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# STABILITY OF THE SOLUTIONS OF INSTANTANEOUS BETHE–SALPETER EQUATIONS WITH CONFINING INTERACTIONS

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### Abstract

For two bound-state equations derived as simplified forms of the Bethe–Salpeter equation with confining interaction, stability of all solutions is rigorously shown.

#### 1 Motivation: Instabilities of Klein-Paradox Type

The Salpeter equation is a frequently applied three-dimensional reduction of the Bethe–Salpeter formalism describing bound states within quantum field theory, derived by assuming all interactions to be instantaneous. For given interactions, encoded in its *integral kernel*  $K(\mathbf{p}, \mathbf{q})$  depending on relative three-momenta  $\mathbf{p}, \mathbf{q}$  of the bound-state constituents, it can be regarded as an eigenvalue equation for the Salpeter amplitude  $\Phi(\mathbf{p})$ , with the mass M of the bound state as eigenvalue. For confining interactions, however, its solutions exhibit in *numerical* studies <sup>1</sup>) certain instabilities, possibly related to Klein's paradox, causing states to decay.

In view of this highly unsatisfactory state of the art, we began a systematic rigorous analysis of the conditions for stability of the energy levels predicted, for confining interactions, within this framework. We regard a bound state as *stable* if its mass eigenvalue M belongs to a *real discrete* spectrum *bounded from below*.

On energetic grounds, any instabilities should manifest themselves first for pseudoscalar bound states; accordingly, we focus to fermion–antifermion bound states characterized by spin-parity-charge conjugation assignment  $J^{PC} = 0^{-+}$ . This allows to take advantage of experience gained in earlier studies 2, 3, 4, 5).

We analyze three-dimensional reductions of the Bethe–Salpeter formalism for increasing complexity of the problem: the *reduced* Salpeter equation  $^{6, 7)}$ , a generalization thereof, proposed in Ref. [8] and applied in Ref. [9], towards exact propagators of the bound-state constituents  $^{10)}$ , and the *full* Salpeter equation.

With Dirac couplings  $\Gamma$ , assumed to be identical for both constituents, and related potential functions  $V_{\Gamma}(\boldsymbol{p}, \boldsymbol{q})$ , the action of some kernel  $K(\boldsymbol{p}, \boldsymbol{q})$  on  $\Phi(\boldsymbol{p})$  is

$$[K(\boldsymbol{p},\boldsymbol{q})\,\Phi(\boldsymbol{q})] = \sum_{\Gamma} V_{\Gamma}(\boldsymbol{p},\boldsymbol{q})\,\Gamma\,\Phi(\boldsymbol{q})\,\Gamma\,.$$

For all interactions of harmonic-oscillator type in configuration space, the above bound-state equations simplify to ordinary differential equations, which may be cast into the form of "tractable" eigenvalue equations for Schrödinger operators, at least in the case of the *reduced* bound-state equations studied in Secs. 2 and 3.

Our primary tool is a (well-known) theorem which states that the spectrum of a Schrödinger Hamiltonian operator with a locally bounded positive potential V rising to infinity in all directions,  $V(x) \to +\infty$  for  $|x| \to \infty$ , is purely discrete.

#### 2 Reduced Salpeter Equation

Approximating the propagation of both bound-state constituents by that of free particles of *constant* effective "constituent" mass m yields the Salpeter equation

$$\Phi(\boldsymbol{p}) = \int \frac{\mathrm{d}^{3}q}{(2\pi)^{3}} \left( \frac{\Lambda^{+}(\boldsymbol{p}) \gamma_{0} \left[ K(\boldsymbol{p}, \boldsymbol{q}) \Phi(\boldsymbol{q}) \right] \Lambda^{-}(\boldsymbol{p}) \gamma_{0}}{M - 2 E(\boldsymbol{p})} - \frac{\Lambda^{-}(\boldsymbol{p}) \gamma_{0} \left[ K(\boldsymbol{p}, \boldsymbol{q}) \Phi(\boldsymbol{q}) \right] \Lambda^{+}(\boldsymbol{p}) \gamma_{0}}{M + 2 E(\boldsymbol{p})} \right), \qquad (1)$$

with one-particle energy  $E(\mathbf{p})$  and energy projection operators  $\Lambda^{\pm}(\mathbf{p})$  defined by

$$E(\boldsymbol{p}) \equiv \sqrt{\boldsymbol{p}^2 + m^2} , \qquad \Lambda^{\pm}(\boldsymbol{p}) \equiv \frac{E(\boldsymbol{p}) \pm \gamma_0 \left( \boldsymbol{\gamma} \cdot \boldsymbol{p} + m \right)}{2 E(\boldsymbol{p})} .$$

Dropping of the second term on its RHS yields the reduced Salpeter equation 11)

$$[M - 2E(\boldsymbol{p})] \Phi(\boldsymbol{p}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \Lambda^+(\boldsymbol{p}) \gamma_0 [K(\boldsymbol{p}, \boldsymbol{q}) \Phi(\boldsymbol{q})] \Lambda^-(\boldsymbol{p}) \gamma_0 .$$
(2)

Because of its projector structure, a pseudoscalar  $\Phi(p)$  has just one independent component. For a large class of kernels, all its solutions prove to be stable 6, 7.

## 3 Instantaneous Bethe–Salpeter Equation with Exact Propagators

By Lorentz covariance, the exact fermion propagator S(p) is fully determined by two real *p*-dependent (Lorentz-scalar) functions, which may be interpreted, e.g., as this fermion's mass function  $m(p^2)$  and wave-function renormalization  $Z(p^2)$ :

$$S(p) = \frac{\mathrm{i} Z(p^2)}{\not p - m(p^2) + \mathrm{i} \varepsilon} , \qquad \not p \equiv p^{\mu} \gamma_{\mu} , \qquad \varepsilon \downarrow 0$$

If these propagator functions are assumed to depend *approximately* only on the three-momentum p, an *exact-propagator bound-state equation* may be found <sup>8</sup>), from which, for free propagators,  $m(p^2) \rightarrow m, Z(p^2) \rightarrow 1$ , Salpeter's equation is recovered, and which may be reduced to the exact-propagator version of Eq. (2):

$$[M - 2E(\mathbf{p})] \Phi(\mathbf{p}) = Z^{2}(\mathbf{p}^{2}) \int \frac{\mathrm{d}^{3}q}{(2\pi)^{3}} \Lambda^{+}(\mathbf{p}) \gamma_{0} [K(\mathbf{p}, \mathbf{q}) \Phi(\mathbf{q})] \Lambda^{-}(\mathbf{p}) \gamma_{0} ; \quad (3)$$

here, one-particle energy  $E(\mathbf{p})$  and energy projection operators  $\Lambda^{\pm}(\mathbf{p})$  now read

$$E(\mathbf{p}) \equiv \sqrt{\mathbf{p}^2 + m^2(\mathbf{p}^2)} , \qquad \Lambda^{\pm}(\mathbf{p}) \equiv \frac{E(\mathbf{p}) \pm \gamma_0 \left[ \mathbf{\gamma} \cdot \mathbf{p} + m(\mathbf{p}^2) \right]}{2 E(\mathbf{p})} ,$$

For reasonably well-behaved  $m(\mathbf{p}^2) > 0$  and  $Z(\mathbf{p}^2) > 0$ , bound states are stable.

## 4 (Full) Salpeter Equation

Trivially, similar considerations may be applied to the full Salpeter equation (1). There any analogous analysis is, however, considerably more complicated for, at least, two reasons. On the one hand, full-Salpeter amplitudes involve more than one independent components. Eq. (1) therefore reduces to a *set* of second-order differential equations, equivalent to a single differential equation of higher order. On the other hand, although all mass eigenvalues squared  $M^2$  are guaranteed to be real, the spectrum of mass eigenvalues M is in general not necessarily real: for the phenomenologically perhaps most relevant interaction kernels this spectrum is a union of real opposite-sign pairs (M, -M) and imaginary points  $M = -M^*$ .

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