New Dirac quantum modes in moving frames of the de Sitter spacetime

Ion I. Cotăescu* and Cosmin Crucean†
West University of Timişoara,
V. Pârvan Ave. 4, RO-300223 Timişoara, Romania

November 4, 2018

Abstract

Recently a new time-evolution picture of the Dirac quantum mechanics was defined in charts with spatially flat Robertson-Walker metrics, under the name of Schrödinger picture [I. I. Cotăescu, gr-qc/0708.0734] . In the present paper new Dirac quantum modes are found in moving charts of the de Sitter spacetime using the technical advantages offered by this picture. The principal result is a new set of energy eigenspinors which behave as polarized plane waves and form a complete system of orthonormalized solutions of the free Dirac equation.

Pacs: 04.62.+v

*E-mail: cota@physics.uvt.ro
†E-mail: crucean@physics.uvt.ro

1 Introduction

The quantum modes of the free Dirac field on de Sitter (dS) backgrounds is well-studied in different local charts (or natural frames) and tetrad gauge fixings.

The first solutions of the free Dirac equation on dS backgrounds are the energy eigenspinors found by Otchik [1] using the diagonal gauge in central charts with spherical coordinates (i.e. static charts with spherical symmetry). A special type of rotation-covariant Cartesian gauge in central charts allowed us to express these eigenspinors in terms of the well-known spherical spinors of special relativity [2]. Other interesting solutions in these charts were derived recently under the null tetrad gauge [3].

In dS spacetimes there are moving charts equipped with Cartesian or spherical coordinates whose line elements are of the Robertson-Walker (RW) type. The first spherical wave solutions of the free Dirac equation in moving charts with spherical coordinates were obtained by Shishkin [4]. We have shown that some linear combinations of these solutions are eigenspinors of the scalar momentum operator [5]. Other solutions are the polarized plane waves derived as eigenspinors of the momentum operator in moving frames with Cartesian coordinates and diagonal gauge [6]. We have correctly normalized these solutions in the momentum scale pointing out that these constitute a complete system of orthonormalized spinors [7]. With their help we quantized the Dirac field in canonical manner writing down the expressions of the most important conserved operators of the field theory [7].

Thus we see that in moving frames we know only Dirac modes with well-defined momentum but in which the energy can not be measured exactly since the momentum operators do not commute with the Hamiltonian one [7]. Consequently, an important non-trivial problem remains open, namely that of finding the Dirac *energy* eigenspinors in moving charts of the dS spacetimes. In this paper we would like to solve this problem using the theory of time evolution pictures of the Dirac quantum mechanics in spatially flat RW geometries we have recently proposed [8].

In the non-relativistic quantum mechanics the time evolution can be studied in different pictures (e.g. Schrödinger, Heisenberg, etc.) which transform among themselves through specific time-dependent unitary transformations. In special and general relativity, despite of its importance, the problem of time-evolution pictures is less studied because of some technical difficulties related to the Klein-Gordon equation which has no Hamiltonian form. However, the Dirac quantum mechanics is a convenient framework for introducing different pictures as auxiliary tools since the Dirac equation can be brought in Hamiltonian form at any time. We have shown that at least two time evolution pictures can be identified in the case of the Dirac theory on backgrounds with spatially flat RW metrics. We considered that the *natural* picture (NP) is that in which the free Dirac equation is written directly as it results from its Lagrangean, in a diagonal gauge and Cartesian coordinates [7]. The second one, called the Schrödinger picture (SP), is a new picture where the free Dirac equation is transformed such that its kinetic part takes the same form as in special relativity while the gravitational interaction is separated in a specific term. In this picture we defined the principal operators of our theory obtaining thus the ingredients we need for determining quantum modes [8].

Here we use the SP for deriving new Dirac quantum modes in moving frames of the dS spacetime. First we study new energy eigenspinors which behave as spherical waves in moving frames but describing similar quantum modes as those derived previously in cental frames [2]. Furthermore, we focus on quite new Dirac quantum modes in moving frames whose spinors are polarized plane waves solutions of the free Dirac equation determined by energy, momentum direction and helicity. These plane waves can be normalized in the energy scale (in generalized sense) and form a *complete* system of energy eigenspinors.

We start in the second section with a brief review of the Dirac quantum mechanics in spatially flat RW backgrounds, including our theory of time evolution pictures. In the next section we derive the mentioned new Dirac quantum modes in moving frames of the dS spacetime where the Dirac equation in SP is analytically solvable either in coordinates or even in momentum representation. The spherical waves are obtained separating the spherical coordinates while the polarized plane waves are derived separating variables in momentum representation. The orthonormalization and completeness properties of the plane wave solutions are also deduced.

2 Dirac fields in spatially flat RW spacetimes

In what follows we present the principal time evolution pictures of the Dirac quantum mechanics on spatially flat RW backgrounds, pointing out the technical advantages of using different pictures.

2.1 The Dirac equation in NP

The relativistic quantum mechanics we discuss here is build in local chart with Cartesian or spherical coordinates, x^μ $(\mu,\nu,...=0,1,2,3)$, of a $(1+3)$-dimensional spatially flat RW manifold. These are the proper time $x^0=t$ and either the Cartesian space coordinates x^i $(i,j,...=1,2,3)$ or the associated spherical ones, $r=|\vec{x}|$, θ and ϕ. In these charts the RW line element

$$ds^2=dt^2-\alpha(t)^2(d\vec{x}\cdot d\vec{x})=dt^2-\alpha(t)^2(dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2) \qquad (1)$$

depends only on the arbitrary function α.

In gauge-covariant field theories, the fields with spin half obey equations whose form strongly depends on the choice of the local orthogonal frames and coframes with respect of which the spin operators are defined. The local frames and coframes are given by the tetrad fields $e_{\hat{\mu}}(x)$ and, respectively, $\hat{e}^{\hat{\mu}}(x)$ [9]. These fields are labeled by the local indices $(\hat{\mu},\hat{\nu},...=0,1,2,3)$ of the Minkowski metric $\eta=$diag$(1,-1,-1,-1)$, satisfy $e^{\alpha}_{\hat{\mu}}\hat{e}^{\hat{\mu}}_{\beta}=\delta^{\alpha}_{\beta}$, $e^{\alpha}_{\hat{\mu}}\hat{e}^{\hat{\nu}}_{\alpha}=\delta^{\hat{\mu}}_{\hat{\nu}}$ and give the metric tensor as $g_{\mu\nu}=\eta_{\hat{\alpha}\hat{\beta}}\hat{e}^{\hat{\alpha}}_{\mu}\hat{e}^{\hat{\beta}}_{\nu}$.

The free Dirac field ψ of mass m, seen as a perturbation that does not affect the geometry, satisfies the free Dirac equation which can be easily written in the diagonal gauge where the tetrad fields have the non-vanishing components [6, 4],

$$e_0^0 = 1\,, \quad e_j^i = \frac{1}{\alpha(t)}\delta_j^i\,, \quad \hat{e}_0^0 = 1\,, \quad \hat{e}_j^i = \alpha(t)\delta_j^i\,. \tag{2}$$

In this gauge one obtains the usual form of the Dirac equation in RW spacetimes [6],

$$\left(i\gamma^0\partial_t + i\frac{1}{\alpha(t)}\gamma^i\partial_i + \frac{3i}{2}\frac{\dot{\alpha}(t)}{\alpha(t)}\gamma^0 - m\right)\psi(x) = 0\,, \tag{3}$$

expressed in terms of Dirac γ-matrices [10], with the notation $\dot{\alpha} = \partial_t\alpha$. The Dirac quantum modes are described by solutions of Eq. (3) that behave as tempered distributions or square integrable spinors with respect to the relativistic scalar product [11],

$$\langle\psi, \psi'\rangle = \int_D d^3x \sqrt{g(t)}\,\bar{\psi}(x)\gamma^0\psi'(x)\,, \tag{4}$$

where $\sqrt{g(t)} = \sqrt{|\det g_{\mu\nu}(t)|} = \alpha(t)^3$ play the role of a weight function. The field $\bar{\psi} = \psi^+\gamma^0$ is the Dirac adjoint of ψ and D is the space domain of the chart we use.

The above choice of the local chart and tetrad fields leads to the NP in which the time evolution is governed by the Dirac equation (3) resulted from the standard formalism of the gauge-covariant field theories, without other transformations or artifices. The principal operators of this picture, the energy $\hat{H}$, momentum $\vec{\hat{P}}$ and coordinate $\vec{\hat{X}}$, can be defined as in special relativity,

$$(\hat{H}\psi)(x) = i\partial_t\psi(x)\,, \quad (\hat{P}^i\psi)(x) = -i\partial_i\psi(x)\,, \quad (\hat{X}^i\psi)(x) = x^i\psi(x)\,. \tag{5}$$

The operators $\hat{X}^i$ and $\hat{P}^i$ are time-independent and satisfy the well-known canonical commutation relations

$$\left[\hat{X}^i, \hat{P}^j\right] = i\delta_{ij}I\,, \quad \left[\hat{H}, \hat{X}^i\right] = \left[\hat{H}, \hat{P}^i\right] = 0\,, \tag{6}$$

where I is the identity operator. Other operators are formed by orbital parts and suitable spin parts that may be point-dependent too. In general, the orbital terms are freely generated by the basic orbital operators $\hat{X}^i$ and $\hat{P}^i$. An example is the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ where $\vec{L} = \vec{\hat{X}} \times \vec{\hat{P}}$ and $\vec{S}$ is the spin operator. We specify that the operators $\hat{P}^i$ and J^i are generators of the spinor representation of the group $\tilde{E}(3)$ defined as the universal covering group of the isometry group $E(3)$ of the spatially flat RW manifolds [7]. Therefore, these operators are conserved in the sense that they commute with the Dirac operator [12, 13].

2.2 The Dirac equation in SP

The NP can be changed using point-dependent operators which could be even non-unitary operators since the relativistic scalar product does not have a direct physical meaning as that of the non-relativistic quantum mechanics. We exploited this opportunity for defining the SP in coordinate representation [8] but in this picture the momentum representation is also efficient for studying quantum modes.

Let us start with the coordinate representation where we defined the SP as the picture in which the kinetic part of the Dirac operator takes the standard form $i\gamma^0\partial_t + i\gamma^i\partial_i$ [8]. The transformation $\psi(x) \to \psi_S(x) = W(x)\psi(x)$ leading to the SP is produced by the operator of time dependent dilatations

$$W(x) = \exp\left[-\ln(\alpha(t))(\vec{x}\cdot\vec{\partial})\right]\,, \tag{7}$$

which has the remarkable property [1]

$$W(x)^{\dagger} = \sqrt{g(t)}\,W(x)^{-1}\,, \tag{8}$$

and the following convenient action

$$W(x)F(\vec{x})W(x)^{-1} = F\left(\frac{1}{\alpha(t)}\vec{x}\right)\,, \quad W(x)G(\vec{\partial})W(x)^{-1} = G\left(\alpha(t)\vec{\partial}\right)\,, \tag{9}$$

upon any analytical functions F and G. Performing this transformation we obtain the free Dirac equation of SP

$$\left[i\gamma^0\partial_t + i\vec{\gamma}\cdot\vec{\partial} - m + i\gamma^0\frac{\dot{\alpha}(t)}{\alpha(t)}\left(\vec{x}\cdot\vec{\partial} + \frac{3}{2}\right)\right]\psi_S(x) = 0\,, \tag{10}$$

and the new form of the relativistic scalar product,

$$\langle\psi_S, \psi'_S\rangle = \langle\psi, \psi'\rangle = \int_D d^3x\,\bar{\psi}_S(x)\gamma^0\psi'_S(x)\,, \tag{11}$$

calculated from Eqs. (4) and (8). We observe that this is no more dependent on $\sqrt{g(t)}$, taking the same form from as in special relativity.

The specific operators of SP, denoted by H_S, P^i_S and X^i_S, are defined in usual manner as

$$(H_S\psi_S)(x) = i\partial_t\psi_S(x)\,, \quad (P^i_S\psi_S)(x) = -i\partial_i\psi_S(x)\,, \quad (X^i_S\psi_S)(x) = x^i\psi_S(x)\,, \tag{12}$$

obeying commutation relations similar to Eqs. (6). With their help the Dirac equation (10) can be put in Hamiltonian form,

$$i\partial_t\psi_S(x) = \mathcal{H}_S\psi_S(x) \tag{13}$$

[1] We denote by $(\ )^{\dagger}$ the adjoint operators with respect to the scalar product (4) and by $(\ )^{+}$ the adjoint matrices.

where the Dirac Hamiltonian operator $\mathcal{H}_S = \mathcal{H}_0 + \mathcal{H}_{int}$ has the standard kinetic term $\mathcal{H}_0 = \gamma^0\vec{\gamma}\cdot\vec{P}_S + \gamma^0 m$ and the interaction term with the gravitational field,

$$\mathcal{H}_{int} = \frac{\dot{\alpha}(t)}{\alpha(t)}\left(\vec{X}_S\cdot\vec{P}_S - \frac{3i}{2}I\right) = \frac{\dot{\alpha}(t)}{\alpha(t)}\left(\vec{\hat{X}}\cdot\vec{\hat{P}} - \frac{3i}{2}I\right), \tag{14}$$

that is proportional just to the Huble function $\dot{\alpha}/\alpha$. In these circumstances, we assume that the correct quantum observables are the operators defined by Eqs. (12). Performing the inverse transformation we find that in NP these operators become new interesting time-dependent operators,

$$H(t) \;=\; W(x)^{-1}H_S W(x) = \hat{H} + \frac{\dot{\alpha}(t)}{\alpha(t)}\vec{\hat{X}}\cdot\vec{\hat{P}}\,, \tag{15}$$

$$X^i(t) \;=\; W(x)^{-1}X_S^i W(x) = \alpha(t)\hat{X}^i\,, \tag{16}$$

$$P^i(t) \;=\; W(x)^{-1}P_S^i W(x) = \frac{1}{\alpha(t)}\hat{P}^i\,, \tag{17}$$

which satisfy usual commutation relations as those given by Eqs. (6). The angular operators, $\vec{J}$ and K, as well as the operator (14) have the same expressions in both these pictures since they commute with $W(x)$.

In NP the eigenvalues problem $H(t)f_E(t,\vec{x}) = Ef_E(t,\vec{x})$ of the Hamiltonian operator (15) leads to energy eigenfunctions of the form

$$f_E(t,\vec{x}) = F[\alpha(t)\vec{x}]e^{-iEt} \tag{18}$$

where F is an arbitrary function. This explains why in this picture one can not find energy eigenstates separating variables. However, in SP these eigenfunctions become the new functions

$$f_E^S(t,\vec{x}) = W(x)f_E(t,\vec{x}) = F(\vec{x})e^{-iEt} \tag{19}$$

which have separated variables. This means that in SP new quantum modes could be derived using the method of separating variables in coordinates or even in momentum representation.

3 New energy eigenspinors in moving frames of dS spacetimes

Now we shall see how can be used the SP for solving the mentioned problem of finding Dirac energy eigenspinors in moving frames of the dS spacetime. There are two types of solutions of the Dirac equation, determined by different sets of commuting operators which include the Hamiltonian operator. We show that these behave either as spherical waves or as polarized plane waves.

3.1 New spherical waves

In the particular case of the dS spacetime there is a moving chart $\{t, r, \theta, \phi\}$ with spherical coordinates which has the line element (1) with $\alpha(t) = e^{\omega t}$. Another important chart with spherical coordinates is the central chart $\{t_s, r_s, \theta, \phi\}$ having the line element

$$ds^2 = (1-\omega^2 r_s^2)dt_s^2 - \frac{dr_s^2}{1-\omega^2 r_s^2} - r_s^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{20}$$

In each of these charts the observers which stay at $\vec{x} = 0$ have specific event horizons. Thus an observer A situated at the point $r = 0$ of the moving chart and following the trajectory of ∂_t has an event horizon at $r = 1/\omega$. In the central chart, another observer, B, at $r_s = 0$ with the trajectory along to the direction of the Killing vector field ∂_{t_s}, can observe events up to $r_s = 1/\omega$ [11]. This means that the radial domain of the moving chart where we have to investigate quantum properties may be $D_r = [0, 1/\omega)$.

These observers measure quantum modes of the free Dirac field using not only different coordinates but different local frames too. We assume that B observes the quantum modes of Ref. [2], using the Cartesian gauge defined therein, while in the moving frame the observer A chooses the present diagonal gauge (2). These tetrad gauge fixings are different since the time axis of the local frame of B is along to ∂_{t_s} but the time axis of the local frame of A is along to ∂_t. However, despite of these differences, both these observers recognize the same physical quantities globally defined as conserved operators produced by the Killing vectors of the dS geometry. The Hamiltonian operator (15) is just the conserved operator $H = i\partial_{t_s}$ produced by the time-like Killing vector. In dS moving frames and NP this has the form $H = i\partial_t + \omega\vec{\hat{X}}\cdot\vec{\hat{P}}$ [7] while in SP its action is given by the first of Eqs. (12). Other conserved operators are the components of the total angular momentum $\vec{J}$ which has the same form for both the observers if they keep unchanged the gauge fixings specified above [13, 7]. We note that the operators $\hat{P}^i$ are also conserved and more three other operators fill out the set of ten conserved generators of the spinor representation of the universal covering group of the dS isometry group $SO(1,4)$ [7, 13].

Our purpose is to derive the Dirac energy eigenspinors observed by A as spherical waves in dS moving frames. This can be achieved only in SP where the free Dirac equation,

$$\left[i\gamma^0\partial_t + i\vec{\gamma}\cdot\vec{\partial} - m + i\gamma^0\omega\left(\vec{x}\cdot\vec{\partial} + \frac{3}{2}\right)\right]\psi_S(x) = 0\,, \tag{21}$$

does not depend explicitly on time. Therefore, in this picture we can separate the spherical variables of the Dirac equation as in the problems with spherical symmetry of special relativity. The starting point is the Dirac equation put in the form (13) with the Dirac Hamiltonian operator of the SP written in terms of radial and angular operators as,

$$\mathcal{H}_S = -\frac{i}{r^2}\left[\gamma^0(\vec{\gamma}\cdot\vec{x})\left(\vec{x}\cdot\vec{\partial} + 1\right) + (\vec{\gamma}\cdot\vec{x})\,K\right] + \gamma^0 m - i\omega\left(\vec{x}\cdot\vec{\partial} + \frac{3}{2}\right)\,, \tag{22}$$

where $K = \gamma^0(2\vec{L}\cdot\vec{S}+1)$ is the Dirac angular operator. Furthermore, we have to look for particular solutions of the Dirac equation defined as common eigenspinors of the complete set of commuting operators $\{\mathcal{H}_S, \vec{J}^2, K, J_3\}$ corresponding to the set of eigenvalues $\{E, j(j+1), -\kappa_j, m_j\}$ formed by energy, E, angular quantum numbers, j and m_j, and $\kappa_j = \pm(j+1/2)$ [10]. The particular solutions of particle type are the positive frequency spinors,

$$\begin{aligned} & U^S_{E,\kappa_j,m_j}(t,r,\theta,\phi) \\ & \qquad = \frac{1}{r}\left[f^{(+)}_{E,\kappa_j}(r)\Phi^+_{m_j,\kappa_j}(\theta,\phi) + f^{(-)}_{E,\kappa_j}(r)\Phi^-_{m_j,\kappa_j}(\theta,\phi)\right]e^{-iEt}\,, \end{aligned} \tag{23}$$

which depend on a pair of radial functions, $f^{(\pm)}_{E,\kappa_j}$, and the usual spherical spinors $\Phi^{\pm}_{m_j,\kappa_j}$ which completely solve the angular eigenvalues problem [10]. Taking into account that these spinors are orthonormalized with respect to the angular scalar product [10] we find that the scalar product of two spinors of the form (23) reads

$$\begin{aligned} \left\langle U^S_{E,m_j,\kappa_j}, U^S_{E',m'_j,\kappa'_j}\right\rangle &= \delta_{\kappa_j,\kappa'_j}\delta_{m_j,m'_j} \\ & \times \int_{D_r} dr\left\{[f^{(+)}_{E,\kappa_j}(r)]^* f^{(+)}_{E',\kappa_j}(r) + [f^{(-)}_{E,\kappa_j}(r)]^* f^{(-)}_{E',\kappa_j}(r)\right\}\,. \end{aligned} \tag{24}$$

After the separation of the angular variables of the Dirac equation we are left with the pair of radial equations

$$\left(\pm m - E - \frac{i\omega}{2} - i\omega r\frac{d}{dr}\right) f^{(\pm)}_{E,\kappa_j}(r) = \left(\pm\frac{d}{dr} - \frac{\kappa_j}{r}\right) f^{(\mp)}_{E,\kappa_j}(r) \tag{25}$$

that can be rewritten as

$$\begin{aligned} & \left[(1-\omega^2 r^2)\frac{d}{dr} \pm \frac{\kappa_j}{r} + i\omega r\left(E \mp m + \frac{i\omega}{2}\right)\right] f^{(\pm)}_{E,\kappa_j}(r) \\ & \qquad = \left[\pm E + m + i\omega\left(\kappa_j \pm \frac{1}{2}\right)\right] f^{(\mp)}_{E,\kappa_j}(r)\,. \end{aligned} \tag{26}$$

This system has to be analytically solved in the radial domain D_r looking for tempered distributions corresponding to a continuous energy spectrum. This means that these solutions should not have singularities at $r=0$. When $\kappa_j = j + \frac{1}{2} > 0$ we find regular solutions of the form

$$f^{(\pm)}_{E,\kappa_j}(r) = N^{(\pm)}_{E,\kappa_j}(\omega r)^{j+1\pm\frac{1}{2}} F\left(a_{E,j}\,,\, b_{E,j} \pm \tfrac{1}{2}\,;\, j + \tfrac{3}{2} \pm \tfrac{1}{2}\,;\, \omega^2 r^2\right)\,, \tag{27}$$

where the Gauss hypergeometric functions F depend on the parameters

$$a_{E,j} = -\frac{i}{2\omega}(E-m) + \frac{j}{2} + 1\,, \tag{28}$$

$$b_{E,j} = -\frac{i}{2\omega}(E+m) + \frac{j}{2} + 1\,, \tag{29}$$

while the normalization factors obey the condition

$$\frac{N^{(+)}_{E,\kappa_j}}{N^{(-)}_{E,\kappa_j}} = \frac{E+m}{2(j+1)} + \frac{i\omega}{2}\,. \tag{30}$$

In the case of $\kappa_j = -(j+\frac{1}{2}) < 0$ the regular solutions are

$$f^{(\pm)}_{E,-|\kappa_j|}(r) = N^{(\pm)}_{E,-|\kappa_j|}(\omega r)^{j+1\mp\frac{1}{2}} F\left(b_{E,j}\,,\, a_{E,j}\mp\tfrac{1}{2}\,;\, j+\tfrac{3}{2}\mp\tfrac{1}{2}\,;\, \omega^2 r^2\right)\,, \tag{31}$$

having normalization factors which satisfy

$$\frac{N^{(-)}_{E,-|\kappa_j|}}{N^{(+)}_{E,-|\kappa_j|}} = -\frac{E-m}{2(j+1)} + \frac{i\omega}{2}\,. \tag{32}$$

All these solutions are regular on the domain D_r but for $r \to 1/\omega$ these are divergent. Indeed, the parameters of the above hypergeometric functions, $F(a,b;c;\omega^2 r^2)$, do not satisfy the condition $\Re(c-a-b) > 0$ which assures the convergence of these functions for $\omega r \to 1$ [14]. The next step might be the normalization in the energy scale with respect to the scalar product (24) but, unfortunately, this can not be done since the radial functions are too complicated.

The results obtained in SP must be rewritten in NP where our observer A measures the energy eigenstates. This will see the particle-like energy eigenspinors

$$U_{E,\kappa_j,m_j}(t,r,\theta,\phi) = W(x)^{-1} U^S_{E,\kappa_j,m_j}(t,r,\theta,\phi) = U^S_{E,\kappa_j,m_j}(t,e^{\omega t}r,\theta,\phi)\,, \tag{33}$$

which have no longer separated variables. The antiparticle energy eigenspinors can be derived directly using the charge conjugation as in [7]. All these spinors are regular in the domain where the condition $r_s = re^{\omega t} < 1/\omega$ is fulfilled.

It is important to specify that the quantum modes we derived above (observed by A) and those reported in Ref. [2] (measured by B) are *equivalent*, since in both these cases the solutions of the Dirac equation are eigenspinors of the *same* system of commuting operators, $\{H, \vec{J}^2, K, J_3\}$. For this reason these modes are defined on the same domain of the dS manifold. In this situation, one may ask how these observers could compare their results, concluding that they measure the same quantum modes, when they are using different coordinates and local frames. Obviously, a transformation between these two types of modes must exist but this can not be reduced to a simple coordinate transformation since one has to change simultaneously the positions of the local frames. This can be done with the help of a *combined* transformation involving, beside the coordinate transformation, a suitably associated gauge transformation [13]. Our preliminary calculations indicate that this is a local Lorenz boost of parameter $\mathrm{argtanh}(\omega r_s)$, along the direction of $\vec{x}$. However, details concerning this topics will be discussed elsewhere.

3.2 New polarized plane waves

The above arguments show that the spherical waves we obtained here describe in fact quantum modes discovered formerly in central charts. Therefore, if this would be the only new result due to our SP then the effort of studying new time evolution pictures might appear as useless or at most of academic interest. Fortunately, other quite new energy eigenspinors can be found in this framework. We show that these are polarized plane wave solutions in helicity basis that can be derived in momentum representation.

For solving the Dirac equation (21) in momentum representation we assume that the spinors of the SP may be expanded in terms of plane waves of positive and negative frequencies as,

$$\begin{aligned}\psi_S(x) &= \psi_S^{(+)}(x) + \psi_S^{(-)}(x)\\ &= \int_0^\infty dE \int_{\hat{D}} d^3p\ \left[\hat{\psi}_S^{(+)}(E,\vec{p})\, e^{-i(Et-\vec{p}\cdot\vec{x})} + \hat{\psi}_S^{(-)}(E,\vec{p})\, e^{i(Et-\vec{p}\cdot\vec{x})}\right] \end{aligned} \tag{34}$$

where $\hat{\psi}_S^{(\pm)}$ are spinors which behave as tempered distributions on the domain $\hat{D} = \mathbb{R}_p^3$ such that the Green theorem may be used. Then we can replace the momentum operator $\vec{P}_S$ by $\vec{p}$ and the coordinate operator $\vec{X}_S$ by $i\vec{\partial}_p$ obtaining the free Dirac equation of the SP in momentum representation,

$$\left[\pm E\gamma^0 \mp \vec{\gamma}\cdot\vec{p} - m - i\gamma^0\omega\left(\vec{p}\cdot\vec{\partial}_p + \frac{3}{2}\right)\right]\hat{\psi}_S^{(\pm)}(E,\vec{p}) = 0\,, \tag{35}$$

where E is the energy defined as the eigenvalue of H_S. Denoting $\vec{p} = p\,\vec{n}$ with $p = |\vec{p}|$, we observe that the differential operator of Eq. (35) is of radial type and reads $\vec{p}\cdot\vec{\partial}_p = p\,\partial_p$. Therefore, this operator acts on the functions which depend on p while the functions which depend only on the momentum direction $\vec{n}$ behave as constants.

Following the method of Ref. [7], we should like to derive the fundamental solutions in *helicity* basis using the standard representation of the γ-matrices (with diagonal γ^0) [10]. We start with the general solutions,

$$\hat{\psi}_S^{(+)}(E,\vec{p}) = \sum_\lambda u^S(E,\vec{p},\lambda)\, a(E,\vec{n},\lambda)\,, \tag{36}$$

$$\hat{\psi}_S^{(-)}(E,\vec{p}) = \sum_\lambda v^S(E,\vec{p},\lambda)\, b^*(E,\vec{n},\lambda)\,, \tag{37}$$

involving spinors of helicity $\lambda = \pm\frac{1}{2}$ and the wave functions a and b^* which play the role of constants since they do not depend on p. According to our previous results [7], the spinors of the momentum representation must have the form

$$u^S(E,\vec{p},\lambda) = \begin{pmatrix} \frac{1}{2} f_E^{(+)}(p)\,\xi_\lambda(\vec{n}) \\ \lambda g_E^{(+)}(p)\,\xi_\lambda(\vec{n}) \end{pmatrix}, \tag{38}$$

$$v^S(E,\vec{p},\lambda) = \begin{pmatrix} -\lambda g_E^{(-)}(p)\,\eta_\lambda(\vec{n}) \\ \frac{1}{2} f_E^{(-)}(p)\,\eta_\lambda(\vec{n}) \end{pmatrix}, \tag{39}$$

where $\xi_\lambda(\vec{n})$ and $\eta_\lambda(\vec{n}) = i\sigma_2[\xi_\lambda(\vec{n})]^*$ are the Pauli spinors of helicity basis which satisfy the eigenvalue equations

$$(\vec{n}\cdot\vec{\sigma})\,\xi_\lambda(\vec{n}) = 2\lambda\xi_\lambda(\vec{n})\,, \quad (\vec{n}\cdot\vec{\sigma})\,\eta_\lambda(\vec{n}) = -2\lambda\eta_\lambda(\vec{n})\,, \tag{40}$$

and the othonormalization conditions

$$[\xi_\lambda(\vec{n})]^+\xi_{\lambda'}(\vec{n}) = [\eta_\lambda(\vec{n})]^+\eta_{\lambda'}(\vec{n}) = \delta_{\lambda\,\lambda'}\,. \tag{41}$$

It remains to derive the radial functions solving the system

$$\left[i\omega\left(p\,\frac{d}{dp}+\frac{3}{2}\right)\mp(E-m)\right]f_E^{(\pm)}(p) = \mp p\,g_E^{(\pm)}(p)\,, \tag{42}$$

$$\left[i\omega\left(p\,\frac{d}{dp}+\frac{3}{2}\right)\mp(E+m)\right]g_E^{(\pm)}(p) = \mp p\,f_E^{(\pm)}(p)\,, \tag{43}$$

resulted from Eq. (35). Denoting by $k=\frac{m}{\omega}$ and $\epsilon=\frac{E}{\omega}$, we find the solutions

$$f_E^{(+)}(p) = [-f_E^{(-)}(p)]^* = Cp^{-1-i\epsilon}e^{\pi k/2}H^{(1)}_{\nu_-}(\tfrac{p}{\omega})\,, \tag{44}$$

$$g_E^{(+)}(p) = [-g_E^{(-)}(p)]^* = Cp^{-1-i\epsilon}e^{-\pi k/2}H^{(1)}_{\nu_+}(\tfrac{p}{\omega})\,, \tag{45}$$

expressed in terms of Hankel functions of indices $\nu_\pm = \frac{1}{2}\pm ik$ whose properties are briefly presented in the Appendix A. The normalization constant C has to assure the normalization in the energy scale.

Collecting all the above results we can write down the final expression of the Dirac field (34) as

$$\psi_S(x) = \int_0^\infty dE\int_{S^2}d\Omega_n\,\sum_\lambda\left[U^S_{E,\vec{n},\lambda}(t,\vec{x})a(E,\vec{n},\lambda)\right.$$
$$\left.+V^S_{E,\vec{n},\lambda}(t,\vec{x})b^*(E,\vec{n},\lambda)\right]\,, \tag{46}$$

where the integration covers the sphere $S^2\subset\hat{D}$. The notation U^S and V^S stands for the *fundamental* spinor solutions of positive and, respectively, negative frequencies, with energy E, momentum direction $\vec{n}$ and helicity λ. According to Eqs. (38), (39), (44) and (45), these are

$$U^S_{E,\vec{n},\lambda}(t,\vec{x})$$
$$= iNe^{-iEt}\int_0^\infty s\,ds\left(\begin{array}{c}\frac{1}{2}\,e^{\pi k/2}H^{(1)}_{\nu_-}(s)\,\xi_\lambda(\vec{n})\\ \lambda\,e^{-\pi k/2}H^{(1)}_{\nu_+}(s)\,\xi_\lambda(\vec{n})\end{array}\right)e^{i\omega s\vec{n}\cdot\vec{x}-i\epsilon\ln s}\,, \tag{47}$$

$$V^S_{E,\vec{n},\lambda}(t,\vec{x})$$
$$= iNe^{iEt}\int_0^\infty s\,ds\left(\begin{array}{c}-\lambda\,e^{-\pi k/2}H^{(2)}_{\nu_-}(s)\,\eta_\lambda(\vec{n})\\ \frac{1}{2}\,e^{\pi k/2}H^{(2)}_{\nu_+}(s)\,\eta_\lambda(\vec{n})\end{array}\right)e^{-i\omega s\vec{n}\cdot\vec{x}+i\epsilon\ln s}\,, \tag{48}$$

where we denote the dimensionless integration variable by $s=\frac{p}{\omega}$ and take

$$N = \frac{1}{(2\pi)^{3/2}}\,\frac{\omega}{\sqrt{2}}\,. \tag{49}$$

Then it is not hard to verify that these spinors are charge-conjugated to each other,

$$V^S_{E,\vec{n},\lambda}=(U^S_{E,\vec{n},\lambda})^c=\mathcal{C}(\overline{U}^S_{E,\vec{n},\lambda})^T\,,\quad \mathcal{C}=i\gamma^2\gamma^0\,,\tag{50}$$

and satisfy the orthonormalization relations

$$\begin{aligned}\left\langle U^S_{E,\vec{n},\lambda},U^S_{E,\vec{n}\,',\lambda'}\right\rangle&=\left\langle V^S_{E,\vec{n},\lambda},V^S_{E,\vec{n}\,',\lambda'}\right\rangle\\&=\delta_{\lambda\lambda'}\delta(E-E')\,\delta^2(\vec{n}-\vec{n}\,')\,,\end{aligned}\tag{51}$$

$$\left\langle U^S_{E,\vec{n},\lambda},V^S_{E,\vec{n}\,',\lambda'}\right\rangle=\left\langle V^S_{E,\vec{n},\lambda},U^S_{E,\vec{n}\,',\lambda'}\right\rangle=0\,.\tag{52}$$

deduced as in the Appendix B. However, the most important result is that in SP these fundamental spinors accomplish the condition

$$\begin{aligned}\int_0^\infty dE\int_{S^2}d\Omega_n\sum_\lambda\big\{&U^S_{E,\vec{n},\lambda}(t,\vec{x})[U^S_{E,\vec{n},\lambda}(t,\vec{x}\,')]^+\\&+V^S_{E,\vec{n},\lambda}(t,\vec{x})[V^S_{E,\vec{n},\lambda}(t,\vec{x}\,')]^+\big\}=\delta^3(\vec{x}-\vec{x}\,')\,,\end{aligned}\tag{53}$$

which means that they form a *complete* system of solutions.

The last step is to rewrite all these results in NP where the Dirac field,

$$\psi(x)=\psi_S(t,e^{\omega t}\vec{x})=\int_0^\infty dE\int_{S^2}d\Omega_n\,\sum_\lambda[U_{E,\vec{n},\lambda}(t,\vec{x})a(E,\vec{n},\lambda)\tag{54}$$

$$+V_{E,\vec{n},\lambda}(t,\vec{x})b^*(E,\vec{n},\lambda)]\,,\tag{55}$$

depends on the fundamental solutions in NP,

$$U_{E,\vec{n},\lambda}(t,\vec{x})=U^S_{E,\vec{n},\lambda}(t,e^{\omega t}\vec{x})\,,\quad V_{E,\vec{n},\lambda}(t,\vec{x})=V^S_{E,\vec{n},\lambda}(t,e^{\omega t}\vec{x})\,,\tag{56}$$

which satisfy orthonormalization relations similar to Eqs. (51) and (52) but a different completness relation,

$$\begin{aligned}\int_0^\infty dE\int_{S^2}d\Omega_n\sum_\lambda\big\{&U_{E,\vec{n},\lambda}(t,\vec{x})[U_{E,\vec{n},\lambda}(t,\vec{x}\,')]^+\\&+V_{E,\vec{n},\lambda}(t,\vec{x})[V_{E,\vec{n},\lambda}(t,\vec{x}\,')]^+\big\}=e^{-3\omega t}\delta^3(\vec{x}-\vec{x}\,')\,,\end{aligned}\tag{57}$$

similar to that of Ref. [7].

An interesting problem is to find the set of commuting operators which determine these quantum modes. Apart from the Hamiltonian operator H we must take into account the conserved operator $\vec{\mathcal{N}}$ of the momentum direction, defined in the momentum representation as $(\mathcal{N}^i\hat{\psi}_S)(\vec{p})=n^i\hat{\psi}_S(\vec{p})$, and the operator $\mathcal{W}=\vec{\mathcal{N}}\cdot\vec{S}$ which is a version of the Pauli-Lubanski operator. The set of fundamental spinors (47) and (48) are common eigenspinors of the set of commuting operators $\{H,\vec{\mathcal{N}},\mathcal{W}\}$. More specific, the concrete eigenvalue problems in NP read

$$H\,U_{E,\vec{n},\lambda}=E\,U_{E,\vec{n},\lambda}\,,\qquad H\,V_{E,\vec{n},\lambda}=-E\,V_{E,\vec{n},\lambda}\,,\tag{58}$$

$$\mathcal{N}^i\,U_{E,\vec{n},\lambda}=n^i\,U_{E,\vec{n},\lambda}\,,\qquad \mathcal{N}^i\,V_{E,\vec{n},\lambda}=-n^i\,V_{E,\vec{n},\lambda}\,,\tag{59}$$

$$\mathcal{W}\,U_{E,\vec{n},\lambda}=\lambda U_{E,\vec{n},\lambda}\,,\qquad \mathcal{W}\,V_{E,\vec{n},\lambda}=-\lambda V_{E,\vec{n},\lambda}\,.\tag{60}$$

Hance we get all the ingredients we need for introducing the second quantization in canonical manner as in Ref. [7] and calculate the conserved operators of the quantum field theory. Let us observe that the wave functions a and b^* (which have to become field operators) are the same in both the picture considered here which means that the second quantization is independent on the picture choice. A further paper will be devoted to this problem.

4 Concluding remarks

We derived new Dirac quantum modes in moving frames of the dS spacetime using the SP which allows us to separate the coordinate or momentum variables of the free Dirac equation. This new picture is related to the NP through a non-unitary transformation which preserves the eigenvalues equations but changes the form of the relativistic scalar product of the SP, eliminating from Eq. (11) the weight function $\sqrt{g(t)}$ of the original scalar product (4). However, this is not an impediment since the relativistic scalar product has no more the same immediate physical interpretation as that of the non-relativistic quantum mechanics. Moreover, we believe that the elimination of this weight function is a remarkable advantage since in this way the scalar product (11) becomes just that of special relativity. Nevertheless, if one does not agree with these arguments then one can use our SP only as a tool for finding new solutions, following to carry out the results in NP where the physical interpretation is evident. Anyway, it is obvious that this new picture helps one to derive new quantum modes which never could be found in NP.

The principal result obtained here is the complete system of orthonormalized energy eigenspinors determined in dS moving charts by the set of commuting operators $\{H, \vec{\mathcal{N}}, \mathcal{W}\}$. We remind the reader that in Ref. [7] we found another set of Dirac solutions which form a complete system of orthonormalized eigenspinors of the commuting operators $\{\hat{P}^i, \mathcal{W}\}$ in the same dS chart. Since H does not commute with $\hat{P}^i$ it results that these two sets of quantum modes are quite different. Therefore, we have now a complete theory of free Dirac quantum modes in dS moving frames helping us to understand how can be measured the momentum in states of given energy and, reversely, the energy in states having a well-determined momentum. We hope that in this way a coherent relativistic quantum mechanics has to be built as the starting point to a successful theory of Dirac quantum fields on dS backgrounds.

Finally, we note that the quantum theory on dS manifolds we try to develop takes into account quantum modes *globally* defined as eigenstates of different sets of commuting conserved operators. These can be chosen from the large algebra of conserved observables produced by the high symmetry of the dS geometry. Physically speaking this means that we use a global apparatus providing the same type of measurements on the whole domain of the observer's chart. In our opinion, this attitude does not contradict the general concept of local measurements [11] which is the only possible option when the symmetries are absent and the global apparatus does not work.

Acknowledgments

We are grateful to Mihai Visinescu for interesting and useful discussions on closely related subjects.

Appendix A: Some properties of Hankel functions

According to the general properties of the Hankel functions [14], we deduce that those used here, $H^{(1,2)}_{\nu_\pm}(z)$, with $\nu_\pm=\frac{1}{2}\pm ik$ and $z\in\mathbb{R}$, are related among themselves through

$$[H^{(1,2)}_{\nu_\pm}(z)]^*=H^{(2,1)}_{\nu_\mp}(z)\,, \tag{61}$$

satisfy the equations

$$\left(\frac{d}{dz}+\frac{\nu_\pm}{z}\right)H^{(1)}_{\nu_\pm}(z)=ie^{\pm\pi k}H^{(1)}_{\nu_\mp}(z) \tag{62}$$

and the identities

$$e^{\pm\pi k}H^{(1)}_{\nu_\mp}(z)H^{(2)}_{\nu_\pm}(z)+e^{\mp\pi k}H^{(1)}_{\nu_\pm}(z)H^{(2)}_{\nu_\mp}(z)=\frac{4}{\pi z}\,. \tag{63}$$

Appendix B: Normalization integrals

In spherical coordinates of the momentum space, $\vec{n}\sim(\theta_n,\phi_n)$, and the notation $\vec{p}=\omega s\vec{n}$, we have $d^3p=p^2dp\,d\Omega_n=\omega^3\,s^2ds\,d\Omega_n$ with $d\Omega_n=d(\cos\theta_n)d\phi_n$. Moreover, we can write

$$\delta^3(\vec{p}-\vec{p}\,')=\frac{1}{p^2}\,\delta(p-p')\delta^2(\vec{n}-\vec{n}')=\frac{1}{\omega^3s^2}\,\delta(s-s')\delta^2(\vec{n}-\vec{n}')\,, \tag{64}$$

where we denoted $\delta^2(\vec{n}-\vec{n}')=\delta(\cos\theta_n-\cos\theta_n')\delta(\phi_n-\phi_n')\,.$

Then the scalar products of the fundamental spinors of positive frequencies can be calculated according to Eqs. (47), (49), (63), (41) and (64) as

$$\begin{aligned}\left\langle U^S_{E,\vec{n},\lambda},U^S_{E,\vec{n}\,',\lambda'}\right\rangle &=\int_D d^3x\,[U^S_{E,\vec{n},\lambda}(t,\vec{x})]^+U^S_{E,\vec{n}\,',\lambda'}(t,\vec{x})\\ &=e^{i(E-E')t}\left[\frac{1}{2\pi\omega}\int_0^\infty\frac{ds}{s}\,e^{i(\epsilon-\epsilon')\ln s}\right]\delta_{\lambda\lambda'}\,\delta^2(\vec{n}-\vec{n}\,')\\ &=\delta_{\lambda\lambda'}\delta(E-E')\,\delta^2(\vec{n}-\vec{n}\,')\,. \end{aligned}\tag{65}$$

The properties (51) - (53) are deduced in the same manner.

References

[1] V. S. Otchik, *Class. Quantum Grav.* **2**, 539 (1985);

[2] I. I. Cotăescu, *Mod. Phys. Lett. A* **13**, 2991 (1998).

[3] A. Lopez-Ortega, *Gen. Rel. Grav.* **38**, 743 (2006).

[4] G. V. Shishkin, *Class. Quantum Grav.* **8**, 175 (1991).

[5] I. I. Cotăescu, R. Racoceanu and C. Crucean, *Mod. Phys. Lett. A* **21**, 1313 (2006).

[6] A. O. Barut and I. H. Duru, *Phys. Rev. D* **36**, 3705 (1987); F. Finelly, A. Gruppuso and G. Venturi, *Class. Quantum Grav.* **16**, 3923 (1999).

[7] I. I. Cotăescu, *Phys. Rev. D* **65**, 084008 (2002).

[8] I. I. Cotăescu, `gr-qc/0708.0734`

[9] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[10] B. Thaller, *The Dirac Equation* (Springer Verlag, Berlin Heidelberg, 1992)

[11] N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge 1982).

[12] B. Carter and R. G. McLenaghan, *Phys. Rev. D* **19**, 1093 (1979).

[13] I. I. Cotăescu, *J. Phys. A: Math. Gen.* **33**, 1977 (2000).

[14] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, 1964)