/\sqrt{B(R_0)}}\right]^2 - 1} = \sqrt{\left(\frac{nr}{n_0 r_0}\right)^2 - 1}. \tag{A.2}$$

At the infinite point P_∞, the space-time becomes flat. So we have

$$\sqrt{A(R_\infty)} = 1, \tag{A.3}$$
$$\sqrt{B(R_\infty)} = 1, \tag{A.4}$$
$$\frac{dR}{dr}|_\infty = 1, \tag{A.5}$$
$$\frac{R_\infty}{r_\infty} = 1, \tag{A.6}$$
$$n_\infty = 1. \tag{A.7}$$

According to Eqs. (A.1), (A.3), (A.5) and (A.6), we have

$$f(R_\infty) = R_\infty/\sqrt{A(R_\infty)}. \tag{A.8}$$

Applying Eq. (A.2) to the infinite point P_∞ reads

$$\frac{R_\infty/\sqrt{A(R_\infty)}}{f(R_\infty)}\sqrt{\left[\frac{R_\infty/\sqrt{B(R_\infty)}}{R_0/\sqrt{B(R_0)}}\right]^2 - 1} = \sqrt{\left(\frac{n_\infty r_\infty}{n_0 r_0}\right)^2 - 1}. \tag{A.9}$$

Substituting Eqs. (A.4), (A.6), (A.7), (A.8) into the above equation gives

$$n_0 r_0 = R_0/\sqrt{B(R_0)}. \tag{A.10}$$

Next we consider another light path $P_0 P_0^{'}$ in the gravitational field. Now the closest point to the gravitational center is not P_0 but $P_0^{'}$, then Eq. (A.2) becomes

$$\frac{R/\sqrt{A(R)}}{f(R)}\sqrt{\left[\frac{R/\sqrt{B(R)}}{R_0^{'}/\sqrt{B(R_0^{'})}}\right]^2 - 1} = \sqrt{\left(\frac{nr}{n_0^{'} r_0^{'}}\right)^2 - 1}. \tag{A.11}$$

Similar to Eq. (A.10), we can get

$$n_0^{'} r_0^{'} = R_0^{'}/\sqrt{B(R_0^{'})}. \tag{A.12}$$

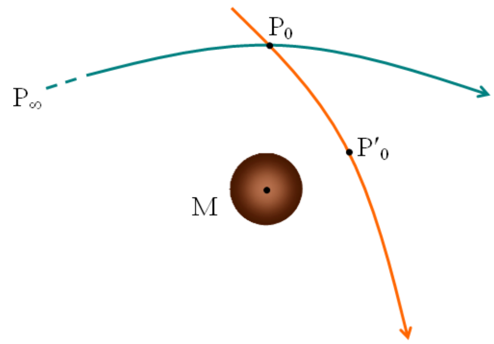

FIG. 5: Two light paths in a gravitational field.

Applying Eq. (A.11) to the point P_0 reads

$$\frac{R_0/\sqrt{A(R_0)}}{f(R_0)}\sqrt{\left[\frac{R_0/\sqrt{B(R_0)}}{R_0^{'}/\sqrt{B(R_0^{'})}}\right]^2 - 1} = \sqrt{\left(\frac{n_0 r_0}{n_0^{'} r_0^{'}}\right)^2 - 1}. \tag{A.13}$$

Combining Eqs. (A.10), (A.12) and (A.13) gives

$$f(R_0) = R_0/\sqrt{A(R_0)}. \tag{A.14}$$

In the same way, we can obtain the following relation for every point P in the gravitational field:

$$f(R) = R/\sqrt{A(R)}. \tag{A.15}$$

Substituting the above equation into Eq. (A.1) gives Eq. (11). Combining Eqs. (6), (10), (A.10) and (11) gives Eq. (12).

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