# Testing the general relativistic "no-hair" theorems using the galactic center black hole SgrA<sup>\*</sup>

Clifford M. Will

McDonnell Center for the Space Sciences, Department of Physics, Washington University, St. Louis, Missouri 63130; cmw@wuphys.wustl.edu

## ABSTRACT

If a class of stars orbits the central black hole in our galaxy in short period (~ 0.1 year), high eccentricity (~ 0.9) orbits, they will experience precessions of their orbital planes induced by both relativistic frame-dragging and the quadrupolar gravity of the hole, at levels that could be as large as 10 µarcseconds per year, if the black hole is rotating faster than 1/2 of its maximum rotation rate. Astrometric observations of the orbits of at least two such stars can in principle lead to a determination of the angular momentum vector **J** of the black hole and its quadrupole moment  $Q_2$ . This could lead to a test of the general relativistic no-hair theorems, which demand that  $Q_2 = -J^2/M$ . Future high-precision adaptive infrared optics instruments may make such a fundamental test of the black-hole paradigm possible.

Subject headings: galactic center, black hole, general relativity, no-hair theorem

### 1. Introduction

The discovery, using infrared telescopes, of stars orbiting within an arcsecond of the central object SgrA<sup>\*</sup> in our galaxy, together with accurate determinations of their orbits, has provided strong evidence for the existence there of a massive black hole (MBH) of around  $3.6 \times 10^6 M_{\odot}$  (see Alexander (2005) for a review). In addition to opening a window on the innermost region of the galactic center, the discovery of these stars has made it possible to contemplate using orbital dynamics to probe the curved spacetime of a rotating black hole.

The orbital periods of these stars are on the scale of tens of years, and thus most relativistic effects, such as the advance of the pericenter, are too small to be observed at present (see, however Zucker et al. (2006)). Nevertheless, there seems to be every expectation that, with improved observing capabilities, a population of stars significantly closer to the hole will eventually be discovered, making orbital relativistic effects detectable (Jaroszyński 1998; Fragile & Mathews 2000; Rubilar & Eckart 2001; Weinberg et al. 2005; Kraniotis 2007). Furthermore, plans are being developed to achieve infrared astrometry on such objects at the level of 10  $\mu$ arcseconds (Eisenhauer et al. 2008). Highprecision Doppler measurements may also be possible (Zucker et al. 2006).

This makes it possible to consider doing more than merely detect relativistic effects, but rather to provide the first test of the black hole "no-hair" or uniqueness theorems of general relativity. According to those theorems, an electrically neutral black hole is completely characterized by its mass M and angular momentum J. As a consequence, all the multipole moments of its external spacetime are functions of M and J, specifically, the quadrupole moment  $Q_2 = -J^2/M$  (in units where G = c = 1).

If the black hole were non-rotating (J = 0), then its exterior metric would be that of Schwarzschild, and the most important relativistic effect would be the advance of the pericenter. If it is rotating, then two new phenomena occur, the dragging of inertial frames and the effects of the hole's quadrupole moment, leading not only to an additional pericenter precession, but also to a precession of the orbital plane of



Fig. 1.— Orbital periods vs. eccentricity required to give measurable relativistic precession rates. Dotted curves show minimum periods vs. *e* that avoid tidal disruption, for various stellar masses.

the star. These precessions are smaller than the Schwarzschild effect in magnitude because they depend on the dimensionless angular momentum parameter  $\chi \equiv J/M^2$ , which is always less than one, and because they fall off faster with distance from the black hole. However, accumulating evidence suggests that MBH should be rather rapidly rotating, with  $\chi$  larger than 0.5 and possibly as large as 0.9, so these effects could be significant.

The purpose of this paper is to point out that, if a class of stars were to be found with orbital periods of fractions of a year, and with sufficiently large orbital eccentricities, then the quadrupoleinduced precessions could be as large as 10  $\mu$ as per year. Figure 1 illustrates this: assuming a black hole with  $\chi = 0.7$ , it shows the orbital period required as a function of eccentricity, for the rates of precessions due to Schwarzschild (S), framedragging (J) and quadrupole (Q<sub>2</sub>) terms to be as large as 10, 5, and 1  $\mu$ as per year.

Figure 2 shows the effect of black hole spin on the amplitudes of the relativistic effects. For orbits with eccentricity 0.9 and periods of one year and 0.1 years, the amplitudes of the three effects are plotted in  $\mu$ as per year.

The precession of the orbital plane is the most important effect here, because it depends only on J and  $Q_2$ ; the Schwarzschild part of the metric affects only the pericenter advance. The orbital plane is determined by its inclination angle *i* rel-



Fig. 2.— Relativistic precession amplitudes vs. black hole spin parameter  $\chi$ .

ative to the plane of the sky and by the angle of nodes  $\Omega$  between a reference direction and the intersection of the two planes. Standard astrometric and Doppler observations can determine  $\Omega$ , *i*, the pericenter angle  $\omega$ , the semimajor axis *a* and the orbital eccentricity *e*, and, given sufficient observation time, the secular rates of change  $d\Omega/dt$ , di/dt, and  $d\omega/dt$ .

However, in order to test the no-hair theorems, one must determine five parameters: the mass of the black hole, the magnitude and two angles of its spin, and the value of the quadrupole moment. The "Kepler-measured" mass is determined from the orbital periods of stars, but may require data from a number of stars to fix it separately from any extended distribution of mass. Then, to measure  $\mathbf{J}$  and  $Q_2$ , it is necessary and sufficient to measure  $d\Omega/dt$  and di/dt for two stars in non-degenerate orbits. A test of the no-hairness of the central object in our galaxy would be compelling evidence that it is truly a black hole of general relativity.

## 2. Orbit perturbations in the field of a rotating black hole

For the purpose of this rough analysis, it suffices to work in the post-Newtonian limit. The equation of motion of a body of negligible mass in the field of a body with mass M, angular momentum **J** and quadrupole moment  $Q_2$  is given by

$$\mathbf{a} = -\frac{M\mathbf{x}}{r^3} + \frac{M\mathbf{x}}{r^3} \left(4\frac{M}{r} - v^2\right) + 4\frac{M\dot{r}}{r^2}\mathbf{v}$$

$$-\frac{2J}{r^3}[2\mathbf{v}\times\hat{\mathbf{J}}-3\dot{r}\mathbf{n}\times\hat{\mathbf{J}}-3\mathbf{n}(\mathbf{h}\cdot\hat{\mathbf{J}})/r] +\frac{3}{2}\frac{Q_2}{r^4}[5\mathbf{n}(\mathbf{n}\cdot\hat{\mathbf{J}})^2-2(\mathbf{n}\cdot\hat{\mathbf{J}})\hat{\mathbf{J}}-\mathbf{n}], \quad (1)$$

where  $\mathbf{x}$  and  $\mathbf{v}$  are the position and velocity of the body,  $\mathbf{n} = \mathbf{x}/r$ ,  $\dot{r} = \mathbf{n} \cdot \mathbf{v}$ ,  $\mathbf{h} = \mathbf{x} \times \mathbf{v}$ , and  $\mathbf{\hat{J}} = \mathbf{J}/|J|$  (see, eg. Will (1993)). The first line of Eq. (1) corresponds to the Schwarzschild part of the metric (at post-Newtonian order), the second line is the frame-dragging effect, and the third line is the effect of the quadrupole moment (formally a Newtonian-order effect). For an axisymmetric black hole, the symmetry axis of the hole's quadrupole moment coincides with its rotation axis, given by the unit vector  $\mathbf{\hat{J}}$ . The magnitude of the quadrupole moment will be left free.

Using standard orbital perturbation theory, we find that the precessions per orbit of the orientation variables are given by

$$\Delta \varpi = A_S - 2A_J \cos \alpha$$
$$-\frac{1}{2} A_{Q_2} (1 - 3\cos^2 \alpha), \qquad (2a)$$

$$\sin i\Delta\Omega = \sin \alpha \sin \beta (A_J - A_{Q_2} \cos \alpha), (2b)$$

$$\Delta i = \sin \alpha \cos \beta (A_J - A_{Q_2} \cos \alpha), (2c)$$

where

$$A_S = 6\pi M/p, \qquad (3a)$$

$$A_J = 4\pi J/(mp^3)^{1/2}$$
, (3b)

$$A_{Q_2} = 3\pi Q_2 / m p^2,$$
 (3c)

where  $\Delta \varpi = \Delta \omega + \cos i \Delta \Omega$  is the precession of pericenter relative to the fixed reference direction, and  $p = a(1 - e^2)$  is the semilatus rectum. The quantities  $\alpha$  and  $\beta$  are the polar angles of the black hole's angular momentum vector with respect the star's orbital plane defined by the line of nodes  $\mathbf{e}_p$ , and the vector in the orbital plane  $\mathbf{e}_q$  orthogonal to  $\mathbf{e}_p$  and  $\mathbf{h}$ .

The structure of Eqs. (2b) and (2c) can be understood as follows: Eq. (1) implies that the orbital angular momentum **h** precesses according to  $d\mathbf{h}/dt = \boldsymbol{\omega} \times \mathbf{h}$ , where the orbit-averaged  $\boldsymbol{\omega}$  is given by  $\boldsymbol{\omega} = \mathbf{\hat{J}}(A_J - A_{Q_2} \cos \alpha)$ ; the orbit element variations are given by  $di/dt = \boldsymbol{\omega} \cdot \mathbf{e}_p$  and  $\sin i d\Omega/dt = \boldsymbol{\omega} \cdot \mathbf{e}_q$ . As a consequence, we have the purely geometric relationship,

$$\frac{\sin i d\Omega/dt}{di/dt} = \tan\beta.$$
(4)

To get an idea of the astrometric size of these precessions, we define an angular precession rate amplitude  $\dot{\Theta}_i = (a/D)A_i/P$ , where D is the distance to the galactic center and  $P = 2\pi (a^3/M)^{1/2}$ is the orbital period. Using  $M = 3.6 \times 10^6 M_{\odot}$ , D = 8 kpc, we obtain the rates, in microarcseconds per year

$$\dot{\Theta}_S \approx 13.3 P^{-1} (1 - e^2)^{-1},$$
 (5)

$$\dot{\Theta}_{I} \approx 0.847 \, \chi P^{-4/3} (1-e^2)^{-3/2} \,, \qquad (6)$$

$$\dot{\Theta}_{Q_2} \approx 9.68 \times 10^{-3} \chi^2 P^{-5/3} (1-e^2)^{-2}, (7)$$

where we have assumed  $|Q_2| = M^3 \chi^2$ . The observable precessions will be reduced somewhat from these raw rates because the orbit must be projected onto the plane of the sky. For example, the contributions to  $\Delta i$  and  $\sin i \Delta \Omega$  are reduced by a factor  $\sin i$ ; for an orbit in the plane of the sky, the plane precessions are unmeasurable.

For the quadrupole precessions to be observable, it is clear that the black hole must have a decent angular momentum ( $\chi > 0.5$ ) and that the star must be in a short period high-eccentricity orbit. Figures 1 and 2 show the quantitive requirements, based on these rate amplitudes.

#### 3. Testing the no-hair theorems

Although the pericenter advance is the largest relativistic orbital effect, it is *not* the most suitable effect for testing the no-hair theorems. The frame-dragging and quadrupole effects are small corrections of the leading Schwarzschild pericenter precession, and thus one would need to know M, a and e to sufficient accuracy to be able to subtract that dominant term to reveal the smaller effects of interest. Furthermore, the pericenter advance is affected by a number of complicating phenomena. (i) For such relativistic orbits, Schwarzschild contributions to the pericenter precession at the *second* post-Newtonian order may be needed. (ii) Any distribution of mass (such as dark matter or gas) within the orbit, even if it is spherically symmetric, will generally contribute to the pericenter advance. (iii) Tidal distortions of the stars are likely to occur near the pericenters of the highly eccentric orbits, leading to additional contributions to the pericenter advance of the form  $30\pi (M/m)(R/a)^5 k_2(1+3e^2/2+e^4/8)/(1-e^2)^5,$ where m, R and  $k_2$  are the mass, radius and "apsidal constant", or Love number of the star, respectively. Tidal contributions could be significant for sufficiently close and eccentric orbits.

Of course, if a star gets too close to the black hole, it could be tidally disrupted. This possibility sets a lower bound on the orbital period  $P_{\rm min} \sim 2\sqrt{3}\pi (R^3/m)^{1/2}(1-e)^{-3/2}$ , set by requiring that the pericenter distance exceed the Roche radius of the star. This is illustrated by the dotted curves in Fig. 1.

By contrast, the precessions of the node and inclination are relatively immune from such effects. Any spherically symmetric distribution of mass has no effect on these orbit elements. As long as any tidal distortion of the star is quasiequilibrium with negligible tidal lag, the resulting perturbing forces are purely radial, and thus have no effect on the node or inclination.

From the measured orbit elements and their drifts for a given star, Eq. (4) gives the angle  $\beta$ , independently of any assumption about the no-hair theorems. This measurement then fixes the spin axis of the black hole to lie on a plane perpendicular to the star's orbital plane that makes an angle  $\beta$  relative to the line of nodes. The equivalent determination for another stellar orbit fixes another plane; as long as the two planes are not degenerate, their intersection determines the direction of the spin axis, modulo a reflection through the origin.

This information is then sufficient to determine the angles  $\alpha$  and  $\beta$  for each star. Then, from the magnitude

$$\left(\left[\sin i\frac{d\Omega}{dt}\right]^2 + \left[\frac{di}{dt}\right]^2\right)^{1/2} = \sin\alpha(A_J - A_{Q_2}\cos\alpha)\,,$$
(8)

determined for each star, together with the orbit elements, one can solve for J and  $Q_2$ .

In practice, of course, the analysis of the astrometric data will be carried out in a more sophisticated, if less transparent manner. Using data from all detected stars, one carries out a multiparameter least-squares fit, standard in solarsystem celestial mechanics, to determine their orbit elements. Their motions would be based on Eq. (1) but with M, **J** and  $Q_2$  treated as parameters to be fit along with the orbit elements of each star. If necessary, the model can be extended to include effects of an additional matter distribution, tidal effects, and so on.

### 4. Concluding remarks

We have shown that a class of stars orbiting a rotating central black hole in our galaxy in short period, high eccentricity orbits, will experience precessions of their orbital planes induced by both frame dragging and the quadrupolar gravity of the hole, at levels that could be as large as 10  $\mu$ arcseconds per year. Observations of the orbits of at least two such stars can in principle lead to a determination of the angular momentum vector **J** and quadrupole moment  $Q_2$  of the black hole, and could provide a test of the no-hair theorems of general relativity.

Alternative possibilities for no-hair tests involve timing measurements of pulsars orbiting black-hole companions (Wex & Kopeikin 1999), gravitational-wave measurements of compact objects spiralling into massive black holes (Ryan 1997; Glampedakis & Babak 2006; Hughes 2006), or detection of quasi-normal "ringdown" gravitational radiation of perturbed black holes (Dreyer, et al. 2004; Berti, et al. 2006).

Detecting such stars so close to the black hole, and carrying out infrared astrometry to  $10 \,\mu$ arcsecond accuracy will be a challenge. However, if this challenge can be met with future improved adaptive optics systems currently under study, such as GRAVITY (Eisenhauer et al. 2008), it could lead to a powerful test of the blackhole paradigm.

In future work, we plan to study in detail such complicating effects as second-post-Newtonian (2PN) corrections to the Schwarzschild part of the pericenter advance, tidal effects, effects of unseen mass distributions within the observed stellar orbits, and light deflection and Shapiro time delay effects (Rubilar & Eckart 2001; Weinberg et al. 2005). For example, a torus of matter of mass m orbiting the black hole at a distance R will induce fractional changes in the apparent angular momentum and quadrupole moment of order  $\delta J/J \sim (m/M)(R/M)^{1/2}(1/\chi)$  and  $\delta Q/Q \sim$  $(m/M)(R/M)^2(1/\chi)^2$ , so only a very massive and/or very distant torus will be relevant. We also plan to carry out covariance analyses to obtain more realistic estimates of the accuracies that might be obtained for the no-hair test for given raw astrometric accuracies, and for a range of observing schedules.

This work was supported in part by the National Science Foundation under grant No. PHY 06-52448. We are grateful to the Groupe Gravitation et Cosmologie (GR $\varepsilon$ CO), Institut d'Astrophysique de Paris, Université Pierre et Marie Curie for their hospitality during the initial stages of this work. Peter Ronhovde made useful contributions at an early phase of this study. We thank Andrea Ghez and Robert Reasenberg for helpful comments.

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