Semiclassical Horizons

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Abstract: The entropy of apparent horizons is derived using coherent states or semiclassical states in quantum gravity. The leading term is proportional to area for large horizons, and the correction terms differ according to the details of the graph which is used to regularise the quantum gravity phase space variables.

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1 Introduction

A black hole space-time is characterised by parameters like mass, angular momentum and charge. This behaviour is similar to thermodynamic systems like that of a ideal gas, where details of the dynamics of the molecules is averaged over and the entire system is described by macroscopic parameters like temperature, volume, pressure etc. Further, the laws of black hole mechanics are very similar to laws of thermodynamics, including a second law which says the area of horizon increases in any physical process. This led to the conjecture that horizons have entropy like a thermodynamic system, and the entropy is equal to area of the horizon divided by four Planck length squared $(A_H/4l_p^2)$. This conjecture is not proven experimentally though for theoretical consistency, a quantum theory of gravity is searching for a microscopic origin of horizon entropy.

Black holes like the Schwarzschild space-time are vacuum solution of Einstein's equation. One cannot say that these are comprised of fundamental constituents like the gas molecules and the averaging over these degrees of freedom give the macroscopic entropy and temperature. Even if the fundamental structure of space-time is quantised, then it has to be explained why the black hole is not a pure 'condensate' of such fundamental degrees of freedom. Thus it appears that a plausible explanation is the entropy has originated from tracing over a part of the system. There clearly is a loss of information for 'a part of the system' for the classical outside observer, due to the presence of the horizon. In a quantum mechanical description of the black hole space-time, if the outside and the inside of the horizon are described by two different Hilbert spaces, and the outside observer, has access to one of them, one has to trace over the Hilbert space inside the horizon. A density matrix then describes the system outside the horizon. If the two Hilbert spaces inside and outside the horizon are entangled, then the reduced density matrix is a mixed one and a entropy results. Thus the task would be to find a 'quantum mechanical' wavefunction for the horizon and trace over the part of the wavefunction within the horizon. Since horizons are macroscopic and lightcones which determine the causal properties of any space-time and are classically well defined, any attempt to obtain a 'exact quantum black hole' would require non-perturbative 'solitonic' sectors to exist in the quantum theory, or they should be recovered from semi-classical states just as light waves emerge from coherent states which are condensates of photons described by the electromagnetic 'quantum' states.

Loop quantum gravity (LQG) is a approach to quantum gravity, where the canonical metric variables and the extrinsic curvature is redefined as a SU(2) gauge connection, and it's conjugate momentum or electric field. A kinematic Hilbert space can be identified for this theory, though the constraints have to be solved yet. A coherent state can be described in the kinematic Hilbert space, which have all the properties of the usual coherent states including minimum uncertainty, overcompleteness, and peakedness about classical solutions [1, 2, 8]. While this is not a complete description, as the coherent states are defined in the kinematic Hilbert space, certain questions about quantum fluctuations about the classical geometry, resolution of the singularity at the center of the black hole, and correlations across the horizon can be answered.

The LQG Hilbert space uses regularised canonical variables, defined along one dimensional piecewise analytic edges (e) of a graph (Γ). The configuration space variables are the holonomy h_e of a SU(2) gauge connection comprising of the tangent space 'spin connection' and the extrinsic curvature of the spatial slice. The 'dual' momentum P^I_e (I=1,2,3 is the SU(2) index) is the set of triads describing the intrinsic geometry of the spatial slice, smeared in two dimensional surfaces, which the edges intersect. The pair h_e, P^I_e comprise the phase space variables, and one can define a coherent state for each edge in the graph $\psi_t(g_e h_e^{-1})$, peaked at the classical values of the complexified version of these variables $g_e = \exp(iT^I P^I_e) h_e$, where T^I are the generators of SU(2) [2]. The semiclassicality parameter t controls the quantum fluctuations. This parameter is fixed as $t = \frac{l_p^2}{A_H}$ where l_p is the Planck length, and A_H is the area of the horizon associated with a black hole. Thus for large black holes t is small, and the semi-classical approximation is better, and for Planck size black holes, the semiclassicality parameter t is order 1, and hence one has to fully quantise the system in that regime. Once a graph is obtained discretising the entire spatial slice, as observed in [3, 4, 5], the regularised variables are well defined even at the central singularity of the black hole, and smooth across the horizon for a particular spatial slicing.

2 Entropy of Horizons

The coherent state is defined for a graph which when embedded in the classical space-time comprises of edges along the coordinate lines of spherically symmetric axis (r, θ, ϕ). The apparent horizon equation $\nabla_a S^a - K_{ab} S^a S^b -$

$K=0$ (S^a is the normal to the apparent horizon,K_{ab} the extrinsic curvature of the spatial surface and K, the trace of the extrinsic curvature) is re-written in the regularised 'classical variables' is satisfied in the classical limit, and encodes correlation across the horizon [4]. Written in terms of the holonomies at a vertex v_1 outside the horizon, and v_2 inside the horizon, one obtaines

$$4P_{e_\theta}^2\left[\mathrm{Tr}\left(T^J h_{e_\theta}^{-1}V^{1/2}h_{e_\theta}\right)_{v_1}-\mathrm{Tr}\left(T^J h_{e_\theta}^{-1}V^{1/2}h_{e_\theta}\right)_{v_2}\right]\mathrm{Tr}\left(T^J h_{e_\theta}^{-1}V^{1/2}h_{e_\theta}\right)_{v_1}$$
$$-\frac{1}{\sqrt{\beta}}\frac{\partial}{\partial\beta}\mathrm{Tr}\left(T^{I\ \beta}h_{e_\theta}\right)P^I_{e_\theta\ v_1}=0 \tag{1}$$

e_θ denotes a edge along the coordinate lines of θ, β denotes the Immirzi parameter, h_{e_θ} and $P^I_{e_\theta}$ denote the holonomy and momenta along the edge e_θ, one set for a edge beginning/ending at vetex v_1 another set for a edge beginning/ending at vertex v_2. Symbolically it has the following form

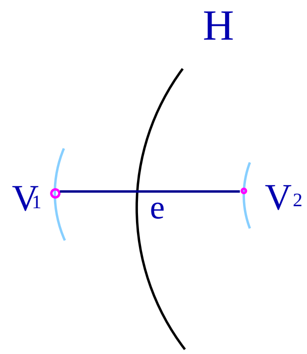
$$\hat{A}[\hat{B}^I(v_1)-\hat{B}^I(v_2)]\hat{C}^I(v_1)-\hat{D}=0 \tag{2}$$

where $\hat{A},\hat{B},\hat{C},\hat{D}$ are operators given as functions of h_e, P^I_e of the angular edges which begin/end at a vertex v_1 outside the horizon and those angular edges which begin/end at a vertex inside the horizon. In the coherent states this can be written as

$$\mathrm{Limit}_{t\to 0} < \Psi|\hat{H}|\Psi >= 0, \tag{3}$$

The expectation value of the horizon operator is thus vanishing only in the semi-classical limit, realised as $t\to 0$. where $\hat{H}=\hat{A}\hat{B}^I(v_1)\hat{C}^I(v_1)-\hat{A}\hat{B}^I(v_2)\hat{C}^I(v_1)-\hat{D}$.

Note that this set of equations show that the coherent state wavefunction within the horizon must be correlated with the coherent state outside the horizon. The correlation was encoded in a conditional probability function to derive what the zeroeth order entropy should be, but the actual computation of the correlations and the density function should be obtained using physical coherent states, a first step towards which has been taken in a recent paper by Thiemann et al [8]



In the above, the graph at the horizon is taken to simply comprise of radial edges linking vertices inside the horizon to those outside the horizon. Further the coherent states of the angular edges (along the θ, ϕ coordinates) ending/beginning at vertices v_1 outside the horizon get correlated with coherent states of angular edges ending/beginning at vertices v_2 inside the horizon. Note v_1 and v_2 are linked by a radial edge e_H. Thus the coherent state is written thus

$$|\Psi >= \prod f(v_1, v_2)|\psi_{v_1} > |\psi_H > |\psi_{v_2} > \tag{4}$$

A tracing over of the edges inside the horizon (or the set of coherent states in the above tensor product state labelled by v_2), gives a mixed density matrix ρ, which in the classical limit is diagonal [5]. The entropy obtained as $-\text{Tr}(\rho \ln \rho)$ of this density matrix, is the number of ways to induce the horizon area as the degeneracy due to the horizon state $|\psi_H >$ contributes to the trace (The degeneracy of a horizon state which induces a area $(j_e + 1/2)t$ is $2j_e + 1$. Thus given a set of radial edges inducing the horizon with area, one simply obtaines the degeneracy associated with the coherent state for those edges, given the total area of the horizon, and a log of that gives the entropy.

Note that this way of deriving entropy of log of the spins at the horizon has been used, and is similar to the 'it for bit' formulation introduced by Bekenstein, however, this is a quantum mechanical derivation of the similar principle using a coherent state wavefunction. Also this differs from the derivations of [7] as there is no additional constraint to restrict the degeneracy of area at the horizon. In my formulation, the horizon wavefunction is free as per the apparent horizon equation (1) in the semiclassical limit.

Thus the degeneracy of the horizon coherent state at the horizon is determined by the number of edges, as well as the spins associated with those edges. This brings us to the *choice of the graph at the horizon*, which will determine the number of ways to induce the horizon area. Having only one graph, with the edges each carrying spin j_e, with the area equally distributed on the horizon sphere one can count the entropy as simply by summing the degeneracy $(2j_e+1)$ associated with each edge. The constraint is that if N is the number of edges, $(j_e+1/2)N = \frac{A_H}{l_p^2}$. The log of the degenracy is thus

$$S_{BH} = \frac{A_H}{l_p^2}\ln(2j_e+1) \tag{5}$$

The entropy calculation is exact, and there are no corrections to the Bekenstein-Hawking term.

In the situation one fixes the number of edges a priori, to be a number N and these edges are allowed to be distributed asymmetrically, i.e. the spins j_e of the edges need not be equal, the constraint is $\sum_{j_e}(j_e+1/2) = \frac{A_H}{l_p^2}$ the entropy is

$$S_{BH} = \frac{A_H}{l_p^2}\left(\frac{3}{2}\ln 3 - \ln 2\right) - \frac{1}{2}\ln(\frac{A_H}{l_p^2}) + .. \tag{6}$$

As seen above, the log area correction appears here with the coefficient 1/2, and this type of correction has been obtained in other derivations of entropy [7]. However as we show below, this is not unique, and entropy corrections will differ if the number of edges is allowed to vary.

We then generalised the case of one graph coherent state to the sum over graphs [6] 'generalised coherent state'. Keeping the spherical symmetry in place, the graph at the horizon can vary as per (i) the number of edges crossing the horizon (ii) the distribution of the edges across the horizon. These different graphs are labeled as 'minimal graphs', as one graph *cannot* be obtained by subdividing the edges of the other graph. The generalised LQG Hilbert space can be written as a direct sum of Hilbert spaces corresponding to each minimal graph.

$$H = \oplus_\Gamma H_\Gamma \tag{7}$$

In a sum over graphs situation, where the number of edges is not fixed a priori, and the (j_e) are arbitrary, subject only to the constraint that $\sum_{j_e}(j_e+1/2) = A_H/l_p^2$, the entropy is not Bekenstein-Hawking anymore. Two different answers are obtained, as per the two restrictions in the tracing procedure to obtain the generalised density matrix. (i)The generalised density matrix is a tensor sum over the density matrices for each Hilbert space, the entropy is given by

$$S_{BH} = (2\frac{A_H}{l_p^2} - 1)\ln 2 + \exp(-(2\frac{A_H}{l_p^2} - 1)\ln 2)\ln(\frac{A_H}{l_p^2}) \tag{8}$$

where, the entropy is Bekenstein-Hawking with corrections. As the area of the horizon increases, the corrections decrease, and the leading term indeed is a log area correction term. However, the complete correction, is decreasing *exponentially in area.*

(ii)The second situation arises, when the entropy is obtained, from a 'generalised coherent state' which is a superposition of all the orthogonal graph Hilbert, coherent states. This superposition allows for transition from one graph-Hilbert space to another. The entropy in this case is

$$S_{BH} = \ln\left(\frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^{2A_H/l_p^2} - \left(\frac{3-\sqrt{5}}{2}\right)^{2A_H/l_p^2}\right) \tag{9}$$

For large areas the leading term is indeed Bekenstein-Hawking, and the corrections are all exponentially decreasing in area.

$$S_{BH} = 2\frac{A_H}{l_p^2}\ln\left(\frac{3+\sqrt{5}}{2}\right) + (6.854)^{-2A_H/l_p^2} + . \tag{10}$$

Thus, even semiclassically, the different ways of counting give different corrections,though the leading term is universally acknowledged to be Bekenstein-Hawking term. In particular for this derivation, the entropy is different, as per the 'coherent state' used to describe the same classical space-time. If the coherent state is taken from one-graph Hilbert space, there are no correction terms (5). If the coherent state is in a superposed state as a sum over different graph coherent states, then the correction term is exponentially decreasing in area (10).

Thus the correction terms and the proportionality constant (which is fixed to $1/4l_p^2$ using the Immirzi parameter, and can be done in this formalism also) in the entropy counting remain ambiguous. A experimental verification of the correction term will determine the specific 'generalised coherent state' the system is in.

3 Quantum fluctuations

In the discussion of the above sections Bekenstein-Hawking entropy is shown to originate from the $t \to 0$ limit of the coherent state. Hence the corrections to the 'classical metric' have not been included in the analysis. What happens when one starts including corrections to the classical metric? Obviously, the horizon is not a impervious membrane any more. The order t corrections are just proportional to t multiplied by the classical values of h_e, P_e^I [3]. The apparent horizon equation is no longer satisfied, and the apparent horizon operator is not vanishing in the coherent states at this order. The apparent horizon operator gets corrected thus:

$$< \psi|\hat{H}|\psi >= t \; F(h(v_1, v_2), P(v_1, v_2)) + O(t^2) \tag{11}$$

where F is a function of the classical holonomy and momenta of edges linked at both the vertices v_1 (outside the horizon) and v_2 (vertices inside the horizon). Thus, information about the phase space variables will start to emerge from behind the horizon. These fluctuations can be included in the classical Hamiltonian, and the density matrix evolved in time. Such time evolutions usually lead to a thermalisation of the density matrix, and hence we are discovering the semi-classical origin of Hawking radiation.

4 Discussions

The perspective described in this paper, where entropy of horizons is attributed to a 'entanglement' entropy when one traces over the semi-classical wavefunction inside the horizon appears to be a correct way to search for the origin of black hole entropy. Further work is in progress to obtain the coherent state in the physical Hilbert space.
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