Geodesic motion in the neighbourhood of submanifolds embedded in warped product spaces

Fábio Dahia[a], Carlos Romero[b], Lúcio F. P. da Silva,[b]
Reza Tavakol[c]
[a]Departamento de Física, Universidade Federal
de Campina Grande, 58109-970
Campina Grande, Pb, Brazil
[b]Departamento de Física, Universidade Federal da Paraíba,
Caixa Postal 5008, 58059-979 João Pessoa, Pb, Brazil
[c]School of Mathematical Sciences , Queen Mary, University of London,
London E1 4NS, UK

October 23, 2018

Abstract

We study the classical geodesic motions of nonzero rest mass test particles and photons in $(3+1+n)$-dimensional warped product spaces. An important feature of these spaces is that they allow a natural decoupling between the motions in the $(3+1)$-dimensional spacetime and those in the extra n dimensions. Using this decoupling and employing phase space analysis we investigate the conditions for confinement of particles and photons to the $(3+1)$- spacetime submanifold. In addition to providing information regarding the motion of photons, we also show that these motions are not constrained by the value of the extrinsic curvature. We obtain the general conditions for the confinement of geodesics in the case of pseudo-Riemannian manifolds as well as establishing the conditions for the stability of such confinement. These results also generalise a recent result of the authors concerning the embeddings of hypersurfaces with codimension one.

1 Introduction

An intriguing idea in modern cosmology is the possibility that the universe may have a higher number of dimensions than the classically observed (3+1). There has been a number of motivations for this idea, mainly related to attempts at constructing a fundamental theory of physical interactions. These range from the original attempts at unification of gravity with electromagnetism by Kaluza and Klein [1] to String/M-theory [2, 4, 5].

An immediate task within this higher dimensional framework is how to explain the four dimensionality of the observed universe. An original idea - which dates back to the work of Kaluza and Klein - assumes these extra dimensions to be compact and very small. In most recent braneworld models, on the other hand, stringy effects are invoked to argue that in low energy regimes particles are restricted to a special $3 + 1$ *brane* hypersurface, which is embedded in a higher-dimensional *bulk*, while the gravitational field is free to propagate in the bulk [6].

The intense recent interest in the higher-dimensional scenarios has also provided strong motivation to look for geometrical mechanisms which could allow confinement and this has led to the investigation of the so-called warped product spaces [7] and their geometrical properties.

Originally most braneworld and higher dimensional studies concentrated on $(4 + 1)$ scenarios, i.e. co-dimensional one models. However, given the prediction of the string theory, according to which spacetime is 10-dimensional, the study of higher co-dimensional models is urgently called for. Increasing effort has recently gone into the study of such models (see for e.g. [9]).

An important question concerning such models, specially from an observational point of view, is the behaviour of the geodesics in these models, and in particular the relation between the geodesics of the higher-dimensional space and those belonging to the hypersurface. A great deal of effort has recently gone into the study of geodesic motions in five-dimensional spaces [12]. Recently the authors studied the ability of five-dimensional warped product spaces to provide a mechanism for geodesic confinement on co-dimension

one hypersurfaces, purely classically and based on gravitational effects [11]. The aim of this article is to generalise that analysis to the case of co-dimension n warped product spaces [10]. We do this by studying the classical geodesic motions of nonzero rest mass test particles and photons in $(3 + 1 + n)$-dimensional warped product spaces. We show that it is possible to obtain a general picture of these motions, using the natural decoupling that occurs in such spaces between the motions in the n extra dimensions and the motion in the $4D$ submanifolds. This splitting allows the use of phase space analysis in order to investigate the possibility of confinement[1] and the stability of motion of particles and photons to submanifolds in such $(3 + 1 + n)$-dimensional spaces.

The paper is organised as follows. In Section 2 we write the geodesic equations for warped product spaces and consider the parts due to 4D and the higher dimensions separately. We then show that the equations that describe the motion in higher dimensions decouples from the rest. We proceed in Section 3 to find the mathematical conditions that must be satisfied by the warping function f in order for timelike and null geodesics of the higher-dimensional space to be confined to M^4. In Section 4 we rewrite the geodesic equations in the higher dimensions as an autonomous $2n$-dimensional dynamical system. We then employ phase plane analysis to study the motion of particles with nonzero rest mass (timelike geodesics) and photons (null geodesics) respectively. Such qualitative analysis allows a number of conclusions to be drawn about the possible existence of confined motions and their nature in the neighbourhood of hypersurfaces. In Section 5 we give an analysis of the motion in the extra dimensions by reducing the problem to the motion of a particle subjected to the action of an effective potential and illustrate the method with an example from the literature. We conclude in Section 6 with some final remarks.

2   Warped product spaces and geodesic motion

In general a warped product space is defined in the following way. Let (M^m, h) and (N^n, k) be two Riemannian (or pseudo-Riemannian) manifolds of dimension m and n, with metrics h and k, respectively. Let $f : N^n \to \mathbb{R}$ be a smooth function (which we shall refer to as the warping function). We can then construct a new warped product Riemannian (pseudo-Riemannian) space by setting $M = M^m \times N^n$ and defining a metric $g = e^{2f}h \oplus k$. In this paper we shall take $m = 4$ and identify M^4 with the (3+1)-dimensional *spacetime,* a four-dimensional Lorentzian manifold with signature $(+ - - -)$. Therefore the class of warped geometries which we shall consider can be characterised by the following line element

$$dS^2 = e^{2f}h_{\alpha\beta}dx^\alpha dx^\beta - k_{ab}dy^a dy^b, \tag{1}$$

where $f = f(y^1, ..., y^n)$, $h_{\alpha\beta} = h_{\alpha\beta}(x)$ and k_{ab} is in general an n-dimensional Riemannian metric[2].

Let us now consider the equations of geodesics in the $(n + 4)$-dimensional space M

$$\frac{d^2 Z^A}{d\lambda^2} + \Gamma^A_{BC}\frac{dZ^B}{d\lambda}, \frac{dZ^C}{d\lambda} = 0, \tag{2}$$

where λ is an affine parameter and Γ^a_{bc} are the Christoffel symbols of the second kind defined by $\Gamma^A_{BC} = \frac{1}{2}g^{AD}(g_{DB,C} + g_{DC,B} - g_{BC,D})$. After some algebra it is not difficult to show that the geodesic equations for the 4D part can be written in the form

$$\frac{d^2 x^\mu}{d\lambda^2} + {}^{(4)}\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \phi^\mu, \tag{3}$$

where

$$\phi^\mu = -\Gamma^\mu_{ab}\frac{dy^a}{d\lambda}\frac{dy^b}{d\lambda} - 2\Gamma^\mu_{\alpha b}\frac{dx^\alpha}{d\lambda}\frac{dy^b}{d\lambda}, \tag{4}$$

and ${}^{(4)}\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$. Likewise the extra-dimensional part of the geodesic equations (2) may be written as

$$\frac{d^2 y^a}{d\lambda^2} + {}^{(n)}\Gamma^a_{bc}\frac{dy^b}{d\lambda}\frac{dy^c}{d\lambda} = \psi^a \tag{5}$$

where

$$\psi^a = -\Gamma^a_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} - 2\Gamma^a_{\alpha b}\frac{dx^\alpha}{d\lambda}\frac{dy^b}{d\lambda}$$

---

[1]Throughout by confinement of photons we mean to say that their motion is constrained to lie in the brane, rather than being bounded.

[2]Throughout capital Latin indices take value in the range $(0, 1, ...(3 + n))$, lower case Latin indices take values in the range $(4, ...(3 + n))$ while Greek indices run over $(0, 1, 2, 3)$. Thus, the coordinates of a generic point P of the manifold M will be denoted by $Z^A = (x^\alpha, y^a)$, where x^α denotes the 4D spacetime coordinates and y^a refers to the n extra coordinates of P.

We now assume that our spacetime M^4 corresponds in this scenario to a particular submanifold defined by the n equations $y^a = y_o^a$ =constant. The geometry of M^4 is then determined by the induced metric

$$ds^2 = g_{\alpha\beta}(x, y_o^1, ..., y_o^n)dx^\alpha dx^\beta.$$

Therefore the quantities $^{(4)}\Gamma^\mu_{\alpha\beta}$ which appear on the left-hand side of Eq.(3) may be identified with the Christoffel symbols associated with the metric induced on the leaves of the foliation defined above.

For the warped product space (1) the quantities ϕ^μ and ψ^a reduce respectively to

$$\phi^\mu = -2f_{,a}\dot{x}^\mu\dot{y}^a,$$

and

$$\psi^a = -f^{,a}e^{2f}h_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta,$$

where $f_{,a} = \frac{\partial f}{\partial y^a}$, a dot denotes differentiation with respect to the affine parameter λ and we are using the notation $f^{,a} = k^{ab}f_{,b}$. Note that if the warping function f is constant the equations (2) which describe the geodesics of the entire warped product M separates into the geodesic equations for the two submanifolds M^4 and N^n. On the other hand if we restrict ourselves to the class of timelike or null geodesics of M we have the following first integral

$$e^{2f}h_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - k_{ab}\dot{y}^a\dot{y}^b = \epsilon, \tag{6}$$

where $\epsilon = 1$ in the case of particles with nonzero rest mass and $\epsilon = 0$ in the case of photons. With the help of the above first integral it is now possible to decouple the motion in the extra dimensions from the spacetime. Thus the geodesic equations (2) finally take the form

$$\ddot{x}^\mu +^{(4)}\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -2f_{,a}\dot{x}^\mu\dot{y}^a, \tag{7}$$

$$\ddot{y}^a + \Gamma^a_{bc}\dot{y}^b\dot{y}^c = -f^{,a}(\epsilon + k_{bc}\dot{y}^b\dot{y}^c). \tag{8}$$

These equations constitute a system of second-order ordinary differential equations which can, in principle, be solved once the function $f = f(y^1, ..., y^n)$ is given.

3 Confinement of the motion in four-dimensional spacetime

In this section we wish to investigate the possibility of confinement of massive particles and photons in the spacetime manifold M^4. This amounts to finding the mathematical conditions required to be satisfied by the warping function f such that timelike and null geodesics of M coincide with those corresponding to M^4. In general such submanifolds of a Riemannian (or pseudo-Riemannian) manifold M are referred to as totally geodesic. In the case where M is a Riemannian space, then according to a theorem of differential geometry the submanifold M^4 is totally geodesic if and only if all normal curvatures (or extrinsic curvatures) of M^4 vanish [13]. If the geometry of M is pseudo-Riemannian (in our case M is a Lorentzian warped product space) then, as we shall show below, the theorem is still valid, however the confinement of null geodesics does not depend on the value of the normal curvatures of the submanifold M^4. From a physical standpoint it is interesting that for the large class of warped product spaces defined by the equation (1) the motion of photons is not constrained by the extrinsic (or normal) curvature of the spacetime. This result generalises a recent result of the authors concerning the embeddings of hypersurfaces with codimension one [11].

To show this, consider equations (7) and (8). Let γ be a timelike (or spacelike) geodesic curve of the submanifold M^4. Since $\gamma \in M^4$ we must have $\dot{y}^a = 0$, which implies that (7) is identically satisfied. Now if $f^{,a}(y_o^1, ...y_o^n) = 0$, then (8) also holds. Therefore Eq. (2) is satisfied, implying that γ is a geodesic of M. Conversely, if any geodesic γ of M with parametric equations $(x^\alpha = x^\alpha(\lambda), y^a = y_o^a)$ is a geodesic of M^4 then from (8) we must have $f^{,a}(y_o^1, ...y_o^n) = 0$. Let us now calculate the normal curvatures of M^4. From (1) we see that the vectors $N_{(a)} = \frac{1}{\sqrt{k_{aa}}}\frac{\partial}{\partial y^a}$ (where no summation is implied over the index a) are normal to the submanifold M^4. Let γ be a curve of M^4 with tangent vector given by $V = (\frac{dx^\alpha}{d\lambda}, 0)$. The normal curvature of M^4, at a point $p \in M^4$, in the direction of $N_{(a)}$ is given by the inner product $\Omega_a = g(N_{(a)}, \frac{DV}{d\lambda})$ at p, where $\frac{DV}{d\lambda}$ denotes the covariant derivative of V with respect to the Levi-Civita connection determined by g. After computing $\frac{DV}{d\lambda}$ from (1) we can easily show that

$$\Omega_a = g\left(\frac{1}{\sqrt{k_{aa}}}\frac{\partial}{\partial y^a}, \frac{DV}{d\lambda}\right) = \frac{k_{ab}}{\sqrt{k_{aa}}}\Gamma^b_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \frac{f_{,a}}{\sqrt{k_{aa}}}e^{2f}h_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta,$$

where, again, no summation over index a is intended. Thus the coefficients of the normal curvature $\Omega_a = \Omega_{\alpha\beta a}\dot{x}^\alpha \dot{x}^\beta$ are given by $\Omega_{\alpha\beta a} = \frac{f_{,a}}{\sqrt{k_{aa}}} e^{2f} h_{\alpha\beta}$. It is clear that $\Omega_{\alpha\beta a} = 0$ if and only if $f_{,a} = 0$. Therefore, in view of the above we conclude the following: *timelike or spacelike geodesics can be confined to the submanifold M^4 if and only if the normal curvature of M^4 vanishes*. It is interesting to ask what happens in the case of null geodesics. Now since for a curve γ to be a null geodesic of M^4 ϵ needs to be zero, equations (7) and (8) imply that γ would also be a geodesic of M, irrespective of the value of $f_{,a}$, and hence irrespective of the value of the normal curvatures of M^4. Thus the motion of photons in this setting is not constrained by the extrinsic curvature of the hypersurface.

The discussion of the geodesics in the previous section allows this theorem to be generalised to cases where the geometry of the ambient space M is Riemannian and the hypersurface under consideration is replaced by a submanifold of codimension n. This can be readily seen from Eqs. (8) by noting that the condition for the vanishing of the extrinsic (normal) curvatures in this case is given by $f_{,a} = 0$ which ensures that the 4D part of the geodesic Eqs. (8) is geodesic.

The above results give the conditions for the geodesics of the higher dimensional space M to be confined to the co-dimension n hypersurface, but do not give any information concerning the stability of such a confinement. We shall consider this question in the following sections, using a phase space analysis. This generalises a recent result of the authors concerning the embeddings of hypersurfaces with codimension one [11].

4 Motion in the extra dimensions: an analysis of the phase space

Defining $\frac{dy^a}{d\lambda} = z^a$ allows the equations (8) to be expressed as an autonomous dynamical system given by

$$\frac{dy^a}{d\lambda} = z^a \tag{9}$$
$$\frac{dz^a}{d\lambda} = P^a(z, y),$$

where $P^a(z, y) = -f^{,a}(\epsilon + k_{bc} z^b z^c) - \Gamma^a_{bc} z^b z^c$, with $\epsilon = 0, 1$. Writing the equations as a dynamical system allows the equilibrium points (given here by $\frac{dy^a}{d\lambda} = 0 = \frac{dz^a}{d\lambda}$) as well as their stabilities to be studied. These in turn allow a great deal of information to be gained regarding the types of behaviour allowed by the system (see e.g. [14]).

5 Motion of massive particles in the neighbourhood of the sub-manifold $\mathbf{M}^4$

Let us start with the investigation of the motion of massive particles. For simplicity we shall first consider the case where the metric k_{ab} is Euclidean, i.e. $k_{ab} = diag(+1, ... + 1)$. It is then easy to see that the equilibrium points exist if the simultaneous equations $f_{,a} = 0$ have real roots (which we denote by $y_o = (y_o^1, ..., y_o^n)$). The equilibrium points are then given by $(z^a = 0, y = y_o)$.

The nature as well as the stability of these equilibrium points can be obtained by linearising equations (9) and studying the eigenvalues of the corresponding Jacobian matrix about the equilibrium points. In this case this is a $2n \times 2n$ matrix given by

$$\Omega = \begin{bmatrix} 0_{n\times n} & 1_{n\times n} \\ -(f_{ab})_{n\times n} & 0_{n\times n} \end{bmatrix}, \tag{10}$$

where $f_{,ab} = \frac{\partial^2 f}{\partial y^a \partial y^b}$, and $0_{n\times n}$ and $1_{n\times n}$ are zero and unit $n \times n$ matrices respectively. It is not difficult to see that the determinant and the trace of Ω ($\det \Omega$ and I) are given, respectively, by $\det \Omega = (-1)^n \det f_{,a}$ and $I = 0$.

For general forms of the warping function the eigenvalues of the system will satisfy a polynomial of order $2n$ which would be difficult to analyse analytically. Our primary aim here, however, is see whether the system is in principle capable of providing confinement of particles in the neighbourhood of the spacetime $(3 + 1)$ hypersurface in the co-dimension n setting. Thus rather than pursuing the general case we shall ask whether there are classes of special warping functions f for which such confinement is possible.

As an example we shall consider the cases where f_{ij} computed at the equilibrium points is zero for all $i \neq j$ and real positive numbers for $i = j$. In such cases the matrix simplifies and the eigenvalues (all of which turn out to be pure imaginary) can be readily found:

$$\lambda = \pm i\sqrt{f_{,aa}}, \quad a = 1....2n, \tag{11}$$

where no summation is intended over a. This condition would clearly be satisfied for warping functions of the type

$$f = \sum_a c_a(y^a)^n, \tag{12}$$

where n is a positive integer.

Writing the system (9) with (10) in terms of normal coordinates, allows it to be expressed in terms of de-coupled harmonic oscillators:

$$\frac{d^2 y^j}{dt^2} = -\omega_j^2 y^j, \tag{13}$$

where $\omega_j = f_{jj}$ and again no summation is intended over j.

Thus in such co-dimension n cases, one can view the motion in extra dimensions as confined to n-tori. This generalises the case of the centre equilibrium point that was found in the co-dimension one case recently [11] and amounts to a *toroidal confinement* of particles in the neighbourhood of the spacetime hypersurface.

6 Confinement through an effective potential

An analysis of the motion in the extra dimensions can also be carried out by reducing the problem to the motion of a particle subjected to the action of an *effective potential* $V = V(y)$. For generality in this section we shall assume that the metric k_{ab} in Eq. (1) is a Riemannian (positive-definite) metric.

The geodesic equations for the extra dimensions part are given by

$$\frac{d\dot{y}_a}{d\lambda} - \frac{1}{2}k_{bc,a}\dot{y}^b\dot{y}^c + f_{,a}e^{2f}h_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{14}$$

In these coordinates equation (6) becomes

$$e^{2f}h_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} - k_{ab}\frac{dy^a}{d\lambda}\frac{dy^b}{d\lambda} = \epsilon, \tag{15}$$

which again allows the decoupling of the motion in the extra dimensions with equations

$$\frac{d\dot{y}_a}{d\lambda} - \frac{1}{2}k_{bc,a}\dot{y}^b\dot{y}^c + f_{,a}\left(\epsilon + k_{bc}\dot{y}^b\dot{y}^c\right) = 0. \tag{16}$$

A first integral of the above equation may be found by multiplying (16) by the factor $2\dot{y}^a e^{2f}$ to give

$$2e^{2f}\left(\dot{y}^a\frac{d\dot{y}_a}{d\lambda} - \frac{1}{2}\dot{y}^a k_{bc,a}\dot{y}^b\dot{y}^c\right) + 2\dot{y}^a e^{2f}f_{,a}\left(\epsilon + k_{bc}\dot{y}^b\dot{y}^c\right) = 0,$$

which, in turn, gives

$$e^{2f}\frac{d}{d\lambda}\left(k_{bc}\dot{y}^b\dot{y}^c\right) + \frac{de^{2f}}{d\lambda}\left(k_{bc}\dot{y}^b\dot{y}^c\right) + \epsilon\frac{de^{2f}}{d\lambda} = 0.$$

Integrating we obtain

$$e^{2f}\left(k_{bc}\dot{y}^b\dot{y}^c\right) = C - \epsilon e^{2f}.$$

Now, if we assume that $f(0,...,0) = 0$, then it follows that

$$C = \dot{y}_0^2 + \epsilon,$$

where $\dot{y}_0^2 \equiv k_{bc}\dot{y}_0^i\dot{y}_0^j$ is related to the initial kinetic energy corresponding to the motion in the extra dimensions. Thus, we have

$$e^{2f}\left(k_{bc}\dot{y}^b\dot{y}^c\right) = \dot{y}_0^2 - V(y), \tag{17}$$

where $V(y)$ can be treated as an effective *potential energy* given by

$$V = \epsilon^2(e^{2f} - 1).$$

Given that the metric k_{ab} is positive-definite the motion is not allowed in the region where $V(y) > \dot{y}_0^2$. On the other hand, the particle may be bound to Σ in the neighbourhood of $y = 0$ if $y = 0$ is a point of minimum of $V(y)$. We note, however, that in cases where the embedding of the submanifold M^4 has codimension greater than one we can have bounded motion without the particle crossing the brane.

As a simple example we consider the case corresponding to an embedding of M^4 with codimension two, that is, $M = M^4 \times N^2$. Let us assume the geometry of N^2 is given by the line element

$$dl^2 = u^2(r)(dr^2 + r^2 d\theta^2). \tag{18}$$

Clearly we can interpret the *radial* coordinate r as related to the distance between the points of M and the submanifold M^4, while the function $u(r)$ gives a measure of the non-Euclideanicity of the metric k_{ab}. Also since two-dimensional Riemannian manifolds are conformally flat, any two-dimensional metric k_{ab} can locally be put in the form (18), but in general with $u = u(r, \theta)$ [15]. We now make the assumption, which in this case seems to be rather natural, that the warping function f is also a function of r, that is, $f = f(r)$. With these assumptions the equation of the motion (17) yields

$$e^{2f} u^2 \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) = \dot{y}_0^2 - V(r).$$

From the equation of motion (16) for θ, we get the following constant of the motion:

$$u^2 r^2 \dot{\theta} = L = const.$$

In this way, the equation for the radial motion reduces to

$$e^{2f} u^2 \dot{r}^2 = \dot{y}_0^2 - \left(V(r) + L^2 \frac{e^{2f}}{u^2 r^2} \right).$$

Here L can be thought of as being related to the angular momentum of the motion of the particle around the submanifold M^4. Clearly the existence of bounded states depends upon the behaviour of the effective potential

$$V_{eff} = \left(V(r) + L^2 \frac{e^{2f}}{u^2 r^2} \right).$$

We shall end this section by deducing some general properties of the motion. Let us assume that $f(0) = 0$ and that $u(0)$ is regular and does not vanish at the origin, since otherwise the metric g would not be well defined at the spacetime M^4. Now if $L \neq 0$ then the dominant term in the potential V_{eff} in the limit $r \to 0$ is $L^2 \frac{e^{2f}}{u^2 r^2} \to \infty$. We therefore have our first result (which is physically obvious) thus: if $L \neq 0$, then the particle cannot return to the submanifold, whether or not its motion corresponds to a bound state. On the other hand, for $L = 0$, V_{eff} acts as confining potential (for any value of $\dot{y}_0^2$) if asymptotically (as $r \to \infty$), we have $f(r) \to \infty$.

As a simple application of this result, let us consider a particular case found in the literature [16], in which the functions $f(r)$ and $u(r)$ are explicitly given by

$$e^{2f} = \frac{c^2 + ar^2}{c^2 + r^2}$$

$$u^2 = \frac{c^4}{(c^2 + r^2)^2},$$

where a and c are real constants. In this case the effective potential is given by

$$V_{eff} = \left[\epsilon^2 \left(\frac{c^2 + ar^2}{c^2 + r^2} \right) - 1 \right] + L^2 \frac{(c^2 + ar^2)(c^2 + r^2)}{c^4 r^2}$$

The behaviour of V_{eff} is displayed in the Fig. 1, for two different values of the parameter L. The upper curve (with $L = 1$) corresponds to the case where $V_{eff} \to \infty$ as $r \to 0$ and hence the particle cannot cross the submanifold M^4, despite the fact that it might be in a bound state. The lower curve (with $L = 0$), on the other hand, corresponds to the case where the particle is confined to M^4.

As can be seen, V_{eff} will act as a confining potential if $a > 1$ and $L = 0$. In such cases, there exists a limiting value of the kinetic energy above which the particle will escape to infinity.
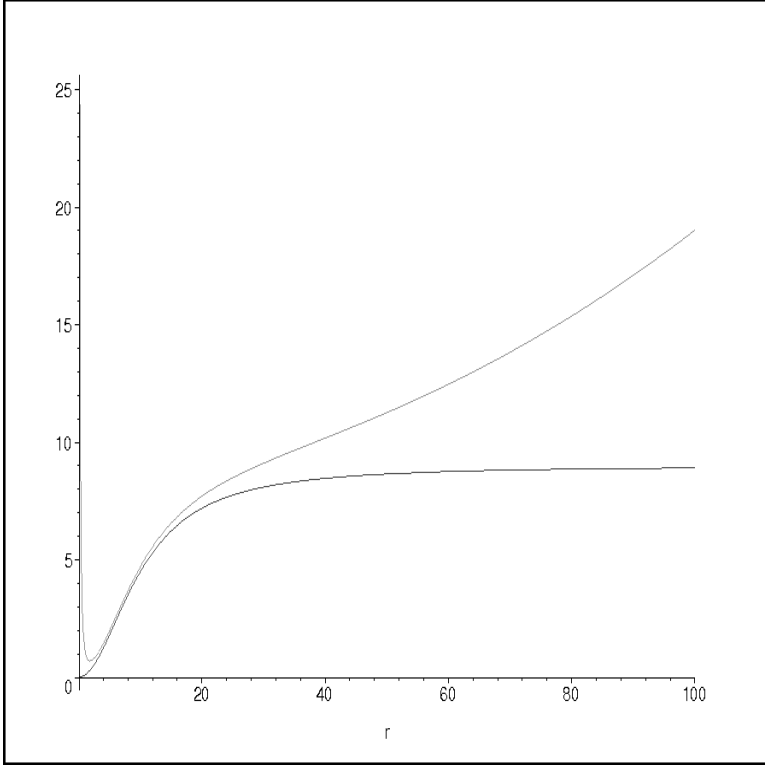
Figure 1: Figure showing the behaviour of the effective potential V_{eff} for different values of L. For the upper curve (with $L = 1$), $V_{eff} \rightarrow \infty$ as $r \rightarrow 0$, implying that the particle cannot cross the submanifold M^4. For the lower curve (with $L = 0$), on the other hand, the particle is confined to M^4. The value of the parameter c was taken to be $c = 100$.

7 Conclusions

In this paper we have examined some aspects of the motion of massive particles and photons in a $(3+1+n)$-dimensional warped product spaces. Spaces of this type, where the codimension of the embedding is one or two, have received a great deal of attention over the recent years mainly in connection with the so-called braneworld scenarios. Our treatment has been geometrical and classical in nature.

We have derived the conditions under which timelike and spacelike geodesics in the full space M coincide with those on the codimension n hypersurface. We also have shown that the motion of photons does not depend on the extrinsic curvature. Employing the splitting that naturally occurs in such spaces between the motion in the hypersurface and the remaining dimensions, and using plane analysis, further allows the stability of such a confinement to be also studied. Using this approach, we have found a novel form of quasi-confinement (namely toroidal confinement) which is neutrally stable. The importance of such confinements is that they are due purely to the classical gravitational effects, without requiring the presence of brane type confinement mechanisms.

Finally in connection with our results regarding the conditions that need to be satisfied by a warped product space in order to ensure the confinement and stability of geodesic motions, we would like to mention a recent work [17], in which the author considers an analogous question. However, although this analysis is quite general, the author restricts himself to the case of codimension one embedding. Both approaches would essentially lead to same results when the warped space is taken to be five-dimensional.

Acknowledgement

References

[1] See e.g T. Appelquist, A. Chodos and P.G.O. Freund, Modern Kaluza–Klein Theories (Addison–Wesley, Reading MA. 1987).

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999); L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999).

[3] P.S. Wesson and J. Ponce de Leon, J. Math. Phys. 33 3883 (1992); J.M. Overduin and P.S. Wesson, Phys. Rept. 283 303 (1997), gr-qc/9805018; P.S. Wesson, Space-Time-Matter (World Scientific, Singapore 1999).

[4] T. Shiromizu, K-i. Maeda and M. Sasaki, Phys. Rev. D62 024012 (2000)

[5] R. Maartens, Living Reviews Relativity 7 (2004), gr-qc/0312059.

[6] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B 436, 257 (1998)

[7] G. N. Felder, A. Frolov and L. Kofman, Class. Quantum Grav. 19, 2983 (2002); M. Arik, A. Baykal, M. C. Çalik, D. Çiftçi, and Ö. Delice, Phys. Rev. D 68, 123503 (2003).

[8] R. Koley, S. Kar, Class.Quant.Grav. **24**, 79-94 (2007), hep-th/0611074.

[9] N. Chatillon, C. Macesanu and M. Trodden, Phys. Rev. D74, 124004 (2006); R. Koley, S. Kar, Class. Quant. Grav. **24**, 79 (2007); C. de Rham and A. J. Tolley JCAP 0602, 003 (2006); C. Charmousis and R. Zegers, Phys. Rev. D72, 064005 (2005); J. Vinet and J. M. Cline, Phys. Rev. D71, 064011 (2005).

[10] R. L. Bishop and B. O. O´Neill, Trans. Am. Math. Soc. 145, 1 (1969); See also J. K. Beem, P. E. Ehrlich and K. Easley, Global Lorentzian geometry (Marcel Dekker, New York, 1996); J. K. Beem and P. E. Ehrlich, Math. Proc. Camb. Phil. Soc. 85, 161 (1979); J. Carot and J. da Costa, Class. Quantum Grav. 10, 461 (1993).

[11] F. Dahia, C. Romero, L. F. P. Silva, R. Tavakol, J. Math. Phys. 48, 072501 (2007).

[12] P. S. Wesson and J. Ponce de Leon, Astron. Astrophys. 294, 1 (1995); P. S. Wesson, B. Mashhoon, H. Liu, and W. N. Sajko, Phys. Lett. B 456, 34 (1999); H. Ishihara, Phys. Rev. Lett. 86, 381 (2001); F. Dahia, E. M. Monte and C. Romero, Mod. Phys. Lett. A 18, 1773 (2003); E. Anderson and R. Tavakol, JCAP 510, 17 (2005); M. E. Kahil, J. Math. Phys. 47, 052501 (2006); D. Youm, Phys. Rev. D62, 084002 (2000); S. Jalalzadeh, B. Vakili, F. Ahmadi, H.R. Sepangi, Class. Quant. Grav, 23, 6015 (2006); P. de Leon, Gen. Rel. Grav, 36, 1333 (2004); Gen. Rel. Grav, 36, 1335 (2004).

[13] L. P. F. Eisenhart, Riemannian Geometry (Princeton University Press, Princeton, 1997).

[14] M. W. Hirsch, S. Smale, I. I. Gordon, A. G. Maier, *Differential Equations, Dynamical Systems, and an Introduction to Chaos* (Academic Press, New York, 2007).

[15] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern Geometry - Methods and Applications: Part I: The Geometry of Surfaces, Transformation Groups, and Fields* (Springer, New York, 1991).

[16] M. Gogberashivilli and D. Singleton, Phys. Lett. B **582**, 95 (2004). See, also, V.Dzhunushaliev, V. Folomeev, D. Singleton, and S. Aguilar-Rudametkin, hep-th/0703043.

[17] S. S. Seahra, Phys. Rev. D68, 104027 (2003)