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The principle of least action for test particles in a four-dimensional spacetime embedded in $5D$

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Abstract

It is well known that, in the five-dimensional scenario of braneworld and space-time-mass theories, geodesic motion in $5D$ is observed to be non-geodesic in $4D$. Usually, the discussion is purely geometric and based on the dimensional reduction of the geodesic equation in $5D$, without any reference to the test particle whatsoever. In this work we obtain the equation of motion in $4D$ directly from the principle of least action. So our main thrust is not the geometry but the particle observed in $4D$. A clear physical picture emerges from our work. Specifically, that the deviation from the geodesic motion in $4D$ is due to the variation of the rest mass of a particle, which is induced by the scalar field in the $5D$ metric and the explicit dependence of the spacetime metric on the extra coordinate. Thus, the principle of least action not only leads to the correct equations of motion, but also provides a concrete physical meaning for a number of algebraic quantities appearing in the geometrical reduction of the geodesic equation.

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1 Introduction

The possibility that our world may be embedded in a $(4 + d)$-dimensional universe with more than four large dimensions has attracted the attention of a great number of researchers. There are several motivations for the introduction of large extra dimensions. Among them to resolve the differences between gravity and quantum field theory; provide possible solutions to the hierarchy and the cosmological constant problems [1]-[5] and ultimately unify all forces of nature.

The idea of large extra dimensions is also inspired by the vision that matter in $4D$ is purely geometric in nature. In space-time-matter theory (STM) one large extra dimension is needed in order to get a consistent description, at the macroscopic level, of the properties of the matter as observed in $4D$ [6]-[10]. The mathematical support of this theory is given by a theorem of differential geometry due to Campbell and Magaard [11]-[14].

In the Randall & Sundrum braneworld scenario, STM and other higher-dimensional theories, the main attempt is to reproduce the physics of four-dimensional gravity up to higher-dimensional modifications to general relativity. From a mathematical viewpoint, this means that the equations in $4D$ are projections of the $5D$ equations on $4D$-hypersurfaces orthogonal to some vector field ψ^A, which is identified with the "extra" dimension. By an appropriate choice of coordinates one can remove the spacetime part of this vector and put it in the form

$$\psi_A = (0, 0, 0, 0, \Phi). \tag{1}$$

In such coordinates the most general line element can be written as[1]

$$dS^2 = g_{\mu\nu}(x^\rho, y)dx^\mu dx^\nu + \epsilon\Phi^2(x^\rho, y)dy^2, \tag{2}$$

where $g_{\mu\nu}$ is the metric induced in $4D$; x^μ denote the coordinates of the spacetime; y represents the extra coordinate, and the factor ϵ can be -1 or $+1$ depending on whether the extra dimension is spacelike or timelike, respectively, viz., $\psi_A\psi^A = \epsilon$.

A possible way of testing for new physics coming from extra dimensions is to examine the dynamics of test particles. In practice this means to search for deviations from the universal "free fall" in $4D$. The question of how an observer in $4D$, who is confined to making physical measurements in our ordinary spacetime, perceives the motion of test particles governed by the geodesic equation in $5D$

$$\frac{d^2x^A}{dS^2} + \Gamma^A_{BC}U^B U^C = 0, \quad U^A = \frac{dx^A}{dS}, \tag{3}$$

has widely been discussed in the literature [15]-[26]. The discussion is typically based on the dimensional reduction of geodesics in $5D$, which involves subtle technical details as the choice of adequate affine parameters for the motion in $5D$ and $4D$.

After a long and sophisticated calculation, the dimensional reduction of (3) yields

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = \left(\frac{1}{2}u^\mu u^\rho - g^{\mu\rho}\right)u^\lambda\frac{\partial g_{\rho\lambda}}{\partial y}\left(\frac{dy}{ds}\right) + \epsilon\Phi\left[\Phi^{;\mu} - u^\mu u^\rho\Phi_{;\rho}\right]\left(\frac{dy}{ds}\right)^2, \tag{4}$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols calculated with the spacetime metric $g_{\alpha\beta}$; $ds = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$, and u^μ is the usual four-velocity, $u^\mu = dx^\mu/ds$. The equation for the covariant components u_μ, looks a little simpler, namely,

$$\frac{du_\mu}{ds} - \Gamma^\beta_{\mu\alpha}u^\alpha u_\beta = \frac{1}{2}u_\mu u^\lambda u^\rho\frac{\partial g_{\lambda\rho}}{\partial y}\left(\frac{dy}{ds}\right) + \epsilon\Phi\Phi_{;\rho}\left[\delta^\rho_\mu - u_\mu u^\rho\right]\left(\frac{dy}{ds}\right)^2. \tag{5}$$

The aim of this paper is twofold. First, to derive these equations from the principle of least action. Second, to give the most general expression for the rest mass of a particle observed in $4D$ in terms of the metric and momentum along the extra dimension.

[1] Lowercase Greek letters go from 0 to 3; x^0 is time like, x^1, x^2, x^3 are space like. Capital Latin letters A, B denote indexes in $5D$.

2 The principle of least action

The principle of least action is defined by the statement that for each system there exist an integral I, called the action, which has a minimum value for the actual motion, so that its variation δI is zero [27]-[28]. In classical mechanics, the action for a free material point of mass m is the integral ($c = 1$)

$$I_{Mech} = -m \int_a^b ds, \tag{6}$$

along the world line of the particle between two particular events represented by a and b, for an initial and final position, respectively.

Among the possible effects of extra dimensions, is the variation of the rest mass m. Therefore, we will assume here that the appropriate action is

$$I = -\int_a^b m ds, \tag{7}$$

where the *function* m is taken at points on the world line of the particle. We now proceed to derive the equations of motion from this action.

The principle of least action states

$$\delta I = -\delta \int_a^b m ds = 0. \tag{8}$$

Noting that m as well as the metric are allowed to depend on *all* five coordinates, we have

$$\delta m = \left(\frac{\partial m}{\partial x^\mu}\right) \delta x^\mu + \left(\frac{\partial m}{\partial y}\right) \delta y, \tag{9}$$

and

$$\frac{\delta(ds)}{ds} = \frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial x^\rho} \delta x^\rho + u_\nu \frac{d\delta x^\nu}{ds} + \frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y} \delta y. \tag{10}$$

Substituting these expressions into (8) and using that

$$m u_\mu \frac{d\delta x^\mu}{ds} = \frac{d}{ds}\left(m u_\mu \delta x^\mu\right) - \left(m \frac{du_\mu}{ds} + u_\mu \frac{dm}{ds}\right) \delta x^\mu, \tag{11}$$

we obtain

$$\delta S = -\int_a^b \left\{ \left(\frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial x^\rho} - \frac{du_\rho}{ds} - \frac{u_\rho}{m} \frac{dm}{ds} + \frac{1}{m} \frac{\partial m}{\partial x^\rho}\right) \delta x^\rho + \left(\frac{1}{m} \frac{\partial m}{\partial y} + \frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y}\right) \delta y \right\} m ds. \tag{12}$$

In integrating by parts, we have used the fact that $\delta x^\mu = 0$ at the limits. In view of the arbitrariness of δx^μ and δy, it follows that the integrand is zero, that is

$$\frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial x^\rho} - \frac{du_\rho}{ds} - \frac{u_\rho}{m} \frac{dm}{ds} + \frac{1}{m} \frac{\partial m}{\partial x^\rho} = 0, \tag{13}$$

and

$$\frac{1}{m} \frac{\partial m}{\partial y} + \frac{1}{2} u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y} = 0. \tag{14}$$

Rearranging terms in (13) it can be written as

$$\frac{du_\rho}{ds} - \Gamma^\beta_{\rho\alpha} u^\alpha u_\beta = \frac{1}{m} \frac{\partial m}{\partial x^\rho} - \frac{u_\rho}{m} \frac{dm}{ds}. \tag{15}$$

From which it follows that

$$\frac{du_\rho}{ds} - \Gamma^\beta_{\rho\alpha} u^\alpha u_\beta = -\frac{u_\rho}{m} \frac{\partial m}{\partial y} \frac{dy}{ds} + \frac{1}{m} \frac{\partial m}{\partial x^\mu} \left(\delta^\mu_\rho - u^\mu u_\rho\right). \tag{16}$$

Now using (14) we obtain

$$\frac{du_\rho}{ds} - \Gamma^\beta_{\rho\alpha} u^\alpha u_\beta = \frac{1}{2} u_\rho u^\mu u^\nu \frac{\partial g_{\mu\nu}}{\partial y} \left(\frac{dy}{ds}\right) + \frac{1}{m}\frac{\partial m}{\partial x^\mu}\left(\delta^\mu_\rho - u^\mu u_\rho\right). \tag{17}$$

Except for the second term in the r.h.s, this equation is *identical* to (5). Notice that so far we have not specified the function m.

In order to recover (5), the spacetime derivative of m should satisfy the relation[2]

$$\frac{1}{m}\frac{\partial m}{\partial x^\mu} = \epsilon \Phi \frac{\partial \Phi}{\partial x^\mu}\left(\frac{dy}{ds}\right)^2. \tag{18}$$

For a particle at rest, with respect to the system of coordinates ($dx^i = 0$), the four-velocity becomes

$$u^\mu = \frac{\delta^\mu_0}{\sqrt{g_{00}}}. \tag{19}$$

Integrating (14) with the above u^μ, we get

$$m = \frac{F(x^\rho)}{\sqrt{g_{00}(x^\rho, y)}}, \tag{20}$$

where F is an arbitrary function of spacetime coordinates. We note that m has to be invariant with respect to the transformation,

$$\begin{aligned} \bar{x}^0 &= \bar{x}^0(x^0, x^1, x^2, x^3), \\ \bar{x}^k &= \bar{x}^k(x^1, x^2, x^3), \end{aligned} \tag{21}$$

which leaves u^μ invariant. Therefore, F is not a scalar function, but should transform as $\bar{F}(\bar{x}) = (\partial x^0/\partial \bar{x})F(x)$.

This is consistent with what we obtain from the definition of four-momentum $p_\mu = m u_\mu$. It implies $m = p_\mu u^\mu$, which for a particle at rest yields

$$m = \frac{p_0}{\sqrt{g_{00}}}. \tag{22}$$

Thus, the function of integration F is just p_0. It should be noted that (14) and (18) link the derivatives $\partial g_{\mu\nu}/\partial y$ $\partial\Phi/\partial x^\rho$ to the variation of mass in the respective directions, while $\partial g_{\mu\nu}/\partial x^\rho$ are related to the gravitational field.

3 Formulae for the rest mass

In the original Randall & Sundrum scenario, only gravity is allowed to propagate in the bulk, while all matter fields are confined on the brane. However, the inclusion of matter and gauge fields in the bulk has been extensively treated in the literature (see [29] and references therein). In particular, in models of Universal Extra Dimensions [30] in which *all* of the Standard Model fields are allowed to propagate in the bulk.

Therefore, for generality we do not restrict our discussion to null geodesic motion in $5D$. In this section we show how the observed rest mass in $4D$ is related to the metric and momentum along the extra dimension. With this aim we multiply (3) by g_{AD}

$$g_{AD}\frac{d^2x^A}{dS^2} + g_{AD}\Gamma^A_{BC}U^B U^C = 0, \tag{23}$$

and consider the equation for $D = 0$. After some manipulations we get

$$\frac{d}{f ds}\left(g_{\lambda 0}\frac{dx^\lambda}{f ds}\right) = \frac{1}{2}\frac{\partial g_{BC}}{\partial x^0}U^B U^C, \tag{24}$$

[2] To be more precise, since $(\delta^\mu_\nu - u^\mu u_\nu)$ is the projector onto the tree-space orthogonal to u^μ, adding to (18) an additional function $H u_\mu$, with an arbitrary H, is innocuous.

where we have set

$$dS = f ds, \quad \text{with} \quad f \equiv \sqrt{1 + \epsilon \Phi^2 \left(\frac{dy}{ds}\right)^2} \tag{25}$$

In order to interpret the term inside the bracket in (24), we introduce a 4D timelike unit vector field τ^μ, which is tangential to the time-coordinate x^0, viz.,

$$\tau^\mu = \frac{\delta^\mu_0}{\sqrt{g_{00}}}, \quad \tau_\mu = \frac{g_{0\mu}}{\sqrt{g_{00}}} \tag{26}$$

Also we introduce $\lambda_{\mu\nu}$, the projector onto the 3-space, orthogonal to τ^μ

$$\lambda_{\mu\nu} \equiv \tau_\mu \tau_\nu - g_{\mu\nu}. \tag{27}$$

Since $\lambda_{0j} = \lambda_{00} = 0$, the line element in $4D$ becomes

$$ds^2 = g_{\mu\nu} dx^\mu x^\nu = d\tau^2 - dl^2, \tag{28}$$

where $d\tau$ measures the proper time along an infinitesimal displacement dx^μ and dl is the corresponding spatial length, viz.,

$$d\tau = \tau_\mu dx^\mu, \quad dl = \sqrt{\lambda_{ij} dx^i dx^j}. \tag{29}$$

Thus,

$$\frac{d\tau}{ds} = \frac{1}{\sqrt{1 - v^2}}, \tag{30}$$

where $v^2 \equiv \lambda_{ij} v^i v^j$ represents the square of the spatial three-velocity

$$v^i = \frac{dx^i}{d\tau}. \tag{31}$$

Thus we find

$$g_{0\mu} \frac{dx^\mu}{ds} = \frac{\sqrt{g_{00}}}{\sqrt{1 - v^2}}. \tag{32}$$

Coming back to (24) we obtain

$$\frac{d}{f ds}\left(\frac{\sqrt{g_{00}}}{f\sqrt{1 - v^2}}\right) = \frac{1}{2}\frac{\partial g_{BC}}{\partial x^0} U^B U^C. \tag{33}$$

In general relativity, the quantity inside the round bracket (with $f = 1$) is the energy $\mathcal{E}$ per unit mass

$$\frac{\mathcal{E}}{m} = \frac{\sqrt{g_{00}}}{\sqrt{1 - v^2}}, \tag{34}$$

which is constant when the gravitational field is independent of time. These equations suggest that the quantity

$$m = \frac{M}{f} = \frac{M}{\sqrt{1 + \epsilon \Phi^2 (dy/ds)^2}}, \tag{35}$$

where M is an arbitrary constant with the appropriate units, can be interpreted as the mass of a particle in $4D$. Notice that $m = M$ when the motion is confined to hypersurfaces $y = $ constant.

In order to justify this interpretation, we have to show that our definition of mass satisfies (14) and (18) so we recover (5). From (35) we find

$$\frac{dm}{m} = -\frac{df}{f}. \tag{36}$$

From the definition of f in (25) it follows that

$$f\frac{df}{ds} = \epsilon\Phi\frac{d\Phi}{ds}\left(\frac{dy}{ds}\right)^2 + \epsilon\Phi^2\frac{dy}{ds}\left(\frac{d^2y}{ds^2}\right). \tag{37}$$

Now, from (3) with $A = 4$ we obtain

$$\frac{d^2y}{ds^2} = \frac{1}{f}\left(\frac{df}{ds}\right)\frac{dy}{ds} - \Gamma^4_{AB}\frac{dx^A}{ds}\frac{dx^B}{ds}. \tag{38}$$

Substituting into (37) we get

$$\frac{1}{f}\frac{df}{ds} = \epsilon\Phi\left(\frac{dy}{ds}\right)^2 - \epsilon\Phi^2\left(\frac{dy}{ds}\right)\Gamma^4_{AB}\frac{dx^A}{ds}\frac{dx^B}{ds}. \tag{39}$$

After a simple calculation, and using (36) we find,

$$\frac{dm}{m} = \left[-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial y}u^\mu u^\nu\right] dy + \left[\epsilon\Phi\frac{\partial\Phi}{\partial x^\mu}\left(\frac{dy}{ds}\right)^2\right] dx^\mu. \tag{40}$$

Consequently,

$$\frac{1}{m}\frac{\partial m}{\partial y} = -\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial y}u^\mu u^\nu, \quad \text{and} \quad \frac{1}{m}\frac{\partial m}{\partial x^\mu} = \epsilon\Phi\frac{\partial\Phi}{\partial x^\mu}\left(\frac{dy}{ds}\right)^2. \tag{41}$$

Thus, substituting these expressions into (16) we get exactly the equation of motion (5), obtained from the dimensional reduction of the geodesic equation in $5D$. In addition, we note that (35) is invariant under spacetime coordinate transformations $\bar{x}^\mu = \bar{x}^\mu(x^\rho)$, which corresponds to the notion that $m = \sqrt{p_\mu p^\mu}$ is a scalar in $4D$.

3.1 The rest mass in $4D$ for null geodesic motion in $5D$

In the above discussion it is clear that $f \neq 0$, i.e., $dS \neq 0$. The question arises of how the motion along a null geodesic ($dS = 0$) is observed in $4D$. The discussion is relevant to the original Randall & Sundrum braneworld scenario and other Kaluza-Klein models which postulate that the motion in $5D$ is along null geodesics [21]. In this case the geodesic equation becomes

$$\frac{d^2x^A}{d\lambda^2} + \Gamma^A_{BC}U^BU^C = 0, \quad \text{with} \quad U^A = \frac{dx^A}{d\lambda}, \tag{42}$$

where λ is a parameter along the null geodesic in $5D$. Again, we can introduce a quantity $\bar{f}$ such that

$$d\lambda = \bar{f}ds. \tag{43}$$

It should be pointed out that, contrary to quantity f defined in (25), in general we have no formulas relating $\bar{f}$ with other quantities in the theory, except in particular cases (see bellow in section 4.2). However, following the same steps as above we would arrive at (33) with $\bar{f}$ instead of f. So similarly, we can define

$$m = \frac{\bar{M}}{\bar{f}}, \tag{44}$$

where $\bar{M}$ is some constant with the appropriate units.

Since $dS = \sqrt{1 + \epsilon\Phi^2(dy/ds)^2} = 0$, it follows that the extra dimension must be spacelike $\epsilon = -1$, and $ds = \Phi dy$. Consequently, $d\lambda = \bar{f}\Phi dy$ and

$$m = \bar{M}\Phi\frac{dy}{d\lambda}. \tag{45}$$

Taking the differential of this quantity with respect to ds and using (42) with $A = 4$, we easily get

$$\frac{1}{m}\frac{dm}{ds} = -\frac{u^\mu u^\nu}{2\Phi}\frac{\partial g_{\mu\nu}}{\partial y} - \frac{u^\mu}{\Phi}\frac{\partial\Phi}{\partial x^\mu}, \tag{46}$$

which is formally obtained from (40) by setting $(dy/ds) = 1/\Phi$ and $\epsilon = -1$. Clearly, this expression is equivalent to (41). Thus, when the mass defined through (45) is substituted into (16) we obtain the effective equations (5).

4 Specific expressions for m

In order to get a specific form for m one has to know dy/ds. Setting $A = 4$ in (3), after some manipulations we get

$$\frac{d}{fds}\left(\frac{\epsilon\Phi^2}{f}\frac{dy}{ds}\right) = \frac{1}{2}\frac{\partial g_{BC}}{\partial y}U^B U^C. \tag{47}$$

4.1 Metric with no dependence on the extra coordinate

In the case where the r.h.s. is zero, which in particular occurs when the metric is independent of y, we have

$$\frac{\Phi^2}{f}\frac{dy}{ds} = C, \tag{48}$$

where C is a constant. This is an important case because in the literature there are a huge number of solutions of the $5D$ equations with no dependence on the extra coordinate [9], [31], [32]. Thus,

$$\frac{dy}{ds} = \frac{C}{\Phi\sqrt{\Phi^2 - \epsilon C^2}}. \tag{49}$$

Using this expression in (35) we find

$$m = \frac{M\sqrt{\Phi^2 - \epsilon C^2}}{\Phi}. \tag{50}$$

Clearly, $m = M$ for $dy/ds = 0$.

4.2 Null geodesics in $5D$

If $dS = 0$, we should replace $f \to \bar{f}$ in (47). Thus,

$$\frac{\Phi^2}{\bar{f}}\frac{dy}{ds} = \bar{C}, \tag{51}$$

where $\bar{C}$ is a dimensionless constant. Consequently, $dy/d\lambda = \bar{C}/\Phi^2$. For $dS = 0$, it follows that $ds = \Phi dy$ (and $\epsilon = -1$) which gives $\bar{f} = \Phi/\bar{C}$, and therefore $d\lambda = \Phi ds/\bar{C}$. From (44) we find

$$m = \frac{\tilde{M}}{\Phi}, \tag{52}$$

where $\tilde{M} = \bar{M}\bar{C}$. Notice that both (50) and (52) satisfy (18).

When the metric is independent of the extra coordinate and Φ = constant, then m is constant too. As a consequence, the r.h.s. of (16) is zero, meaning that embedding a $4D$ spacetime in a $5D$ manifold as

$$dS^2 = g_{\mu\nu}(x^\rho)dx^\mu dx^\nu \pm dy^2, \tag{53}$$

produces no effects in $4D$.

5 Conclusions

In this paper we have used the principle of least action to obtain the equation of motion for a test particle in a four-dimensional spacetime embedded in a five-dimensional world with metric (2).

From our work emerges a clear physical picture. Specifically, that the deviation from the geodesic motion in $4D$ is due to the variation of the rest mass of a particle, which is induced by an explicit dependence of the spacetime metric on the extra coordinate. More explicitly, the rest mass (35) depends on dy/ds which is governed by

$$\frac{d^2y}{ds^2} = \frac{\epsilon u^\mu u^\nu}{2\Phi^2}\frac{\partial g_{\mu\nu}}{\partial y} + \frac{dy}{ds}\left[\frac{1}{f}\frac{df}{ds} + \frac{2u^\mu}{\Phi}\frac{\partial \Phi}{\partial x^\mu} + \frac{1}{\Phi}\frac{\partial \Phi}{\partial y}\frac{dy}{ds}\right], \tag{54}$$

where (df/fds) is given by (39). This equation indicates that, even if at some initial moment $(dy/ds) = 0$, the non-trivial dependence of the metric on the extra variable implies $(dy/ds) \neq 0$ the next moment, which will be perceived by an observer in $4D$ as a variation in the rest mass of a particle.

Notice that the scalar field Φ by itself does not generate momentum along the extra dimension. Indeed, in the case where the metric is independent of the extra coordinate, if $(dy/ds) = 0$ at some moment, then $d^2y/ds^2 = 0$. Which means that dy/ds will continue to be zero along the motion. Consequently, the geodesic motion is on a hypersurface $y =$ constant, which in $4D$ will be perceived as a particle of constant mass $m = M$. However, one would expect that any small perturbation along y would be enhanced by the scalar field, building up momentum along the extra dimension, which in $4D$ will be perceived as a variation of the rest mass. Clearly, the same reasoning holds for the case of null geodesics in $5D$ where m is given by (44).

The question may arise about the connection between the four momentum and m. In order to see this, let us construct the five-dimensional quantity

$$P^A = M\frac{dx^A}{dS}. \tag{55}$$

The square of this quantity is M^2. Namely,

$$P_A P^A = g_{\mu\nu}P^\mu P^\nu + \epsilon\Phi^2\frac{dy}{dS} = M^2. \tag{56}$$

Using (25) and rearranging terms we obtain

$$g_{\mu\nu}P^\mu P^\nu = m^2, \tag{57}$$

where m is given by (35). This suggests that the spacetime part of P^A, which coincides with the projection onto $4D$, should be identified with the four-momentum p^μ, viz.,

$$p^\mu = P^\mu = M\frac{dx^\mu}{dS} = \left(\frac{M}{f}\right)u^\mu = mu^\mu, \tag{58}$$

in agreement with the usual definition of 4-momentum in relativity.

Some authors postulate that motion in $5D$ is along null geodesics, similarly to braneworld theory. In this case, the extra dimension must be spacelike and the motion along null geodesics is observed in $4D$ as particles with an effective mass, which is given by (44) and (45). In this case, we introduce the quantity

$$\bar{P}^A = \bar{M}\frac{dx^A}{d\lambda}, \tag{59}$$

where $\bar{M}$ is a constant introduced for dimensional consistency, and λ is the affine parameter along the null geodesic introduced in (42). Clearly, in this case $\bar{P}_A\bar{P}^A = 0$. Thus, using (43) we obtain $(\epsilon = -1)$

$$g_{\mu\nu}\bar{P}^\mu\bar{P}^\nu = \frac{\bar{M}\Phi^2}{\bar{f}^2}\left(\frac{dy}{ds}\right)^2. \tag{60}$$

For null geodesics $(dy/ds) = 1/\Phi$. Thus,

$$g_{\mu\nu}\bar{P}^\mu\bar{P}^\nu = m^2, \quad p^\mu = \bar{P}^\mu = mu^\mu, \quad \text{with} \quad m = \bar{M}\Phi\frac{dy}{d\lambda}, \tag{61}$$

as expected. A particular example is provided by (52).

For completeness let us emphasize the main differences and similarities between null and non-null geodesic motion in $5D$, as observed in $4D$: (i) For null geodesics in $5D$, the observed mass in $4D$ is a consequence of non-zero momentum along the extra dimension; it is massless if the motion is confined to a hypersurface y =constant; (ii) For non-null geodesics in $5D$, the observed particle in $4D$ is a massive one, even in the absence of momentum along y; (iii) If dy/ds = constant $\neq 0$ ($dy/d\lambda$ = constant $\neq 0$), then the variation of rest mass is a consequence of the scalar field Φ; (iv) If Φ = constant, then the variation of mass is a consequence of the dependence of the $4D$ metric on the extra coordinate.

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