A Uniqueness theorem for 5-dimensional Einstein-Maxwell black holes

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Abstract

In a previous paper [gr-qc/0707.2775] we showed that stationary asymptotically flat vacuum black hole solutions in 5 dimensions with two commuting axial Killing fields can be completely characterized by their mass, angular momentum, a set of real moduli, and a set of winding numbers. In this paper we generalize our analysis to include Maxwell fields.

1 Introduction

In $n = 4$ spacetime dimensions, asymptotically flat, stationary vacuum or electrovac black hole solutions are completely characterized by their asymptotic charges—mass, angular momentum, and electric charge [3, 17, 14, 2]. The complete classification of stationary black holes in more than $n = 4$ spacetime dimensions is at present an open problem. However, in a recent paper of ours [12], a partial classification was achieved for vacuum solutions under the assumption that the number of commuting axial[1] Killing fields is sufficiently large. The particular case considered there was $n = 5$, and the number of axial Killing fields required was two[2]. Under this hypothesis, we showed how to construct from the given solution a certain set of invariants consisting of a set of real numbers ("moduli") and a collection of integer-valued vectors ("winding numbers"). These data were called the "interval structure" of the

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[1]By this we mean a Killing field whose orbits are periodic.

[2]The higher dimensional rigidity theorem [11] only gives one extra axial Killing field. This is presumably the generic situation.

solution. It determines in particular the horizon topology, which could be either $S^3, S^1 \times S^2$ or a Lens-space $L(p,q)$. We then demonstrated that the interval structure together with the asymptotic charges gives a complete set of invariants of the solutions, i.e., if these data coincide for two given solutions, then the solutions are isometric.

In this paper, we generalize the analysis of our previous paper [12] to include Maxwell fields. We show that, if certain restrictive additional conditions are imposed upon the Maxwell field and the axial Killing fields, then a similar uniqueness theorem as in the vacuum case can be proven. Namely, we find that the solution is now completely characterized by the interval structure, the magnetic charges, as well as the mass and angular momentum. The extra assumptions placed upon the Killing fields imply that the electric charge (but not the magnetic charges), and one of the angular momenta vanishes. They also imply that the possible interval structures are limited. In particular, the horizon topology can only be either S^3 or $S^2 \times S^1$, but not $L(p,q)$.

Non-trivial Einstein-Maxwell black rings (horizon $S^1 \times S^2$) satisfying our assumptions[3] have been found by [5] (see also [22]).

2 Stationary Einstein-Maxwell black holes in n dimensions

Let (M, g_{ab}, F_{ab}) be an n-dimensional, analytic, asymptotically flat, stationary black hole spacetime satisfying the Einstein-Maxwell equations

$$R_{ab} = \frac{1}{2}\left(F_{ac}F_b{}^c - \frac{g_{ab}}{2(n-2)}F_{cd}F^{cd}\right), \tag{1}$$

$$\nabla_a F^{ab} = 0 = \nabla_{[a}F_{bc]}. \tag{2}$$

Let t^a be the asymptotically timelike Killing field, $\pounds_t g_{ab} = 0$, which we assume is normalized so that $\lim g_{ab}t^a t^b = -1$ near infinity. We assume that also the Maxwell tensor is invariant under t^a, in the sense that $\pounds_t F_{ab} = 0$. We denote by $H = \partial B$ the horizon of the black hole, where the black hole B is defined as usual by $B = M \setminus I^-(\mathscr{I}^+)$, with $\mathscr{I}^\pm$ the null-infinities of the spacetime. It is assumed that the latter has topology $\mathbb{R} \times \Sigma_\infty$, where Σ_∞ is metrically and topologically an $(n-2)$-dimensional sphere.[4] We assume that H is "non-degenerate" and that the horizon cross section is a compact connected manifold of dimension $n-2$. Under these conditions, one of the following 2 statements is true: (i) If t^a is tangent to the null generators of H then the spacetime must be static [18]. (ii) If t^a is not tangent to the null generators of H, then the higher dimensional rigidity theorem [11] states that there exist $N \geq 1$ additional linear independent, mutually commuting Killing fields $\psi_1^a, \dots, \psi_N^a$, such that $\pounds_{\psi_1} F_{ab} \dots, \pounds_{\psi_N} F_{ab} = 0$. These Killing fields generate periodic, commuting flows (with period 2π), and there exists a linear combination

$$K^a = t^a + \Omega_1 \psi_1^a + \dots + \Omega_N \psi_N^a, \quad \Omega_i \in \mathbb{R} \tag{3}$$

[3] Note that the Einstein-Maxwell black ring found in [4] has non-vanishing electric charge and hence does not fall into the class studied in the present paper.

[4] In 4 dimensions, Σ_∞ may be *shown* to be an S^2 under suitably strong additional hypothesis. A discussion of the structure of null-infinity in higher dimensions is given in [10].

so that the Killing field K^a is tangent and normal to the null generators of the horizon H, and

$$K_a \psi_i^a = 0 \quad \text{on } H. \tag{4}$$

Thus, in case (ii), the spacetime is axisymmetric, with isometry group $\mathcal{G} = \mathbb{R} \times U(1)^N$. From K^a, one may define the surface gravity of the black hole by $\kappa^2 = \lim_H (\nabla_a f)\nabla^a f / f$, with $f = (\nabla^a K^b)\nabla_a K_b$ the norm, and it may be shown that κ is constant on H [19]. In fact, the non-degeneracy condition implies $\kappa > 0$.

In case (i), one can prove that the spacetime is actually unique, and in fact isometric to the Reissner-Nordström-Tangherlini spacetime [13], for higher dimensions see [8]. In this paper, we will be concerned with case (ii).

Similar to 4 dimensions, the mass and angular momenta of the solution associated with the Killing fields are given, up to irrelevant numerical factors, by the Komar expressions

$$m = -\frac{n-2}{n-3} \int_{\Sigma_\infty} \nabla_a t_b \, dS^{ab} \,, \quad J_i = \int_{\Sigma_\infty} \nabla_a \psi_{i\,b} \, dS^{ab} \tag{5}$$

and we define the electric and magnetic charges of the solution by

$$Q_{\rm E}[\Sigma_\infty] = \int_{\Sigma_\infty} F_{ab} \, dS^{ab} \,, \quad Q_{\rm M}[C_l] = \int_{C_l} *F_{ab\dots c} \, dS^{ab\dots c} \,, \tag{6}$$

where $C_l, l = 1, 2, \dots$ runs through all the topologically inequivalent, non-contractible, closed 2-surfaces in the exterior of the spacetime. These numbers are invariants of the solution, and in 4 dimensions in fact characterize the solution uniquely. However, in higher dimensions this is no longer the case. In fact, we will see that further invariants must be taken into account as well.

We now restrict attention to the exterior of the black hole, $I^-(\mathcal{J}^+)$, which we shall again denote by M for simplicity. We assume that the exterior M is globally hyperbolic. By the topological censorship theorem [7], the exterior M is a simply connected manifold (with boundary $\partial M = H$). To understand better the nature of the solutions, it is useful to first eliminate the coordinates corresponding to the symmetries of the spacetime. More precisely, one considers the factor space $\hat{M} = M/\mathcal{G}$, where $\mathcal{G}$ is the isometry group of the spacetime generated by the Killing fields. Since the Killing fields ψ_i^a in general have zeros, the factor space $\hat{M} = M/\mathcal{G}$ will normally have singularities and is difficult to analyze. However, when the number of axial Killing fields is equal to $N = n-3$, and if there are no points in the exterior M whose isotropy subgroup is discrete, then the factor space can be analyzed by elementary means. This analysis was carried out in [12] for the case of $n = 5$, and a very similar analysis also applies to general n. Since we are assuming that the spacetime is asymptotically flat in the standard sense with spherical infinity $\Sigma_\infty \cong S^{n-2}$, the group of asymptotic symmetries with compact orbits must be isomorphic to a subgroup of $SO(n-1)$, whose maximal torus has dimension $[(n-1)/2]$. Thus $n-3$ axial Killing fields are only possible if either $n = 4$, or if $n = 5$. From now on, we focus on the latter case.

Thus, from now on we assume that the isometry group of the spacetime is $\mathcal{G} = \mathcal{K} \times \mathbb{R}$, where $\mathcal{K} = U(1) \times U(1)$, and we also assume that the action of the isometry group $\mathcal{K}$ generated by the axial symmetries is so that there are no points with discrete isotropy group. We denote the Killing vector fields generating $\mathcal{K}$ by ψ_1^a, ψ_2^a, and we denote the factor space $\hat{M} = M/\mathcal{G}$. The nature of the factor space is described by the following proposition [12]:

Proposition 1: Let (M, g_{ab}) be the exterior of a stationary, asymptotically flat, Einstein-Maxwell black hole spacetime with 2 mutually commuting independent axial Killing fields ψ_1^a, ψ_2^a. Then the orbit space $\hat{M} = M/\mathcal{G}$ by the isometry group is a simply connected, 2-dimensional manifold with boundaries and corners. Points in the interior of $\hat{M}$ correspond to points in M where all Killing fields t^a, ψ_1^a, ψ_2^a are linearly independent. Points on the i-th 1-dimensional boundary segment of $\partial\hat{M}$ correspond to either the horizon of M, or points where a linear combination $v_i^1\psi_1^a + v_i^2\psi_2^a = 0$, where $\mathbf{v}_i = (v_i^1, v_i^2)$ is a vector of integers that is constant on each such segment. Points in the corners of $\partial\hat{M}$ correspond to points in M where $\psi_1^a = 0 = \psi_2^a$. The boundary of $\hat{M}$ is connected.

Away from the boundary of $\hat{M}$, we can define a metric $\hat{g}_{ab}$ by identifying the tangent space $T_{\pi(x)}\hat{M}$ with the subspace H_x of T_xM spanned by the vectors orthogonal to t^a, ψ_1^a, ψ_2^a, where $\pi : M \to \hat{M} = M/\mathcal{G}$ is the projection. We denote this metric by $\hat{g}_{ab}$. It has signature $(++)$. We denote the derivative operator associated with this metric by $\hat{D}_a$. If one defines the 3×3 Gram matrix of the Killing fields by

$$G_{IJ} = g_{ab}X_I^a X_J^b, \quad X_I^a = \begin{cases} t^a & \text{if } I = 0, \\ \psi_i^a & \text{if } I = 1,2, \end{cases} \tag{7}$$

then the Gram determinant

$$r^2 = |\det G| \tag{8}$$

defines a scalar function r on $\hat{M}$ which is harmonic, $\hat{D}^a\hat{D}_a r = 0$, as a consequence of the Einstein-Maxwell equations. Using this, one can show that $r > 0, \hat{D}_a r \neq 0$ on the interior of $\hat{M}$, and one can also show that $r = 0$ on $\partial\hat{M}$. A conjugate harmonic scalar field z may then be defined on $\hat{M}$ by the equation $\hat{D}_a z = \hat{\varepsilon}^b{}_a \hat{D}_b r$. The functions r, z define global coordinates on $\hat{M}$, thus identifying this space with the complex upper half-plane

$$\hat{M} = \{\zeta = z + ir \in \mathbb{C} : \; r \geq 0\},$$

with the boundary segments corresponding to intervals on the real axis. The length $z_i - z_{i+1} = l_i$ of each segment is an invariant of the solution. The induced metric $\hat{g}_{ab}$ is given in these coordinates by

$$d\hat{s}^2 = k(r,z)^2(dr^2 + dz^2) \tag{9}$$

with k^2 a conformal factor.

The set of real "moduli" $\{l_i\}$, and and of the "winding number" vectors $\{\mathbf{v}_i\}$ are global parameters that can be defined in an invariant way for the given solution in addition to the mass m, the two angular momenta J_1, J_2, and the electric and magnetic charges. We refer to these data as the “interval structure” of the solution. As shown in [12], the interval structure determines the structure of M as a fibered space with an action of the torus group $\mathcal{K}$. The winding numbers $\{\mathbf{v}_i\}$ characterize the structure of this fibration near the axis segments. It follows from our analysis in [12] that near such an axis, M locally has the structure of $\mathbb{R}^2 \times \text{Seiffert}(v_i^1, v_i^2)$, i.e., it is a cartesian product of $\mathbb{R}^2$ with a Seiffert torus, i.e., a 3-torus with a twisting characterized by the two winding numbers. The winding numbers on segments adjacent on a corner, respectively adjacent on the horizon have to satisfy the constraint [12]

$\det(\mathbf{v}_j, \mathbf{v}_{j+1}) = \pm 1$	if (z_{j-1}, z_j) and (z_j, z_{j+1}) are not the horizon
$\det(\mathbf{v}_{h-1}, \mathbf{v}_{h+1}) = p$	if (z_h, z_{h+1}) is the horizon

Furthermore, we have the following theorem about the horizon topology [12]:

Proposition 2: In a black hole spacetime of dimension 5 with 2 commuting, independent axial Killing fields, the horizon cross section $\mathcal{H}$ must be topologically either a ring $S^1 \times S^2$, a sphere S^3, or a Lens-space $L(p,q)$, with $p,q \in \mathbb{Z}$, and p as in eq. (2).

Remark: The Lens-spaces $L(p,q)$ (see e.g. [1, Paragraph 9.2]) are the spaces obtained by gluing the boundaries of two solid tori together in such a way that the meridian of the first goes to a curve on the second which wraps around the longitude p-times and which wraps around the meridian q-times. A Lens-space may also be obtained as the quotient of S^3 by a discrete group of isometries.

For illustrative purposes, we list the interval structure for some known solutions [15, 6, 4, 16]:

	Moduli l_i	Vectors $\mathbf{v}_i$	Horizon Topology
Myers-Perry BH	∞, l_1, ∞	$(1,0),(0,0),(0,1)$	S^3
Black Ring	∞, l_1, l_2, ∞	$(1,0),(0,0),(1,0),(0,1)$	$S^2 \times S^1$
Flat Spacetime	∞, ∞	$(1,0),(0,1)$	—

Here we are using the convention that the integer vector $\mathbf{v}_h$ associated with the horizon is taken to be $(0,0)$. Even for a fixed set of of asymptotic charges m, J_1, J_2 the invariant lengths l_1, l_2 may be different for the different Black Ring solutions, corresponding to the fact that there exist non-isometric Black Ring solutions with equal asymptotic charges.

3 Moduli space of Einstein-Maxwell black holes

We would now like to see to what extent the interval structure, and the global charges $m, J_1, J_2, Q_\mathrm{E}, Q_\mathrm{M}$ determine a given black hole solution of the Einstein-Maxwell equations in 5 dimensions. We were not able to analyze this question in generality but only in a simplified case. The simplifying assumptions that we will make in this section in addition to the general hypothesis stated above are the following:

1. About the *spacetime metric* we assume that one of the axial Killing fields, say ψ_1^a, is orthogonal to the other Killing fields, $g_{ab}\psi_1^a\psi_2^b = 0 = g_{ab}t^a\psi_1^b$, and that it is hypersurface orthogonal, $\psi_{1[a}\nabla_b\psi_{1c]} = 0$.

2. About the *Maxwell field* we assume that there is a 1-form ξ_a orthogonal to the Killing fields such that $F_{ab} = \xi_{[a}\psi_{1b]}$. It can easily be shown that, if the Maxwell field arises from a vector potential $F_{ab} = 2\nabla_{[a}A_{b]}$ which is invariant under the Killing fields, then this will be the case if and only if A^a is proportional to ψ_1^a at each point in M. Note, however that we do *not* assume the existence of such a vector potential here.

Let us first point out some simplifications which follow from assumptions 1) and 2). The first immediate consequence of 1) is that $J_1 = 0$. Secondly, because the Killing field ψ_1^a is demanded to be orthogonal to ψ_2^a, if $v^1\psi_1^a + v^2\psi_2^a = 0$ at a point in spacetime, then either $\mathbf{v} = (v^1, v^2) = (0,0)$, or $\mathbf{v} = (0,1), (1,0)$, or both axial Killing fields vanish. Thus, the interval structure (see Prop. 1) of any solution satisfying assumption 1) can only be of the following possibilities (i)—(iv):

	Moduli l_i	Vectors $\mathbf{v}_i$
(i)	$\infty, l_1, \dots, l_p, \infty$	$(1,0),(0,1),\dots(1,0),(0,0),(1,0),(0,1)\dots,(0,1)$
(ii)	$\infty, l_1, \dots, l_p, \infty$	$(1,0),(0,1),\dots(0,1),(0,0),(0,1),(1,0)\dots,(0,1)$
(iii)	$\infty, l_1, \dots, l_p, \infty$	$(1,0),(0,1),\dots(0,1),(0,0),(1,0),(0,1)\dots,(0,1)$
(iv)	$\infty, l_1, \dots, l_p, \infty$	$(1,0),(0,1),\dots(1,0),(0,0),(0,1),(1,0)\dots,(0,1)$

Thus, the possible interval structures are severely restricted by 1). By Prop. 2, it then follows that the only possible horizon topologies are

$$\mathcal{H} \cong S^1 \times S^2 \quad \text{(black ring)}, \quad \mathcal{H} \cong S^3 \quad \text{(black hole)}, \tag{10}$$

with the first case realized when the vectors to the left and right of the horizon $\mathbf{v}_{h-1}, \mathbf{v}_{h+1}$ are equal [i.e., for the interval structures (i) and (ii)] and the second case realized when they are different [i.e., for the interval structures (iii) and (iv)]. In particular, the Lens-spaces $L(p,q)$ are excluded as possible horizon topologies by 1).

From 2), the electric charge vanishes, $Q_\mathrm{E} = 0$, and the Maxwell field is completely characterized by the 1-form

$$f_a = F_{ab}\psi_1^a, \tag{11}$$

which is closed by the equations of motion for the Maxwell field, $\nabla_{[a} f_{b]} = 0$. We define the twist 1-form by

$$\omega_a = \frac{1}{2}\varepsilon_{abcde}\psi_1^b\psi_2^c\nabla^d\psi_2^e\,. \tag{12}$$

Using that ψ_1^a and ψ_2^a are commuting Killing fields, we find that $\nabla_{[a}\omega_{b]}$ is proportional to $\varepsilon_{abcde}\psi_2^c\psi_1^e R^{df}\psi_{2\,f}$. If we now substitute the Einstein-Maxwell equation for the Ricci tensor, and use assumptions 1) and 2), then we see that $\nabla_{[a}\omega_{b]} = 0$. By definition, ω_a and f_a are invariant under the symmetries, so they induce corresponding 1-forms $\hat{\omega}_a$ and $\hat{f}_a$ on the factor space $\hat{M}$, which are still closed. Since the factor space is the upper half plane $\{\zeta = z + ir :\ r \geq 0\}$, i.e. is in particular simply connected, we can define global potentials for these quantities, $\hat{D}_a\chi = \hat{\omega}_a$ and $\hat{D}_a\alpha = \hat{f}_a$. If the Maxwell field arises from a globally defined vector potential, $F_{ab} = 2\nabla_{[a}A_{b]}$—which we do *not* assume—then $\alpha = A_a\psi_1^a$.

Using the potentials α, χ, we can now write down the reduced Einstein-Maxwell equations on the orbit space $\hat{M}$. Let ν, w, u be the functions on $\hat{M}$ be defined by

$$e^{2u} = g_{ab}\psi_1^a\psi_1^b\,, \quad e^{-u+2w} = g_{ab}\psi_2^a\psi_2^b\,, \quad e^{-u+2w+2\nu} = (\nabla_a r)\nabla^a r\,. \tag{13}$$

Then the complete Einstein-Maxwell equations are equivalent to the following set of equations on the upper complex half plane $\hat{M}$ [22] :

$$\begin{aligned} \hat{D}^a\left(r\Phi_1^{-1}\hat{D}_a\Phi_1\right) &= 0\,, \\ \hat{D}^a\left(r\Phi_2^{-1}\hat{D}_a\Phi_2\right) &= 0\,, \end{aligned} \tag{14}$$

together with

$$\begin{aligned} -r^{-1}(\hat{D}^a r)\hat{D}_a\nu &= \left[\frac{3}{8}\mathrm{Tr}\left(\hat{D}^a\Phi_1\hat{D}^b\Phi_1^{-1}\right) + \frac{1}{8}\mathrm{Tr}\left(\hat{D}^a\Phi_2\hat{D}^b\Phi_2^{-1}\right)\right]\cdot\left[\hat{g}_{ab} - 2(\hat{D}_a z)\hat{D}_b z\right] \\ -r^{-1}(\hat{D}^a z)\hat{D}_a\nu &= \left[\frac{3}{4}\mathrm{Tr}\left(\hat{D}^a\Phi_1\hat{D}^b\Phi_1^{-1}\right) + \frac{1}{4}\mathrm{Tr}\left(\hat{D}^a\Phi_2\hat{D}^b\Phi_2^{-1}\right)\right](\hat{D}_a r)\hat{D}_b z\,, \end{aligned} \tag{15}$$

where the matrix fields are defined in terms of u, w, α, χ by

$$\Phi_1 = \begin{pmatrix} e^u + \frac{1}{3}e^{-u}\alpha^2 & \frac{1}{\sqrt{3}}e^{-u}\alpha \\ \frac{1}{\sqrt{3}}e^{-u}\alpha & e^{-u} \end{pmatrix}, \tag{16}$$

and

$$\Phi_2 = \begin{pmatrix} e^{2w} + 4\chi^2 e^{-2w} & 2\chi e^{-2w} \\ 2\chi e^{-2w} & e^{-2w} \end{pmatrix}. \tag{17}$$

The first two equations state that the matrix fields Φ_1 and Φ_2 each satisfy the equations of a 2-dimensional sigma-model. The matrix fields are real, symmetric, with determinant equal to 1 on the interior of $\hat{M}$. We may view them as taking values in the hyperbolic space $\mathbb{H}$. The matrix fields Φ_1, Φ_2 determine the functions α, χ, w, u. The second and third equations (15) are decoupled from the sigma-model equations and determine the function ν.

Using this formulation of the reduced Einstein-Maxwell equations, we will now prove the main result of this paper:

Theorem: Consider two stationary, asymptotically flat, Einstein-Maxwell black hole spacetime of dimension 5, having one time-translation Killing field and two axial Killing fields. We also assume that there are no points with discrete isotropy subgroup under the action of the isometry group in the exterior of the black hole, and we assume that the Killing and Maxwell fields satisfy the assumptions 1) and 2) above, implying that $\mathbf{v}_i = (1,0)$ or $(0,1)$, and $\mathcal{H} = S^3$ or $S^1 \times S^2$, and $Q_{\rm E} = 0 = J_1$ for the solutions. If the two solutions have the same interval structures, the same values of the mass m, same angular momentum J_2, and same magnetic charges $Q_{\rm M}[C_l]$ for all 2-cycles C_l, then they are isometric.

Proof: Consider two solutions (M, g_{ab}, F_{ab}) and $(\tilde{M}, \tilde{g}_{ab}, \tilde{F}_{ab})$ as in the statement of the theorem. As argued in [12], since the interval structures of both solutions are the same, M and $\tilde{M}$ can be identified as manifolds, and the actions of the isometry group $\mathcal{G}$ are conjugate to each other. Thus, we may assume that $\tilde{M} = M$, and that $\tilde{t}^a = t^a$, $\tilde{\psi}^a_i = \psi^a_i$. Furthermore, since the quotient space by the isometries is the upper half plane in both cases, we may assume that $\tilde{r} = r, \tilde{z} = z$ as functions on $\tilde{M} = M$. We now define the 2 by 2 matrix fields as above, which we denote $\tilde{\Phi}_i$ and Φ_i, $i = 1,2$. These functions are mappings $\hat{M} \to \mathbb{H}$ from the upper complex half plane into the 2-dimensional hyperbolic space. We next consider the functions

$$\sigma_1 = {\rm Tr}\Big[\Phi_1^{-1}\tilde{\Phi}_1 - 1\Big] = \frac{(e^u - e^{\tilde{u}})^2}{e^u e^{\tilde{u}}} + \frac{1}{3}\frac{(\tilde{\alpha} - \alpha)^2}{e^u e^{\tilde{u}}} \tag{18}$$

and

$$\sigma_2 = {\rm Tr}\Big[\Phi_2^{-1}\tilde{\Phi}_2 - 1\Big] = \frac{(e^{2w} - e^{2\tilde{w}})^2}{e^{2w} e^{2\tilde{w}}} + 4\frac{(\tilde{\chi} - \chi)^2}{e^{2w} e^{2\tilde{w}}}\,. \tag{19}$$

The quantity σ_1 is a function of the point wise geodesic distance between the maps Φ_1 and $\tilde{\Phi}_1$ in the target space $\mathbb{H}$, and σ_2 similarly between Φ_2 and $\tilde{\Phi}_2$. By a straightforward calculation using the equations (14), one finds that the functions σ_i satisfy the differential inequality

$$\hat{D}^a(r\hat{D}_a\sigma_i) \geq 0\,, \quad \text{for } i = 1,2. \tag{20}$$

It is now convenient to view the maps σ_i not as functions on the complex upper half plane $\hat{M} = \{\zeta = z + ir \in \mathbb{C} : \ r \geq 0\}$, but as axially symmetric functions on $\mathbb{R}^3 \setminus \{z - \text{axis}\}$, by writing points $X = (X_1, X_2, X_3) \in \mathbb{R}^3$ in cylindrical coordinates as $X = (r\cos\varphi, r\sin\varphi, z)$. Eqs. (20) may then be written as

$$\left\{\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \frac{\partial^2}{\partial X_3^2}\right\}\sigma_i(X) \geq 0\,, \quad \text{for } i = 1,2. \tag{21}$$

By a general arguments based on the maximum principle, see e.g. [20, 21], if σ_i are globally bounded above on the entire $\mathbb{R}^3$ including the z-axis and infinity, then they vanish identically. Assuming this has been shown, it follows that the matrix fields must be equal for both solutions $\Phi_i = \tilde{\Phi}_i$ for $i = 1,2$. This may then be used to prove that $\tilde{g}_{ab} = g_{ab}$ and $\tilde{F}_{ab} = F_{ab}$ as follows. First, the equality of the matrix fields immediately implies $\tilde{\chi} = \chi, \tilde{\alpha} = \alpha, \tilde{u} = u, \tilde{w} = w$. If $B = e^{u-2w} g_{ab} t^b \psi^a_2$, then we have $B \to 0$ at infinity and

$$\hat{D}_a B = 2re^{-4w}\,\hat{\varepsilon}_a{}^b\,\hat{D}_b\chi\,, \tag{22}$$

and similarly for the tilda solution. Thus, we have $\tilde{B} = B$. Finally, the norm of the time-like Killing field $N = g_{ab}t^a t^b$ (and similarly for the tilda solution) satisfies

$$N = e^{-u+2w}B^2 - e^{-u-2w}r^2\,, \tag{23}$$

from which it follows that $\tilde{N} = N$. Since ψ_1^a is orthogonal to the other two Killing fields by assumption, we also have $g_{ab}\psi_1^a\psi_2^b = 0 = g_{ab}t^a\psi_1^b$, and likewise for the tilda solution. Hence, the inner products between all Killing fields are equal for both solutions. Finally, it follows from the equations eq. (15) that also $\tilde{\nu} = \nu$, and it follows from $F_{ab}\psi_1^a = \nabla_a\alpha$ and our assumptions about the Maxwell field that $\tilde{F}_{ab} = F_{ab}$. Altogether, this implies that the two solutions coincide, as we desired to show. In fact, the metric and Maxwell field may locally be written as

$$\begin{aligned} ds^2 &= -e^{-u-2w}r^2dt^2 + e^{-u+2w}\left(d\phi_2 + Bdt\right)^2 + e^{-u+2w+2\nu}\left(dr^2 + dz^2\right) + e^{2u}d\phi_1^2 \\ F &= d\alpha \wedge d\phi_1 \end{aligned} \tag{24}$$

in local coordinates such that $t^a = (\partial/\partial t)^a, \psi_i^a = (\partial/\partial\phi_i)^a$.

Thus, what remains is to be shown is that σ_i is bounded. It is at this stage that we must use our assumption that the interval structures and asymptotic charges of both solutions agree. We must consider the behavior of $\sigma_i : \mathbb{R}^3 \setminus \{z - \text{axis}\} \to \mathbb{H}$ on (a) near infinity (b) on the horizon, and (c) on the z-axis for both $i = 1, 2$. We will consider these cases separately.

(a) In order to show that σ_i are bounded near infinity, one uses that both metrics $\tilde{g}_{ab}$ and g_{ab} are asymptotically flat near infinity (in M), with the same asymptotic charges $\tilde{m} = m, \tilde{J}_1 = J_1 = 0, J_2 = \tilde{J}_2$, and the same electric charges $\tilde{Q}_{\rm E} = Q_{\rm E} = 0$. This can be used to show boundedness of σ_i near infinity in $\hat{M}$.

(b) On the open segment corresponding to the horizon, neither e^w nor e^u vanish, since both Killing fields ψ_i^a are non-vanishing by Prop. 2. Thus, $\sigma_i, i = 1, 2$ are bounded on the boundary segment of $\partial\hat{M}$ corresponding to the horizon.

(c) On the boundary segments corresponding to a rotation axis, we must be most careful. We distinguish boundary segments (z_i, z_{i+1}) where $\psi_1^a = 0, \psi_2^a \neq 0$ [corresponding to the vector $\mathbf{v}_i = (1, 0)$], boundary segments where $\psi_1^a \neq 0, \psi_2^a = 0$ [corresponding to the vector $\mathbf{v}_i = (0, 1)$], and corners where $\psi_1^a = 0 = \psi_2^a$.

Near points of the axis where $\psi_1^a = 0, \psi_2^a \neq 0$, we have $e^{2u} \to 0$ and $e^{2w} \to 0$ with e^{2w-u} finite and non-zero, as the latter is the norm of ψ_2^a (and likewise for the tilda quantity). We first focus on this case. We immediately see that we have a potential problem in proving the boundedness of σ_1, see eq. (18), since the second term has $e^u e^{\tilde{u}}$ in the denominator, with no compensating factors in the numerator as in the first term. Clearly, σ_1 can only be finite if and only if $(\alpha - \tilde{\alpha})^2$ goes to zero near such points at least at the same rate as $e^u e^{\tilde{u}}$. Similarly, we also have a potential problem in proving the boundedness of σ_2 see eq. (19), since the second term has $e^{2w}e^{2\tilde{w}}$ in the denominator, with no compensating factors in the numerator as in the first term. Again, σ_2 can only be finite if and only if $(\chi - \tilde{\chi})^2$ goes to zero near such points at least at the same rate as $e^{2w}e^{2\tilde{w}}$.

We first determine the rate at which e^u and e^w tend to zero near the points where $\psi_1^a = 0, \psi_2^a \neq 0$. Since e^{2w-u} is finite and non-zero near such points, it follows that B is finite, too. From the finiteness of N and eq. (23), it then also follows that $e^u = O(r)$, and therefore that $e^{2w} = O(r)$. Thus, in order for σ_1 and σ_2 to be finite near such points, we must have $\tilde{\alpha} = \alpha + O(r)$ and $\tilde{\chi} = \chi + O(r)$. We now prove that this is the case using the equality between the magnetic charges $\tilde{Q}_{\rm M} = Q_{\rm M}$ and the angular momentum $\tilde{J}_2 = J_2$. For this, let ζ_1 and ζ_2 be points on the boundary of the upper half plane $\hat{M}$ corresponding to points in the manifold where $\psi_1^a = 0$. We can calculate the difference between $\alpha(\zeta_1)$ and $\alpha(\zeta_2)$ by chosing an arbitrary path $\hat{\gamma}$ in the interior of the complex upper half plane starting at ζ_1 and ending at ζ_2: Namely, since $f_a = \nabla_a \alpha$, we have, in differential forms notation

$$\alpha(\zeta_1) - \alpha(\zeta_2) = \int_{\hat{\gamma}} \hat{f}\,. \tag{25}$$

Now, it is possible to lift $\hat{\gamma} : [0,1] \to \hat{M}$ to a path $\gamma : [0,1] \to M$, i.e., $\hat{\gamma} = \pi \circ \gamma$, where π is the projection from M to the quotient space $\hat{M}$. Let C be the 2-surface in M that is obtained by acting on points in the image of γ with the isometries generated by ψ_1^a, i.e.,

$$C := \{(e^{2\pi i t}, 0) \cdot \gamma(s) : \; s, t \in [0,1]\}\,. \tag{26}$$

The images of the points $\gamma(0)$ and $\gamma(1)$ under the action of this 1-parameter group isomorphic to $U(1)$ are again points, because $\psi_1^a|_{\gamma(0)} = 0 = \psi_1^a|_{\gamma(1)}$. The image of any other point $\gamma(t), 0 < t < 1$ is a circle. Thus, it follows that the 2-surface C is topologically a 2-sphere. If we now pick a local coordinate system near C such that $\psi_1^a = (\partial/\partial\phi_1)^a$, then we may write

$$\alpha(\zeta_1) - \alpha(\zeta_2) = \int_\gamma f = \frac{1}{2\pi} \int_C f \wedge d\phi_1\,. \tag{27}$$

where $\pi^* \hat{f} = f$, and where we have used in the second step that $\pounds_{\psi_1} f_a = 0$. The term on the right side may now be manipulated using that $f_a = F_{ab} \psi_1^b$, showing that

$$\alpha(\zeta_1) - \alpha(\zeta_2) = \frac{1}{2\pi} \int_C F = \frac{1}{2\pi} Q_{\rm M}[C]\,. \tag{28}$$

We may of course repeat the same argument for the tilda solution. Because the magnetic charges are the same for the two solutions, it follows that $\alpha(\zeta) = \tilde{\alpha}(\zeta)$ up to a constant independent of ζ, for each ζ corresponding to a point where ψ_1^a vanishes. Since that constant vanishes at infinity by asymptotic flatness, it follows that σ_1 is finite near such points.

We would next like to show that the same statement holds true for σ_2. This will follow if we can show that $\tilde{\chi}(\zeta) = \chi(\zeta) + O(r)$ for any $\zeta \in \partial\hat{M}$ not on the horizon segment. To show this, we first note that the twist 1-form ω vanishes on any axis, i.e. any point of $\partial\hat{M}$ not corresponding to the horizon, by Prop. 1. Let $\zeta_1, \zeta_2 \in \partial\hat{M}$, and not on the horizon segment, and take $\hat{\gamma}$ to be the curve $\hat{\gamma}(t) = (1-t)\zeta_1 + t\zeta_2$ in $\hat{M}$. Then we have

$$\chi(\zeta_1) - \chi(\zeta_2) = \int_{\hat{\gamma}} \hat{\omega}\,, \tag{29}$$

where $\pi^* \hat{\omega} = \omega$. If ζ_1, ζ_2 are both to the same side of the horizon, then the above expression vanishes, while if they are on different sides, we find, by the same type of argument as above that

$$\chi(\zeta_1) - \chi(\zeta_2) = \frac{1}{(2\pi)^2} \int_{\mathcal{H}} *(d\psi_2)\,, \tag{30}$$

where ψ_2^a has been identified with a 1-form via g_{ab} and where $\mathcal{H}$ is a horizon cross section in M. We would like to show that the quantity on the right side is proportional to the angular momentum J_2. For this, we pick a spacelike 4-surface Σ in spacetime with interior boundary $\mathcal{H}$ and boundary S^3_∞ at infinity. By Gauss' theorem, we can then write the quantity on the right side as

$$\int_{\mathcal{H}} \nabla_{[a}\psi_{2\,b]}\, dS^{ab} = J_2 + \int_C \nabla^b\nabla_{[a}\psi_{2\,b]}\, dS^a \,. \tag{31}$$

The integrand on the right side may be evaluated standard identities for Killing vectors, the Einstein-Maxwell equations, as well as our assumptions 1) and 2). We have

$$\begin{aligned}
\nabla^b\nabla_{[a}\psi_{2\,b]} &= \frac{1}{2}R_{ab}\psi_2^b \\
&= \frac{1}{4}\left(F_{ac}F_b{}^c - \frac{g_{ab}}{6}F_{cd}F^{cd}\right)\psi_2^b \\
&= -\frac{1}{48}\psi_1^b\psi_{1\,b}\xi^c\xi_c\psi_{2\,a} =: \lambda\psi_{2\,a}\,.
\end{aligned} \tag{32}$$

We may choose Σ to be a surface defined by $T = const.$, where T is a time function that is invariant under the axial Killing fields[5], i.e. in particular $\psi_2^a\nabla_a T = 0$. Choosing now an integration 4-form on Σ by $\varepsilon_{abcde} = 5\nabla_{[a}T\varepsilon_{bcde]}$, and letting dS be the integration element on Σ associated with this 4-form, we see that $\int_\Sigma \lambda\psi_{2\,a}\, dS^a = \int_\Sigma \lambda\psi_2^a\nabla_a T\, dS = 0$, as desired. Since by assumption $\tilde{J}_2 = J_2$, we conclude that $\tilde{\chi}(\zeta) = \chi(\zeta)$ on any rotation axis, i.e. any point of $\partial\hat{M}$ not in the horizon segment. Since the twist potential $\hat{\omega}$ also vanishes on $\partial\hat{M}$ except for the horizon segment, it then follows from eq. (29) that in fact even $\chi - \tilde{\chi} = O(r^2)$ near any boundary segment corresponding to a rotation axis. Thus, in summary, we have now shown that $\sigma_i, i = 1,2$ has a finite limit for any point ζ boundary of $\hat{M}$ where $\psi_1^a = 0, \psi_2^a \neq 0$.

We must now consider the second case, i.e., points where $\psi_2^a = 0, \psi_1^a \neq 0$. For such points, $e^{2w-u} \to 0$, but e^u finite and non-zero, so $e^{2w} \to 0$. From the fact that N is finite and non-zero near such points and eq. (23) it can furthermore be seen that, in fact, $e^{2w} = O(r^2)$. Thus, only σ_2 is potentially unbounded near such points. However, we have already shown that $\tilde{\chi} - \chi = O(r^2)$ near any point in $\partial\hat{M}$ which is not on the horizon segment, so this cannot happen. Thus, $\sigma_i, i = 1,2$ are bounded in that case, too.

Finally, we must consider the corners. Here we may invoke a continuity argument to show that σ_i are bounded. Thus, when viewed as functions on $\mathbb{R}^3$, the functions σ_i are solutions to eq. (21) that are bounded on the entire space $\mathbb{R}^3$, including the z-axis. As we have argued above, this is enough in order to show that the two black hole solutions are identical. □

Remark: The proof shows that the non-trivial 2-cycles [i.e., basis elements of $H_2(M)$] in the exterior of the spacetime may be obtained as follows. We know that the real axis bounding $\hat{M}$ is divided into intervals, each labeled with an integer 2-vector $\mathbf{v}_i = (1,0)$ or $\mathbf{v}_i = (0,1)$. The different possibilities are summarized in the above table. Now consider all possible curves $\hat{\gamma}_p, p = 1,2,\ldots$ in $\hat{M}$ with the property that $\hat{\gamma}_p$ starts on an interval labeled $(1,0)$, and ends on another interval labeled $(1,0)$, with no interval with label $(1,0)$ in between. If we now

[5]Such a function can be obtained from an arbitrary time function $\tilde{T}$ by averaging $\tilde{T}$ over the compact group $\mathcal{K}$ of axial symmetries.

lift $\hat{\gamma}_p$ to a curve γ_p in M, and act with all isometries generated by ψ_1^a on the image of this curve, then we generate a closed 2-surface C_p in M [see eq. (26)], which is topologically a 2-sphere for all p. We may repeat this by replacing $\hat{\gamma}_p, p = 1, 2, \dots$ with a set of curves each starting on an interval labeled $(0,1)$, and ending on another interval labeled $(0,1)$, with no interval with label $(0,1)$ in between. If we again lift these curves to curves in M, and act with all isometries generated by ψ_2^a, then we generate a set of topologically inequivalent closed 2-surfaces $\tilde{C}_q, q = 1, 2, \dots$ in M, each of which is topologically a 2-sphere. It may be seen that the set of 2-surfaces $\{C_p, \tilde{C}_q\}$ forms a basis of $H_2(M)$, and also of $H_2(\Sigma)$, where the 4-manifold Σ is a spatial slice going from infinity to the horizon (so that topologically $M = \mathbb{R} \times \Sigma$). In this 4-manifold, we can compute intersection numbers as $C_p : \tilde{C}_q = \pm 1$ or $= 0$, depending on whether the corresponding curves in $\hat{M}$ intersect or not. The rank of $H_2(\Sigma) = H_2(M)$ in the cases (i) through (iv) in the above table, and the intersection matrix $I_{pq} = C_p : \tilde{C}_q$ is therefore easily computed. This gives invariants of the 4-manifold Σ and hence of the exterior M of the black hole.

Only the magnetic charges $Q_{\rm M}[C_p]$ enter in the proof of the above theorem. The magnetic charges $Q_{\rm M}[\tilde{C}_q]$ are not needed and in fact vanish, due to assumptions 1) and 2) at the beginning of this section. Thus, for the simplest interval structure $(0,1), (0,0), (1,0)$, there are no non-trivial magnetic charges, and the unique solution within the class studied here is completely specified by J_2, m. In fact, this unique solution is the Myers-Perry black hole [15], with vanishing Maxwell field.

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References

[1] Adams, C.C.: *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots* New York, W. H. Freeman (1994)

[2] Bunting, G. L.: Proof of the uniqueness conjecture for black holes, (PhD Thesis, Univ. of New England, Armidale, N.S.W., 1983)

[3] Carter, B.: Axisymmetric black hole has only two degrees of freedom, Phys. Rev. Lett. **26**, 331-333 (1971)

[4] H. Elvang, "A charged rotating black ring," Phys. Rev. D **68**, 124016 (2003) [arXiv:hep-th/0305247].

[5] R. Emparan, "Rotating Circular Strings, and Infinite Non-Uniqueness of Black Rings," JHEP. **0403**, 064 (2004) [arXiv:hep-th/0305247].

[6] Emparan, R. and Reall, H. S.: A rotating black ring in five dimensions. Phys. Rev. Lett. **88**, 101101 (2002)

[7] Galloway, G. J., Schleich, K., Witt, D. M., and Woolgar, E.: Topological censorship and higher genus black holes, Phys. Rev. D **60**, 104039 (1999)

[8] Gibbons G. W., Ida, D., and Shiromizu, T.: Uniqueness and non-uniqueness of static black holes in higher dimensions, Phys. Rev. Lett. **89**, 041101 (2002)

[9] Hawking, S.W.: Black holes in general relativity. Commun. Math. Phys. **25**, 152-166 (1972)

[10] Hollands, S. and Ishibashi, A.: Asymptotic flatness and Bondi energy in higher dimensional gravity, J. Math. Phys. **46**, 022503 (2005) [arXiv:gr-qc/0304054].

[11] Hollands, S., Ishibashi, A. and Wald, R. M.: A higher dimensional stationary rotating black hole must be axisymmetric, Commun. Math. Phys. **271**, 699 (2007) [arXiv:gr-qc/0605106].

[12] S. Hollands and S. Yazadjiev, “Uniqueness theorem for 5-dimensional black holes with two axial Killing fields,” arXiv:0707.2775 [gr-qc].

[13] Israel, W.: Event horizons in static vacuum space-times, Phys. Rev., **164**, 1776-1779 (1967)

[14] Mazur, P. O.: Proof of uniqueness of the Kerr-Newman black hole solution, J. Phys. A, **15**, 3173-3180 (1982)

[15] Myers, R.C. and Perry, M.J.: Black holes in higher dimensional space-times. Annals Phys. **172** 304 (1986)

[16] A. A. Pomeransky and R. A. Sen’kov, “Black ring with two angular momenta,” arXiv:hep-th/0612005.

[17] Robinson, D. C.: Uniqueness of the Kerr black hole, Phys. Rev. Lett. **34**, 905-906 (1975)

[18] Sudarsky, D. and Wald, R.M.: Extrema of mass, stationarity, and staticity, and solutions to the Einstein Yang-Mills equations. Phys. Rev. D **46** 1453-1474 (1992)

[19] Wald, R.M.: *General Relativity*. Chicago: University of Chicago Press, 1984

[20] Weinstein, G.: "Harmonic maps with prescribed singularities into Hadamard manifolds," Math. Research Lett. **3**, 835-844 (1999)

[21] See Lemma 8 in: Weinstein, G.: "On the Dirichlet problem for harmonic maps with prescribed singularities," Duke Math. J. **77** (1995) No.1, 135-165

[22] Yazadjiev, S.: Completely integrable sector in 5D Einstein-Maxwell gravity and derivation of the dipole black ring solutions, Phys. Rev. **D73**, 104007 (2006)