CONFORMAL GEOMETRODYNAMICS: EXACT NONSTATIONARY SPHERICALLY SYMMETRIC SOLUTIONS

M.V. Gorbatenko

Russian Federal Nuclear Center - All-Russian Research Institute of Experimental Physics, Sarov, Nizhnii Novgorod region,

E-mail: gorbatenko@vniief.ru

Abstract

A nonstationary spherically symmetric problem for conformal geometrodynamics equations is considered and general exact solutions in quadratures are obtained. Involvement of Weyl degrees of freedom allows us to consider the problem with arbitrary initial data, as for the conformal geometrodynamics equations the Cauchy problem is set up without connections to initial data. The results of this paper are not confined with the framework of the perturbation theory and open up new avenues for study of the process of space-time singularity evolution in time.

1. Introduction. CGD equations

It is well known that the general relativity equations for the empty Riemannian space can be used to describe dynamic evolution only of those spaces, which on the initial space-like hypersurface satisfy four connections. Given a spherically symmetric (SphS) problem, because of the connections the general relativity equations cannot be used to consider the space evolution with arbitrary spherically symmetric distribution of the metric tensor components at the initial time.

A way out of this situation is sought for by different investigators in different directions: some of them through inclusion of Riemann-tensor quadratic terms in the general relativity equations (see, e.g., [1]) while the others through inclusion of field fluctuations breaking the spherical symmetry and carrying away particle multipole moments near the event horizon in the form of Hawking heat emission (see, e.g., [2]). The typically used method for studying field configurations (including the singular ones) therewith involves the notion of the observer, which is a trial material body fit out with frame attributes and moving freely in a space under consideration. That is the method is used in which the field configuration is considered as the one that has already appeared, its appearance history is not taken into account. One more way-out direction is pursued through the search for the exact general relativity equation solution for the body generating the Schwarzschild field and the particle perturbing the field irrespective of how small the difference from the sphericity would be (see, e.g., [3]).

It may turn out that all or some field configurations considered in all these approaches can never be realized in principle. This can happen, for example, when the dynamic equations lead to appearance and development of discontinuity surfaces near potential points of appearance of the singularities. In these cases the discontinuity surfaces in essence can play the role of field configuration regularizers impeding the singularity formation.

However, the mentioned mechanism of counteraction to the appearance of singularities a fortiori cannot appear when the general relativity equations for empty space are used for the dynamics equations. These equations are known to admit only weak discontinuities (discontinuities only of second normal derivatives of the metric component) and only on light-like surfaces.

As a candidate for more general dynamic equation of particular interest are the conformal geometrodynamics (CGD) equations derived in [4] (without lambda term) and in [5] (with lambda term). The CGD equations describe Weyl empty space dynamics and, as shown in [6], admit setting up the Cauchy problem without connections to initial data. It is this property of the CGD equations that allows us to consider the time evolution of Riemannian space with arbitrary initial data.

In this paper, for simplicity we restrict our consideration to the CGD equations without lambda term. According to [4], in this case the CGD equations are given by

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}, \tag{1}$$

where

$$T_{\alpha\beta} = -2A_\alpha A_\beta - g_{\alpha\beta} A^2 - 2g_{\alpha\beta} A^\nu_{;\nu} + A_{\alpha;\beta} + A_{\beta;\alpha}. \tag{2}$$

As it follows from (2), alongside metric tensor $g_{\alpha\beta}$ vector $A_\alpha$appears in $T_{\alpha\beta}$. CGD equations (1) with tensor $T_{\alpha\beta}$of form (2) differ from the general relativity equations for empty space

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = 0 \tag{3}$$

in two points: first, in the presence of the nonzero energy-momentum tensor $T_{\alpha\beta}$ in the right-hand side; second, in a specific structure of the tensor. As a consequence of the structure the CGD equations possess invariance under conformal transformations

$$\left. \begin{array}{l} g_{\alpha\beta} \to g'_{\alpha\beta} = g_{\alpha\beta} \cdot e^{2\sigma} \\ A_\alpha \to A'_\alpha = A_\alpha - \sigma_{,\alpha} \end{array} \right\}. \tag{4}$$

Here σ is an arbitrary scalar function.

The first difference ($T_{\alpha\beta} \neq 0$) leads to the fact that the dynamic equations begin to admit existence of space-like discontinuity surfaces, for example, in the form of shock waves. At the same time, the dynamic equations used by us describe Weyl empty space dynamics. Thus, the CGD equations are in essence the implementation of Wheeler's idea about the purely geometrodynamic description of matter, but in the Weyl rather than Riemannian space. The second difference (the invariance under conformal transformations) leads to uniqueness of the structure of the tensor $T_{\alpha\beta}$.

We now turn to the discussion of the SphS problem.

First of all recall that in the general relativity the statement known as Israel's theorem (see [7]) has been proved, according to which (see, e.g., [8]-[10]) the solution to the SphS problem in flat asymptotics at infinity is automatically static. Besides, in this problem nothing except Schwarzschild solution or its equivalent (with an accuracy to coordinate transformation on some map) can be obtained whatsoever.

In the case of the CGD equations the SphS problem possesses a great variety of solutions. The static SphS solutions to the CGD equations complemented with a so-called lambda term are discussed in detail in ref. [11].

The solutions are shown to include three branches, each of which is determined by five integration constants. In this paper we will take up the research into another wide class of the SphS solutions to the CGD equations, i.e. the nonstationary SphS solutions. For the general relativity analogs of the nonstationary SphS solutions can be obtained only with nonzero energy-momentum tensors over the entire space. From the standpoint of the general relativity CGD equations (1) are an alternative of the general relativity equations with the energy-momentum tensor of some matter.

2. The spherically symmetric problem for the CGD equations

2.1. The general form of the metric and Weyl vector in the SphS problem

Here we will not derive the spherically symmetric metric. The derivations can be found in many general relativity monographs, see, e.g., [8]. The metric that will be hereinafter referred to as spherically symmetric is given by

$$ds^2 = -e^{\gamma} \cdot dt^2 + e^{\alpha} \cdot dx^2 + e^{\beta} \cdot \left[d\theta^2 + sin^2\theta \cdot d\varphi^2 \right]. \tag{5}$$

Besides assumptions of the metric, certain assumptions of the structure of vector $A_\alpha$should be made when solving the CGD equations. We assume that of 4 independent components of vector A_α in case of SphS as few as two remain, i.e.

$$A_\alpha = (A_0, A_1, A_2, A_3) \quad \rightarrow \quad A_\alpha = (\varphi, f, 0, 0) \, . \tag{6}$$

All the five introduced functions $\alpha,\ \beta,\ \gamma,\ \varphi,\ f$ are functions of time t and radial variable x. The metric in form (5) is considered in detail in ref. [8], which also presents expressions for Christoffel symbols that will be used in what follows as well as for Riemann tensor components.

When reasoning from the metric in form (5), it can be noticed that the radial variable is determined not uniquely by this form. Performing the transformation of the radial variable, we notice that the metric form remains unchanged, but relations between components g_{11} and g_{22} do change. The radial variable can be selected so, that functions α and β coincide,

$$\alpha = \beta. \tag{7}$$

In what follows as assume that the choice of the radial variable is made just in this way, i.e. so that relation (7) hold.

The coordinate transformation spoken about above can be performed both in the general relativity and in CGD. But in the case of the CGD equations the "freedom" of transformation of functions $\alpha,\ \beta,\ \gamma,\ \varphi,\ f$ is not limited to the possibility to attain the fulfillment of relation (7). One more transformation can be performed, viz. the conformal transformation with function σ [see (4)]

$$\sigma = -\frac{1}{2}\beta. \tag{8}$$

Upon the two above transformations for the CGD equations the metric form will include only one function γ,

$$ds^2 = -e^{\gamma} \cdot dt^2 + dx^2 + \left[d\theta^2 + sin^2\theta \cdot d\varphi^2\right], \tag{9}$$

with the form of vector A_α (6) remaining unchanged.

Thus, in what follows we take $\alpha = 0,\ \beta = 0$ in CGD equations, so that of five functions $\alpha,\ \beta,\ \gamma,\ \varphi,\ f$ only three remain: $\gamma,\ \varphi,\ f$. After that the CGD equations for the three functions are written in the following form:

$$-1 = e^{-\gamma} \cdot 3\varphi^2 - f^2 - 2f', \tag{10}$$

$$-1 = e^{-\gamma} \cdot \left[\varphi^2 + 2\dot{\varphi} - \dot{\gamma}\varphi\right] - 3f^2 - \gamma' f, \tag{11}$$

$$0 = -2\varphi f + \varphi' + \dot{f} - \gamma'\varphi, \tag{12}$$

$$\frac{\gamma''}{2} + \frac{\gamma'^2}{4} = e^{-\gamma} \cdot \left[\varphi^2 + 2\dot{\varphi} - \dot{\gamma}\varphi\right] - f^2 - 2f' - \gamma' f. \tag{13}$$

Instead of function γ it is convenient to introduce a new function C defined as

$$C = \sqrt{-g} = \sqrt{e^{\gamma}} = e^{\gamma/2}. \tag{14}$$

We obtain equations (10)-(13) in the new form:

$$-1 = 3\frac{\varphi^2}{C^2} - f^2 - 2f' \tag{15}$$

$$-1 = \frac{1}{C^2}\cdot\left[\varphi^2 + 2\dot{\varphi} - 2\varphi\frac{\dot{C}}{C}\right] - 3f^2 - 2f\frac{C'}{C} \tag{16}$$

$$0 = -2\varphi f + \varphi' + \dot{f} - 2\varphi\frac{C'}{C} \tag{17}$$

$$\frac{C''}{C} = \frac{1}{C^2}\cdot\left[\varphi^2 + 2\dot{\varphi} - 2\varphi\frac{\dot{C}}{C}\right] - f^2 - 2f' - 2f\frac{C'}{C} \tag{18}$$

Having extracted (15) and (16) from (18), we arrive at:

$$\frac{C''}{C} = -2 - 3\frac{\varphi^2}{C^2} + 3f^2. \tag{19}$$

Thus, equation (19) can be considered instead of (18) in what follows.

Ref. [12] shows that from the CGD equations of form (1) with energy-momentum tensor (2) equation

$$F^{\alpha\sigma}_{;\sigma} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\sigma}\left(\sqrt{-g}\; F^{\alpha\sigma}\right) = 0. \tag{20}$$

From (20) it follows that

$$\dot{f} = \varphi' - C\cdot\phi_0. \tag{21}$$

Emphasize that relation (21) follows from CGD equations (15)-(17), (19). The particular attention paid to relation (21) is due to its simplicity.

Rearrange equations (15)-(17), (19), having made one more function substitution,

$$z \equiv \frac{\varphi}{C}, \tag{22}$$

and using relation (21) to eliminate $\dot{f}$. Equation (15) will become

$$f' = \frac{1}{2} + \frac{3}{2}z^2 - \frac{1}{2}f^2. \tag{23}$$

Equation (16) will assume form

$$\frac{\dot{z}}{C} = -\frac{1}{2} - \frac{1}{2}z^2 + \frac{3}{2}f^2 + f \cdot \left(\frac{C'}{C}\right). \tag{24}$$

Equation (17) will take form

$$z' = \frac{\phi_0}{2} + fz. \tag{25}$$

Finally, equation (19) will become

$$\frac{C''}{C} = -2 + 3T, \tag{26}$$

where

$$T \equiv f^2 - z^2. \tag{27}$$

Using definition (27) for function T and formula (21) for $\dot{f}$, equation (24) can be written as

$$\frac{\dot{T}}{C} = (z - \phi_0 f - zT). \tag{28}$$

Equations (21), (23)-(26) [or their equivalent equations (21), (23), (25)-(28)] just make up the equation system, to the solution of which we will pass on right now.

2.2. Determination of desired functions

By direct verification we make sure that from equations (23), (25) relation

$$(z - \phi_0 f - zT)' = 0. \tag{29}$$

follows. Thus, there is relation among functions z, f, T:

$$z - \phi_0 f - zT = F_1(t), \tag{30}$$

where $F_1(t)$ is an arbitrary function of time. Relation (30) is in essence the first integral of equations (23), (25). And it can be used to determine the entire set of the desired functions. It is convenient to perform the further manipulations using the above introduced functions z and T.

Substitute (30) into (28).

$$\dot{T} = F_1(t) \cdot C. \tag{31}$$

Hence it follows that to obtain the stationary SphS solution function $F_1(t)$ should be set identically zero. In other words:

$$Stationary \quad solution \quad \rightarrow \quad F_1(t) \equiv 0. \tag{32}$$

We are seeking the nonstationary solution, therefore hereinafter we assume that $F_1(t) \neq 0$. Under this assumption from equations (31) and (26) it follows that

$$\frac{\dot{T}''}{\dot{T}} = -2 + 3T,$$

i.e.

$$\dot{T}'' + (2 - 3T)\,\dot{T} = 0.$$

If this equation is written as

$$\frac{\partial}{\partial t}\left[T'' + 2T - \frac{3}{2}T^2\right] = 0,$$

then it will follow from this that

$$T'' + 2T - \frac{3}{2}T^2 = F_2(x), \tag{33}$$

where $F_2(x)$ is some function of radial variable. Determine the function. From the definition of T it follows that

$$T' = 2\left(ff' - zz'\right),$$

$$T'' = 2\left(ff'' + f'^2 - zz'' - z'^2\right).$$

The expressions for f' and z' are determined by equations (23) and (25), respectively. Second derivatives f'', z'' will be determined from those same equations by straightforward differentiation. Upon the substitution of the expressions for derivatives in (33) we obtain:

$$F_2(x) = \frac{1}{2}\left(1 - \phi_0^2\right). \tag{34}$$

Here ϕ_0 is the same constant which appeared in relation (21). Substitute (34) into (33).

$$T'' = -2T + \frac{3}{2}T^2 + \frac{1}{2}\left(1 - \phi_0^2\right). \tag{35}$$

Multiply resultant equation (35) by $2T'$.

$$2T''T' = \left[-4T + 3T^2 + \left(1 - \phi_0^2\right)\right] T'. \tag{36}$$

Equation (36) can be integrated.

$$\left(T'\right)^2 = -2T^2 + T^3 + \left(1 - \phi_0^2\right) T + F_3\left(t\right). \tag{37}$$

Here $F_3\left(t\right)$ is some function of time. The function can be determined. To do this replace T' in (37) with expressions in terms of variables f' and z' and for the derivatives use equations (23), (25). As a result we determine that

$$F_3\left(t\right) = F_1^2\left(t\right). \tag{38}$$

Substitute the determined expression for $F_3\left(t\right)$ into equation (37) and arrive at equation

$$\frac{dT}{\sqrt{-2T^2 + T^3 + \left(1 - \phi_0^2\right) T + F_3\left(t\right)}} = dx. \tag{39}$$

Integrate.

$$x + F_4\left(t\right) = \int \frac{dT}{\sqrt{T^3 - 2T^2 + \left(1 - \phi_0^2\right) T + F_3\left(t\right)}}. \tag{40}$$

The integral appearing on the right is expressed in terms of the well-known doubly periodic Weierstrass function (see, e.g., [13]). In the function one period is real, while the other is purely imaginary. The periods depend on $F_1\left(t\right)$ and ϕ_0.

If we transfer to integration variable

$$s = T - \frac{2}{3}, \tag{41}$$

then relation (40) is reduced to form

$$x + F_4(t) = -2 \int\limits_{T-\frac{2}{3}}^{\infty} \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}. \qquad (42)$$

Here:

$$g_2 = \frac{4}{3} + 4\phi_0^2; \quad g_3 = -\frac{8}{27} - 4F_1^2. \qquad (43)$$

In the standard notation $\wp$ for the Weierstrass function we have:

$$T - \frac{2}{3} = \wp\left(-\frac{x + F_4(t)}{2}; g_2; g_3\right). \qquad (44)$$

Function $F_4(t)$ appearing in $\wp$, as well as function $F_1(t)$ and constant ϕ_0 appearing in the expressions for g_2 and g_3 are arbitrary.

Formula (44) gives the answer to the question of the general solution for function T. The expression for function C is determined using formula (31). It remains to determine functions z and f. For this purpose we consider relation (27) and first integral relations (30) as two algebraic equations for determination of functions z and f. Solving these relations, we determine that:

$$z = \frac{F_1 \cdot (1 - T) + \eta\phi_0 \cdot \sqrt{T^3 - 2T^2 + (1 - \phi_0^2)\, T + F_1^2}}{\left[(1 - T)^2 - \phi_0^2\right]}; \qquad (45)$$

$$f = \frac{\phi_0 F_1 + \eta (1 - T) \cdot \sqrt{T^3 - 2T^2 + (1 - \phi_0^2)\, T + F_1^2}}{\left[(1 - T)^2 - \phi_0^2\right]}. \qquad (46)$$

In these expressions functions z and f are expressed only in terms of T, arbitrary time function $F_1(t)$ and some constant ϕ_0. In these expressions η takes on the values of either $\eta = 1$ or $\eta = -1$.

Substituting the resultant expressions for functions z and f into the initial dynamic equations, we can see that these equations get transformed into identities, if function T is determined by relation (44) and function C is calculated using formula (31). These transformations are quite straightforward, but cumbersome, therefore we will not present them here.

Thus, we have found the general solutions for all the functions determining the nonstationary SphS solution.

3. Discussion

This paper has derived the general expressions for functions $T(t,x)$, $C(t,x)$, $z(t,x)$, $f(t,x)$ appearing in the consideration of the nonstationary SphS problem. The expressions for the functions are determined by formulas (44), (31), (45), (46), respectively.

Two special solutions have been constructed[1] . One of them can be associated with a train of spherically symmetric waves converging to center or diverging from it. In either case conformal invariant $\left(F_{\alpha\beta}F_{\mu\nu}g^{\alpha\mu}g^{\beta\nu}\right)$ is nonsingular and nonzero, so that the solutions a fortiori do not refer to the category of conformally flat ones. In either solution this invariant has a finite value.

In connection with the results note that the solutions with spherical symmetry for an empty space refer to simplest exact solutions both in the general relativity and CGD. In the case of the general relativity the SphS solution is unique and reduces to Schwarzschild solution. In the case of CGD the SphS solution set consists of two classes: stationary and nonstationary solutions. The former are analogs of the Schwarzschild solution. As for the nonstationary solutions studied in this paper, in the general relativity there are no analogs of them for the case of empty Riemannian spaces. At the same time it is clear that it is the nonstationary solutions that may be of the greatest interest for the study of the phenomena, such as supernova explosions, Hawking vaporization near event horizons, etc. This supposition can prove valid, if the above phenomena can appear only evolutionarily, i.e. as a result of completion of some phase of the nonstationary solution evolution. Therefore the resort to the nonstationary SphS solutions of the CGD equations opens up new avenues for studying the above-mentioned phenomena.

The appearance of the new class of the nonstationary SphS solutions in the transition from the general relativity to the CGD equations is not occasional. The reason is that in the case of the CGD equations the Cauchy problem is set up without connections to Cauchy data on the initial spacelike hypersurface. The disappearance of four connections to Cauchy data that take place in the case of the general relativity equations unfreezes the degrees of freedom that just make the nonstationary SphS solution appearance possible. In other words, the CGD equations can be used to numerically calculate the metric and Weyl vector evolution under arbitrary initial and

[1] To be published in VANT-TPF.

boundary conditions. This CGD equation property seems to us basically important and serves, in our opinion, a weighty argument in favor of taking into consideration the Weyl degrees of freedom of space-time in cosmological studies.

In conclusion note that using the results of this paper one may try to verify the unbounded energy cumulation instability hypothesis advanced in ref. [14]. In our dynamic equations the cumulation bounding mechanism is associated with the Weyl degrees and appearance of a discontinuity surface in geometrodynamic continuum with the equations of state generated by the equations themselves.

References

[1] R.Bach and H.Weyl. Z. **13**, 134 (1920).

[2] S.W.Hawking. Commun. Math. Phys. **43**, 199 (1975).

[3] L.Herrera. *Possible way out of the Hawking paradox: Erasing the information at the horizon.* arXiv: 0709.4674v1 [gr-qc] 28 Sep 2007.

[4] M.V.Gorbatenko, A.V.Pushkin. *Dynamics of the linear affine connectedness space and conformally invariant extension of Einstein equations.* // VANT-TPF[2] . **2/2**, 40 (1984)

[5] Yu.A.Romanov. *Affine connectedness space dynamics.* // VANT-TPF . **3**, 55 (1996).

[6] M.V.Gorbatenko, Yu.A.Romanov. *Cauchy problem for equations describing the affine connectedness space dynamics.* // VANT-TPF . **2**, 34 (1997).

[7] W.Israel. Phys. Rev. **164**, 1776 (1967).

[8] J.Sing. *The general relativity.* Moscow. Inostrannaya Literatura Publishers (1963).

[9] L.D.Landau, E.M.Lifshits. *The field theory.* Moscow. Nauka Publishers (1988).

[10] A.Z.Petrov. *New methods in the general relativity.* Moscow. Nauka Publishers (1966).

[2] Hereinafter by VANT-TPFis meant Colelction “Voprosy Atomnoy Nauki i Tekhniki. Series: Teoreticheskaya i Prikladnaya Fizika ”.

[11] M.V.Gorbatenko. *Discontinuous solutions in conformally invariant geometrodynamics.* // Advances in Mathematics Research. Vol. 6. Editors: Oyibo, Gabriel. Nova Science Publishers, Inc. (2005).

[12] M.V.Gorbatenko, A.V.Pushkin. *Conformally Invariant Generalization of Einstein Equation and the Causality Principle.* // General Relativity and Gravitation. **34**, No. 2, 175-188 (2002).

[13] E.T.Wittecker, J.N.Watson. *Course of modern analysis.* V. II. Moscow. GIFML Publishers (1963).

[14] E.I.Zababakhin. Pis'ma v ZhETF. **30**, 97 (1979).