

Junction Conditions and Consequences of Quasi-Spherical Space-Time with Electro-Magnetic Field and Vaidya Metric

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In this work the junction conditions between the exterior Reissner-Nordstrom-Vaidya space-time with the interior quasi-spherical Szekeres space-time have been studied for analyzing gravitational collapse in the presence of a magneto-hydrodynamic fluid undergoing dissipation in the form of heat flow. We have discussed about the apparent horizon and have evaluated the time difference between the formation of apparent horizon and central singularity.

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I. INTRODUCTION

In Einstein gravity, gravitational collapse is an important problem for astrophysical objects. Usually, the formation of compact stellar objects such as white dwarf and neutron star are preceded by a period of collapse. Hence, for astrophysical collapse, it is necessary to describe the appropriate geometry of interior and exterior regions and to determine proper junction conditions which allow the matching of these regions.

The study of gravitational collapse was started long ago in 1939 by Oppenheimer and Snyder [1] to solve an idealized problem of gravitational collapse preceded by assuming a spherical symmetric distribution, adiabatic flow and the equation of state in the form of dust. Misner and Sharp [2] have introduced pressure gradient terms in the equation of motion. Outgoing radiation of the collapsing body has been considered by Vaidya [3], Santos [4] and collaborators [5 - 10] and the correspondence to adiabatic distribution of matter has been considered by Misner [11] and collaborators [12, 13]. Models of radiating collapse with viscous fluid were presented by Lake [14] and Santos [4]. Ghosh and Deshkar [15] have considered collapse of a radiating star with a plane symmetric boundary (which has a close resemblance with spherical symmetry [16]) and have concluded with some general remarks.

Most of the studies has considered a collapsing star whose interior geometry is spherical. But in the real astrophysical situation, the geometry of the interior of a star may be quasi-spherical in form. Previously we have discussed [17] junction condition between interior Szekeres model and exterior plane symmetric Vaidya space-time. Oliveira et al [9] have proposed to study the junction conditions of a collapsing non-adiabatic charged body producing radiation. The interior space-time V^- was modeled by a magneto-hydrodynamic fluid undergoing dissipation in the form of a radial heat flow and the exterior space-time V^+ was described by the Reissner-Nordstrom-Vaidya metric which presents a spherically symmetric electric field and a radial flow of unpolarized radiation. Now in this paper we will consider the interior space-time V^- is quasi-spherical Szekeres model and the exterior space-time V^+ is Reissner-Nordstrom-Vaidya metric in the presence of electro-magnetic field.

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II. JUNCTION CONDITIONS

Let Σ be time-like $3D$ hypersurface which separates two $4D$ manifolds V^- and V^+ . The interior $4D$ manifold V^- is given by the quasi-spherical Szekeres space-time with line element

$$V^- : ds_-^2 = -dt^2 + e^{2\alpha} dr^2 + e^{2\beta} (dx^2 + dy^2) \quad (1)$$

where α and β are functions of all space-time co-ordinate variables.

Now for the transformation

$$x = \cot \frac{\theta}{2} \cos \phi, \quad y = \cot \frac{\theta}{2} \sin \phi,$$

the metric (1) in V^- becomes

$$ds_-^2 = -dt^2 + e^{2\alpha} dr^2 + \frac{1}{4} e^{2\beta} \operatorname{cosec} \frac{\theta}{2} (d\theta^2 + \sin^2 \theta d\phi^2) \quad (2)$$

We assume that the source for the Einstein's field equations in V^- is given by

$$G_{\mu\nu} = T_{\mu\nu} + E_{\mu\nu} \quad (3)$$

The stress-energy tensor of a non-viscous heat conducting fluid has the expression

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \quad (4)$$

where ρ , p , q_μ are the fluid density, isotropic pressure and heat flow vector respectively and u_μ is the four velocity. We take the heat flow vector q_μ to be orthogonal to the velocity vector u^μ i.e., $q_\mu u^\mu = 0$. For co-moving co-ordinate system, we choose $u^\mu = (1, 0, 0, 0)$ and $q^\mu = (0, q_r, q_x, q_y)$ with $q_i = q_i(t, r, x, y)$, $i \equiv (r, x, y)$.

The energy-momentum tensor $E_{\mu\nu}$ of the electro-magnetic field is given by

$$E_{\mu\nu} = \frac{1}{4\pi} \left[F_\mu^\nu F_{\nu\mu} - \frac{1}{4} g_{\mu\nu} F^{\nu\delta} F_{\nu\delta} \right] \quad (5)$$

where $F_{\mu\nu}$ is the electro-magnetic field tensor satisfying Maxwell's equations

$$F_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu} \quad (6)$$

and

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu \quad (7)$$

where ϕ_μ is the four-potential and J_μ is the four-current.

Since the charge is at rest in this system, so there will be no magnetic field in this system. Thus, we may choose the four-potential and four-current as

$$\phi_\mu = (\phi(t, r, x, y), 0, 0, 0) \quad (8)$$

and

$$J^\mu = \sigma u^\mu \quad (9)$$

where σ is the charge density. From equation (6) using (8) we have the only non-zero component of the field tensor as

$$F_{01} = -F_{10} = -\frac{\partial\phi}{\partial r} \quad (10)$$

Also from equations (7) and (9) we obtain

$$\frac{\partial^2\phi}{\partial r^2} - \alpha' \frac{\partial\phi}{\partial r} = 4\pi\sigma e^{2\alpha} \quad (11)$$

and

$$\frac{\partial^2\phi}{\partial r\partial t} - \dot{\alpha} \frac{\partial\phi}{\partial r} = 0 \quad (12)$$

where ‘ ’ and ‘ . ’ denote partial derivatives with respect to r and t respectively. Integrating (12), we have

$$\frac{\partial\phi}{\partial r} = K(r, x, y) e^\alpha \quad (13)$$

and then from equation (11) one gets

$$K' = 4\pi\sigma e^\alpha \quad (14)$$

For the interior space-time model (1) with matter field (4) and electromagnetic field (5), the Einstein's equation (3) can be explicitly written as the following :

$$2\dot{\alpha}\dot{\beta} + \dot{\beta}^2 - e^{-2\beta}(\alpha_x^2 + \alpha_y^2 + \alpha_{xx} + \beta_{xx} + \alpha_{yy} + \beta_{yy}) + e^{-2\alpha}(2\alpha'\beta' - 3\beta'^2 - 2\beta'') = \rho - \frac{1}{8\pi}K^2 \quad (15)$$

$$3\dot{\beta}^2 + 2\ddot{\beta} - e^{-2\alpha}\beta'^2 - (\beta_{xx} + \beta_{yy})e^{-2\beta} = -p - \frac{1}{8\pi}K^2 \quad (16)$$

$$\dot{\alpha}^2 + \ddot{\alpha} + \dot{\alpha}\dot{\beta} + \dot{\beta}^2 + \ddot{\beta} + e^{-2\alpha}(\alpha'\beta' - \beta'^2 - \beta'') - e^{-2\beta}[\alpha_x^2 + \alpha_{xx} - \alpha_x\beta_x] - e^{-2\beta}\alpha_y\beta_y = -p + \frac{1}{8\pi}K^2 \quad (17)$$

$$\dot{\alpha}^2 + \ddot{\alpha} + \dot{\alpha}\dot{\beta} + \dot{\beta}^2 + \ddot{\beta} + e^{-2\alpha}(\alpha'\beta' - \beta'^2 - \beta'') - e^{-2\beta}[\alpha_y^2 + \alpha_{yy} - \alpha_y\beta_y] - e^{-2\beta}\alpha_x\beta_x = -p + \frac{1}{8\pi}K^2 \quad (18)$$

$$\alpha_x(-\alpha_y + \beta_y) + \beta_x\alpha_y - \alpha_{xy} = 0 \quad (19)$$

$$\alpha_y(-\alpha_x + \beta_x) + \beta_y\alpha_x - \alpha_{xy} = 0 \quad (20)$$

$$-\dot{\alpha}\beta' + \dot{\beta}\beta' + \dot{\beta}' = \frac{1}{2}q_r e^{2\alpha} \quad (21)$$

$$(\dot{\alpha} - \dot{\beta})\alpha_x + \dot{\alpha}_x + \dot{\beta}_x = q_x e^{2\beta} \quad (22)$$

$$(\dot{\alpha} - \dot{\beta})\alpha_y + \dot{\alpha}_y + \dot{\beta}_y = q_y e^{2\beta} \quad (23)$$

$$\alpha_x \beta' - \beta'_x = 0 \quad (24)$$

$$\alpha_y \beta' - \beta'_y = 0 \quad (25)$$

where subscript stands for partial derivatives with respect to corresponding variables.

For quasi-spherical case i.e., for $\beta' \neq 0$, we have the solutions [17]

$$e^\beta = R(t, r) e^{\nu(r, x, y)} \quad (26)$$

and

$$e^\alpha = \frac{R' + R \nu'}{D(t, r)} \quad (27)$$

where $R(t, r)$ and $D(t, r)$ will satisfy the differential equations

$$2 R \ddot{R} + (\dot{R}^2 - D^2) + \left(p + \frac{K^2}{8\pi}\right) R^2 = f(r) \quad (28)$$

and

$$R \dot{D} = f(r) e^{-2\alpha} \quad (29)$$

with $f(r)$ an arbitrary function of r . The function ν can be written in the form

$$e^{-\nu} = A(r) (x^2 + y^2) + B_1(r) x + B_2(r) y + C(r) \quad (30)$$

where A, B_1, B_2 and C are arbitrary functions of r along with the restriction

$$(B_1^2 + B_2^2) - 4 A C = f(r) - 1 \quad (31)$$

From equations (21), (22) and (23) we have the components of heat flux vector as [17]

$$q_r = \frac{2}{R} \dot{D} e^{-\alpha} \quad (32)$$

$$q_x = -\frac{\dot{D}}{D} \alpha_x e^{-\beta} \quad (33)$$

$$q_y = -\frac{\dot{D}}{D} \alpha_y e^{-\beta} \quad (34)$$

The exterior manifold V^+ is described by the Reissner-Nordstrom-Vaidya space-time having metric

$$V^+ : \quad ds_+^2 = - \left[1 - \frac{2M(v)}{z} + \frac{Q^2}{z^2} \right] dv^2 - 2dv dz + z^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (35)$$

This space-time represents a spherically symmetric field with an outgoing radial flux of unpolarized radiation and v is the retarded time.

The surface of separation Σ between these two manifolds V^\pm is characterized by $r = r_\Sigma$ and the intrinsic metric on it (in the comoving co-ordinates of the interior space-time V^-) is given by

$$ds_\Sigma^2 = -d\tau^2 + R^2(\tau)(d\theta^2 + \sin^2\theta d\phi^2)$$

Now in the description of Santos, the Israel's junction conditions are

(i) The continuity of the line-element

$$(ds_-^2)_\Sigma = (ds_+^2)_\Sigma = ds_\Sigma^2 \quad (36)$$

where $(\)_\Sigma$ means the value of $(\)$ on Σ .

(ii) The continuity of the extrinsic curvature over Σ :

$$[K_{ij}] = [K_{ij}^+] - [K_{ij}^-] = 0, \quad (37)$$

where the expressions of extrinsic curvature in terms of the unit space-like normal vector to Σ , n_α^\pm are given by [18],

$$K_{ij}^\pm = -n_\sigma^\pm \frac{\partial^2 \chi_\pm^\sigma}{\partial \xi^i \partial \xi^j} - n_\sigma^\pm \Gamma_{\mu\nu}^\sigma \frac{\partial \chi_\pm^\mu}{\partial \xi^i} \frac{\partial \xi_\pm^\nu}{\partial \xi^j} \quad (38)$$

Here ξ^i refer to the intrinsic co-ordinates (τ, x, y) on Σ and χ_\pm^μ , $\mu = 0, 1, 2, 3$ stand for the co-ordinates in V^\pm .

The boundary Σ in terms of the co-ordinates of the interior space-time V^- is given by

$$\mathcal{F}(r, t) = r - r_\Sigma = 0 \quad (r_\Sigma, \text{ a constant}) \quad (39)$$

So one can write the normal vector n_μ^- as

$$n_\mu^- = (0, e^\alpha, 0, 0).$$

Now comparing the metric ansatz for Σ and V^- for $dr = 0$, the continuity of the line element gives

$$\frac{dt}{d\tau} = 1, \quad \mathcal{R}(\tau) = e^\beta \quad \text{on} \quad r = r_\Sigma \quad (40)$$

Further, the components of the extrinsic curvature for the interior space-time are

$$\left. \begin{aligned} K_{\tau\tau}^- &= 0, \quad K_{\theta\theta}^- = \text{cosec}^2\theta K_{\phi\phi}^- = \left[\frac{1}{4} \beta'^2 e^{2\beta-\alpha} \text{cosec}^4\left(\frac{\theta}{2}\right) \right]_\Sigma \\ \text{and } K_{ij}^- &= 0, \quad i \neq j \end{aligned} \right\} \quad (41)$$

On the other hand, in terms of the co-ordinates of the Reissner-Nordstrom-Vaidya space-time the surface Σ can be characterized by

$$\mathcal{F}(z, v) = z - z_\Sigma(v) = 0 \quad (42)$$

Hence, as before the unit normal vector to Σ is given by

$$n_\mu^+ = \left(2 \frac{dz_\Sigma}{dv} + 1 - \frac{2M(v)}{z_\Sigma} + \frac{Q^2}{z_\Sigma^2} \right)^{-\frac{1}{2}} \left(-\frac{dz_\Sigma}{dv}, 1, 0, 0 \right) \quad (43)$$

and the components of the extrinsic curvature for the exterior space-time are

$$\left. \begin{aligned} K_{\tau\tau}^+ &= \left[\frac{\ddot{v}}{\dot{v}} - \dot{v} \left(\frac{M(v)}{z^2} - \frac{Q^2}{z^3} \right) \right]_{\Sigma} , \\ K_{\theta\theta}^+ &= \csc^2 \theta K_{\phi\phi}^+ = \left[\dot{v} \left(1 - \frac{2M(v)}{z} + \frac{Q^2}{z^2} \right) z + z\dot{z} \right]_{\Sigma} , \\ K_{ij}^+ &= 0, \quad i \neq j \end{aligned} \right\} \quad (44)$$

(here ‘ . ’ stands for the differentiation with respect to ‘ τ ’).

Now, the continuity of the metric ansatz for Σ and V^+ over $z = z_{\Sigma}(v)$ is given by

$$\left. \begin{aligned} z_{\Sigma}(v) &= R(\tau) , \\ \left[2 \frac{dz}{dv} + 1 - \frac{2M(v)}{z} + \frac{Q^2}{z^2} \right]_{\Sigma} &= \left(\frac{1}{\dot{v}^2} \right)_{\Sigma} \end{aligned} \right\} \quad (45)$$

Also the continuity of the components of the extrinsic curvature across Σ give

$$p = q_r e^{\alpha} + \frac{f(r) - 1}{R^2} + \frac{4}{R^2} \sin^4(\theta/2) e^{-2\nu} - \frac{Q^2}{R^4} - \frac{K^2}{8\pi} \quad (46)$$

and

$$M(v) = \frac{R}{2} + \frac{Q^2}{2R} - \frac{D^2 R}{2} + \frac{R \dot{R}^2}{2} \quad (47)$$

which can be termed as the total energy entrapped inside the surface Σ . We note that on the boundary, the pressure can not be vanish in general. Thus for quasi-spherical shear-free distribution of a collapsing charged fluid undergoing dissipation in the form of heat flow, the isotropic pressure on the surface of discontinuity can not be zero. In particular, if fluid stops dissipation, $q_r = 0$, the pressure can not vanish at the boundary, but in this case radiation can not exist and the exterior space-time V^+ is the Reissner-Nordstrom-Vaidya space-time. But in absence of isotropic pressure there may still be radiation on the boundary and the exterior space-time V^+ will still be Reissner-Nordstrom-Vaidya space-time.

The total luminosity for an observer rest at infinity is [7]

$$L_{\infty} = - \left(\frac{dm}{dv} \right)_{\Sigma} \quad (48)$$

If we now consider an observer on the boundary Σ then the luminosity for that observer is

$$L_{\Sigma} = - \left[\left(\frac{dv}{d\tau} \right)^2 \frac{dm}{dv} \right]_{\Sigma} \quad (49)$$

Now the boundary redshift of the radiation emitted by a star can be written as

$$Z_{\Sigma} = \sqrt{\frac{L_{\Sigma}}{L_{\infty}}} - 1 = \frac{dv}{d\tau} - 1 = \frac{1}{\dot{R} + D} - 1 \quad (50)$$

Hence the luminosity measured by an observer at rest at infinity is reduced by the redshift in comparison to the luminosity observed on the surface of collapsing body. Also when $\dot{R} + D = 0$ then the boundary redshift attains unlimited value (i.e., $Z_{\Sigma} \rightarrow \infty$).

III. APPARENT HORIZON AND ITS TIME OF FORMATION

The first integral of equation (28) is given by

$$\dot{R}^2 = f(r) + \frac{F(r)}{R} - \frac{K^2 R^2}{24\pi} + \frac{1}{R} \int (D^2 - pR^2) dR \quad (51)$$

As D is regular and p blows up at the singularity [19], so let us assume

$$D = \lambda(r)R, \quad p = \frac{f(r)}{R^2} \quad (52)$$

The evolution equation (51) satisfies to

$$\dot{R}^2 = \frac{F(r)}{R} - \frac{K^2 R^2}{24\pi} + \frac{\lambda^2}{3} R^2 \quad (53)$$

Further, the time of formation of apparent horizon $t_{ah}(r)$ is given by

$$\dot{R}^2(t_{ah}(r), r) = 1 + f(r) \quad (54)$$

Hence for equation (53) we have a cubic equation in R as

$$\left(\frac{\lambda^2}{3} - \frac{K^2}{24\pi} \right) R^3(t_{ah}(r), r) - (1 + f(r))R(t_{ah}(r), r) + F(r) = 0 \quad (55)$$

Thus apparent horizon exists if this cubic equation in R has at least one positive root. We shall show below different conditions for existence of positive roots.

I. For $\frac{\lambda^2}{3} - \frac{K^2}{24\pi} > 0$ and $F(r) < \frac{2(1+f)^{\frac{3}{2}}}{\left(\frac{\lambda^2}{3} - \frac{K^2}{24\pi}\right)^{\frac{1}{2}}}$, there are two positive roots of equation (55) and hence there are two apparent horizons, given by

$$R_c(r) = 2\sqrt{\frac{(1+f)^3}{\left(\frac{\lambda^2}{3} - \frac{K^2}{24\pi}\right)}} \cos \left[\frac{1}{3} \cos^{-1} \left(-\frac{1}{2} F(r) \sqrt{\frac{\lambda^2}{3} - \frac{K^2}{24\pi}} \right) \right] \quad (56)$$

$$R_b(r) = 2\sqrt{\frac{(1+f)^3}{\left(\frac{\lambda^2}{3} - \frac{K^2}{24\pi}\right)}} \cos \left[\frac{4\pi}{3} + \frac{1}{3} \cos^{-1} \left(-\frac{1}{2} F(r) \sqrt{\frac{\lambda^2}{3} - \frac{K^2}{24\pi}} \right) \right] \quad (57)$$

which are termed as cosmological and black hole horizons.

II. When $F(r) = \frac{2(1+f)^{\frac{3}{2}}}{\left(\frac{\lambda^2}{3} - \frac{K^2}{24\pi}\right)^{\frac{1}{2}}}$, ($\lambda > \frac{K}{2\sqrt{2\pi}}$) there is only one positive root, which corresponds to a single apparent horizon

$$R_{bc}(r) = \sqrt{\frac{(1+f)^3}{\left(\frac{\lambda^2}{3} - \frac{K^2}{24\pi}\right)}} \quad (58)$$

III. For $F(r) > \frac{2(1+f)^{\frac{3}{2}}}{\left(\frac{\lambda^2}{3} - \frac{K^2}{24\pi}\right)^{\frac{1}{2}}}$, ($\lambda > \frac{K}{2\sqrt{2\pi}}$) there are no positive roots and consequently there are no apparent horizons.

IV. If $\lambda < \frac{K}{2\sqrt{2\pi}}$ then also one positive root exists to have a unique apparent horizon. A detailed study of apparent horizons in quasi-spherical gravitational collapse with a non-zero Λ -term can be found in [20].

Moreover, solving the dynamical equation (53) we obtain

$$R^{\frac{3}{2}} = \frac{2\sqrt{6\pi F}}{\sqrt{8\pi\lambda^2 - K^2}} \sinh \left[\frac{\sqrt{8\pi\lambda^2 - K^2}}{4\sqrt{\frac{2\pi}{3}}} (t - t_s(r)) \right] \quad (59)$$

where $t_s(r)$ is the time of formation of singularity of a particular shell at co-ordinate distance ' r ' (i.e., $R = 0$ at $t_s(r) = 0$). Hence the time difference between the formation of apparent horizon and the singularity formation is given by

$$t_{ah}(r) - t_s(r) = \frac{4\sqrt{\frac{2\pi}{3}}}{\sqrt{8\pi\lambda^2 - K^2}} \sinh^{-1} \left[\frac{\sqrt{8\pi\lambda^2 - K^2}}{2\sqrt{6\pi F}} R_{ah}^{\frac{3}{2}} \right] \quad (60)$$

(One may note that in solving the differential equation (53) it is assumed that $\lambda > \frac{K}{2\sqrt{2\pi}}$. However if $\lambda < \frac{K}{2\sqrt{2\pi}}$ then 'sinh' is to be replaced by 'sin' in the solution (59)). Thus it is to be observed that heat flow and electromagnetic field modify the formation of horizon and also the time difference between the formation of apparent horizon and singularity.

Further, from equation (47) we note that there is dissipation of total energy entrapped inside the surface Σ due to the heat flow while the electromagnetic field favours entrapped energy within the surface Σ . Finally, the contribution to the isotropic pressure due to the heat flow and electromagnetic field is in reverse order (or in the same order) provided D is positive (or negative). Hence for increasing the electromagnetic field tries to diminish the pressure on the boundary which may be balanced by the heat flux term, while for decreasing D both of them try to reduce the pressure. So we conclude that electromagnetic field favours formation of naked singularity while heat flux may or may not be favourable for formation of black hole.

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References:

- [1] J. R. Oppenheimer and H. Snyder, *Phys. Rev.* **56** 455 (1939).
- [2] C. W. Misner and D. Sharp, *Phys. Rev.* **136** B571 (1964).
- [3] P. C. Vaidya, *Proc. Indian Acad. Sci. A* **33** 264 (1951).
- [4] N. O. Santos, *Phys. Lett. A* **106** 296 (1984); *Mon. Not. R. Astr. Soc.* **216** 403 (1985).
- [5] R. W. Lindquist, R. A. Schwartz and C. W. Misner, *Phys. Rev.* **137** B1364 (1965).
- [6] W. Israel, *Nouvo Cimento* **44B** 1 (1966); **48B** 463 (1966); *Phys. Lett.* **24A** 184 (1967).
- [7] A. K. G. de Oliveira, N. O. Santos and C. A. Kolassis, *Mon. Not. R. Astr. Soc.* **216** 1001 (1985).
- [8] A. K. G. de Oliveira, J. A. de F. Pacheco and N. O. Santos, *Mon. Not. R. Astr. Soc.* **220** 405 (1986).
- [9] A. K. G. de Oliveira and N. O. Santos, *Astrophys. J.* **312** 640 (1987).
- [10] A. K. G. de Oliveira, C. A. Kolassis and N. O. Santos, *Mon. Not. R. Astr. Soc.* **231** 1011 (1988).
- [11] C. W. Misner, *Phys. Rev.* **137B** 1360 (1965).
- [12] C. W. Misner and D. H. Sharp, *Phys. Lett.* **15** 279 (1965).
- [13] K. Lake and C. Hellaby, *Phys. Rev. D* **24** 3019 (1981).
- [14] K. Lake, *Phys. Rev. D* **26** 518 (1982).
- [15] S. G. Ghosh and D. W. Deshkar, *Int. J. Mod. Phys. D* **12** 317 (2003).
- [16] S. G. Ghosh and D. W. Deshkar, *Gravitation and Cosmology* **6** 1 (2000).
- [17] U. Debnath, S. Nath and S. Chakraborty, *Gen. Rel. Grav.* **37** 215 (2005).
- [18] L. P. Eisenhart, *Riemannian Geometry*, p. 146 (1949), Princeton.
- [19] S. Chakraborty, S. Chakraborty and U. Debnath, *Int. J. Mod. Phys. D* **14** 1707 (2005).
- [20] U. Debnath, S. Nath and S. Chakraborty, *Mon. Not. R. Astr. Soc.* **369** 1961 (2006).