Quantum FRW cosmological solutions in the presence of Chaplygin gas and perfect fluid

P. Pedram,[*] S. Jalalzadeh[†]

Department of Physics, Shahid Beheshti University, Evin, Tehran 19839, Iran

November 18, 2018

Abstract

We present a Friedmann-Robertson-Walker quantum cosmological model in the presence of Chaplygin gas and perfect fluid for early and late time epoches. In this work, we consider perfect fluid as an effective potential and apply Schutz's variational formalism to the Chaplygin gas which recovers the notion of time. These give rise to Schrödinger-Wheeler-DeWitt equation for the scale factor. We use the eigenfunctions in order to construct wave packets and study the time dependent behavior of the expectation value of the scale factor using the many-worlds interpretation of quantum mechanics. We show that contrary to the classical case, the expectation value of the scale factor avoids singularity at quantum level. Moreover, this model predicts that the expansion of Universe is accelerating for the late times.

Pacs: 98.80.Qc, 04.40.Nr, 04.60.Ds

1 Introduction

Supernova Ia (SNIa) observations show that the expansion of the Universe is accelerating [1], contrary to Friedmann-Robertson-Walker (FRW) cosmological models, with non-relativistic matter and radiation. Also cosmic microwave background radiation (CMBR) data [2, 3] is suggesting that the expansion of our Universe seems to be in an accelerated state which is referred to "dark energy" effect [4]. Cosmological constant, Λ, as the vacuum energy can be responsible for this evolution by providing a negative pressure [5, 6]. Unfortunately, the observed value of Λ is 120 orders of magnitude smaller than the one computed from field theory models [5, 6]. Quintessence is an alternative to consider a dynamical vacuum energy [7], involving one or two scalar fields, some with potentials justified from supergravity theories [8]. However, the fine-tuning problem of these models which arises from cosmic coincidence issue has no satisfactory solution.

The Chaplygin gas model is an interesting proposal [9], describing a transition from a Universe filled with dust-like matter to an accelerating expanding stage. This model was later generalized in Ref. [9, 10]. The

---

[*]Email: pouria.pedram@gmail.com
[†]Email: s-jalalzadeh@sbu.ac.ir

generalized Chaplygin gas model is described by a perfect fluid obeying an exotic equation of state [9, 10]

$$p = -\frac{A}{\rho^\alpha},$$ (1)

where A is a positive constant and $0 < \alpha \leq 1$. The standard Chaplygin gas [9] corresponds to $\alpha = 1$. Some publications [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29] and reviews [23, 24] which studied the Chaplygin gas cosmological models have already appeared in the literature. The Chaplygin gas can be obtained from the string Nambu-Goto action in the light cone coordinate [26]. Since the application of string theory in principle is in very high energy when the quantum effects is important in early Universe, a quantum cosmological study of the Chaplygin gas is also well founded.

Recently, Quantum mechanical description of a FRW model with a generalized Chaplygin gas has been discussed in Ref. [30] in order to retrieve explicit mathematical expressions for the different quantum mechanical states and determine the transition probabilities towards an accelerated stage. Moreover, quantization of FRW model in the presence of Chaplygin gas has been discussed in Ref. [31]. There, we have considered matter as the Chaplygin gas and discussed the early time behavior of expectation value of the scale factor through the application of Schutz's formalism. In this paper, aside from the Chaplygin gas which is coupled to gravity and has an advantage of furnishing a variable connected to matter which can be identified with time, we also include the perfect fluid in this scenario and investigate the analytical solutions in both early and late time Universes. Schutz's formalism [32, 33] gives dynamics to the matter degrees of freedom in interaction with the gravitational field. Using proper canonical transformations, at least one conjugate momentum operator associated with matter appears linearly in the action integral. Therefore, a Schrödinger-like equation can be obtained with the matter variable playing the role of time. The application of Schutz's formalism in Stephani and FRW perfect fluid cosmological models has been discussed in Refs. [34, 35, 36]. Note that our approach in principle is different from Monerat *et. al* [37], where they correspond the dynamical variable to the perfect fluid instead of Chaplygin gas and resort to the numerical methods to obtain the time evolution of an initial wave packet. Note that there is considerable evidence that the early Universe is dominated by radiation. Therefore, a natural setting for quantum cosmology is the one where radiation has the predominant role [38]. On the other hand the Chaplygin gas is dominated by the non-relativistic matter at early times (See the following

2

section). This seems to be in contradiction with our knowledge of baby Universe. According to [39], inflation can be accommodated within the generalized Chaplygin gas scenario. Hence, the way adopted to avoid this inconsistency is that the radiation dominated phase is followed by Chaplygin dominated period so that we have the so-called Chaplygin inflation [39]. Also, it would be more suitable to consider the field theory representation of the Chaplygin gas [40] to describe quantum cosmology. In this way, the Chaplygin gas can be viewed as a modification of gravity as was first pointed out in [40]. Also, the authors of [41] have recently shown that the Chaplygin gas model has a geometrical explanation within the context of brane world theory for any α. Consequently in these models the equation

$$\rho = \left[A + \frac{B}{a(t)^{3(\alpha+1)}} \right]^{\frac{1}{1+\alpha}},$$

(2)

is a consequence of stress-energy conservation for a scalar field on the brane [40], or conservation of induced dark matter on the brane [41, 42]. Here, $a(t)$ is the scale-factor of the universe and B is a positive integration constant. Therefore, it is evocative to view the contribution of Chaplygin gas to the stress-energy tensor as a brane induced modification of gravity. In this article, we used the fluid description for the Chaplygin gas and for Lagrangian formalism, the corresponding pressure. Consequently, if we rely on the model described in [41], we will have covariance in our model.

The paper is organized as follows. In Sec. 2, the quantum cosmological model with a Chaplygin gas, as a portion of the matter content is constructed in Schutz's formalism [32, 33] for early and late time Universes. Then the Schrödinger-Wheeler-DeWitt (SWD) equation in minisuperspace is obtained to quantize the model under the action of a perfect fluid effective potential. The wave function depends on the scale factor a and on the canonical variable associated to the Chaplygin gas, which in the Schutz's variational formalism plays the role of time T. We separate the wave function into two parts, one depends only on the scale factor and the other depends on the time. The time dependent part of the solution is e^{iEt}, where E is the energy. In Sec. 3, we construct wave packets from the eigenfunctions and compute the time-dependent expectation values of the scale factor to investigate the existence of singularities at quantum level. Moreover, we present some analytical solutions in both early and late time epoches. In Sec. 4, we present our conclusions.

2   The Model

The action for gravity plus Chaplygin gas in Schutz's formalism is written as

$$S = \int_M d^4x\sqrt{-g}\,R + 2\int_{\partial M} d^3x\sqrt{h}\,h_{ab}\,K^{ab} + \int_M d^4x\sqrt{-g}\,p_f + \int_M d^4x\sqrt{-g}\,p_c, \tag{3}$$

here, K^{ab} is the extrinsic curvature and h_{ab} is the induced metric over the three-dimensional spatial hypersur-face, which is the boundary ∂M of the four dimensional manifold M. We choose units such that the factor $16\pi G$ becomes equal to one. p_f and p_c denote Chaplygin gas and perfect fluid pressure, respectively. Note that, according to [32] the above action is equivalent to the usual Hawking-Ellis formalism for perfect fluid description [43]. Perfect fluid satisfies the barotropic equation of state

$$p_f = w\rho_f, \qquad w \leq 1. \tag{4}$$

The first two terms were first obtained in [44] and the last term of (3) represents the matter contribution to the total action. In Schutz's formalism [32, 33] the fluid's four-velocity can be expressed in terms of five potentials Φ, ζ, β, θ and S

$$u_\nu = \frac{1}{\mu}(\Phi_{,\nu} + \zeta\beta_{,\nu} + \theta S_{,\nu}) \tag{5}$$

where μ is the specific enthalpy. S is the specific entropy, and the potentials ζ and β are connected with rotation which are absent of models in the Friedmann-Robertson-Walker (FRW) type. The variables Φ and θ have no clear physical meaning. The four-velocity also satisfies the normalization condition

$$u^\nu u_\nu = -1. \tag{6}$$

The FRW metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)g_{ij}dx^i dx^j, \tag{7}$$

can be inserted in the action (3), where $N(t)$ is the lapse function and g_{ij} is the metric on the constant-curvature spatial section. Following the thermodynamic description of Ref. [45], the basic thermodynamic relations take the form

$$\rho_c = \rho_0[1 + \Pi], \quad h = 1 + \Pi + p_c/\rho_0, \tag{8}$$

4

$$\tau dS = d\Pi + p_c\, d(1/\rho_0),$$

$$= \frac{(1 + \Pi)^{-\alpha}}{1 + \alpha} d\left[(1 + \Pi)^{1+\alpha} + \frac{A}{\rho_0^{1+\alpha}}\right]. \tag{9}$$

It then follows that to within a factor

$$\tau = \frac{(1 + \Pi)^{-\alpha}}{1 + \alpha}, \tag{10}$$

$$S = (1 + \Pi)^{1+\alpha} + \frac{A}{\rho_0^{1+\alpha}}. \tag{11}$$

Therefore, the equation of state takes the form

$$p_c = -A\left[\frac{1}{A}\left(1 - \frac{h^{\frac{1+\alpha}{\alpha}}}{S^{1/\alpha}}\right)\right]^{\frac{1+\alpha}{\alpha}}. \tag{12}$$

The particle number density and energy density are, respectively,

$$\rho_c = \left[\frac{1}{A}\left(1 - \frac{h^{\frac{1+\alpha}{\alpha}}}{S^{1/\alpha}}\right)\right]^{\frac{-1}{1+\alpha}}, \tag{13}$$

$$\rho_0 = \frac{\rho + p}{h}, \tag{14}$$

where $h = (\dot{\Phi} + \theta\dot{S})/N$. After dropping the surface terms, the final reduced action takes the form

$$S = \int dt\left\{-6\frac{\dot{a}^2 a}{N} + 6kNa - Na^3\rho_f - Na^3 A\left[\frac{1}{A}\left(1 - \frac{(\dot{\Phi} + \theta\dot{S})^{\frac{1+\alpha}{\alpha}}}{N^{\frac{1+\alpha}{\alpha}} S^{1/\alpha}}\right)\right]^{\frac{1+\alpha}{\alpha}}\right\}. \tag{15}$$

The reduced action may be further simplified using canonical methods [45], resulting in the super-Hamiltonian

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6ka + a^3\rho_f + \left(Sp_\Phi^{1+\alpha} + Aa^{3(1+\alpha)}\right)^{\frac{1}{1+\alpha}}, \tag{16}$$

where $p_a = -12\dot{a}a/N$ and $p_\Phi = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}}$. However, an analytical quantum mechanical treatment of this FRW minisuperspace with the above Hamiltonian does not seem feasible. Therefore, it requires some approximation. We study the Chaplygin gas expression in early and late times limits, namely for small scale factors $Sp_\Phi^{1+\alpha} \gg Aa^{3(1+\alpha)}$ [30, 31] and large scale factors $Sp_\Phi^{1+\alpha} \ll Aa^{3(1+\alpha)}$, separately. So for early Universe, we can use the following expansion

$$\left(Sp_\Phi^{1+\alpha} + Aa^{3(1+\alpha)}\right)^{\frac{1}{1+\alpha}} \approx S^{\frac{1}{1+\alpha}} p_\Phi\left[1 + \frac{1}{1+\alpha}\frac{Aa^{3(\alpha+1)}}{Sp_\Phi^{1+\alpha}} + \frac{1}{2}\frac{1}{1+\alpha}\left(\frac{1}{1+\alpha} - 1\right)\frac{A^2}{S^2 p_\Phi^{2(1+\alpha)}}a^{6(\alpha+1)} + \ldots\right]. \tag{17}$$

Hence, up to the leading order, the super-Hamiltonian takes the form

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6ka + a^3\rho_f + S^{\frac{1}{1+\alpha}} p_\Phi. \tag{18}$$

The following additional canonical transformations

$$T = -(1+\alpha)p_\Phi^{-1}S^{\frac{\alpha}{1+\alpha}}p_S, \qquad p_T = S^{\frac{1}{1+\alpha}}p_\Phi, \tag{19}$$

and use of the explicit form of the energy density of the perfect fluid $\rho_f = \dfrac{B}{a^{3(1+w)}}$, simplify the super-Hamiltonian to

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6ka + Ba^{-3w} + p_T, \tag{20}$$

where B is a constant and the momentum p_T is the only remaining canonical variable associated with matter. It appears linearly in the super-Hamiltonian. The parameter k defines the curvature of the spatial section, taking the values $0, 1, -1$ for a flat, positive-curvature or negative-curvature Universe, respectively.

The classical dynamics is governed by the Hamilton equations, derived from Eq. (20) and Poisson brackets as

$$\begin{cases} \dot{a} = & \{a, N\mathcal{H}\} = -\frac{Np_a}{12a}, \\[2mm] \dot{p}_a = & \{p_a, N\mathcal{H}\} = -\frac{N}{24a^2}p_a^2 + 6Nk + 3wNBa^{-3w-1}, \\[2mm] \dot{T} = & \{T, N\mathcal{H}\} = N, \\[2mm] \dot{p}_T = & \{p_T, N\mathcal{H}\} = 0. \end{cases} \tag{21}$$

We also have the constraint equation $\mathcal{H} = 0$. Choosing the gauge $N = 1$, we have the following solutions for the system

$$T = t, \tag{22}$$

$$p_T = \text{const.}, \tag{23}$$

$$\ddot{a} = -\frac{\dot{a}^2}{2a} - \frac{k}{2a} - \frac{1}{4}wBa^{-3w-2}, \tag{24}$$

$$0 = -6a\dot{a}^2 - 6ka + Ba^{-3w} + p_T. \tag{25}$$

The WD equation in minisuperspace can be obtained by imposing the standard quantization conditions on the canonical momenta $(p_a = -i\frac{\partial}{\partial a}, p_T = -i\frac{\partial}{\partial T})$ and demanding that the super-Hamiltonian operator annihilate the wave function $(\hbar = 1)$

$$\frac{\partial^2\Psi}{\partial a^2} - (144ka^2 - 24Ba^{1-3w})\Psi - i24a\frac{\partial\Psi}{\partial t} = 0. \tag{26}$$

In this equation according to (22), $T = t$ corresponds to the time coordinate. As discussed in [46, 47], in order for the Hamiltonian operator $\hat{H}$ to be self-adjoint the inner product of any two wave functions Φ and Ψ must take the form

$$(\Phi, \Psi) = \int_0^\infty a\, \Phi^* \Psi da, \tag{27}$$

On the other hand, the wave functions should satisfy the following boundary conditions

$$\Psi(0,t) = 0 \quad \text{or} \quad \left.\frac{\partial \Psi(a,t)}{\partial a}\right|_{a=0} = 0. \tag{28}$$

The SWD equation (26) can be solved by separation of variables as follows

$$\psi(a,t) = e^{iEt}\psi(a), \tag{29}$$

where the a dependent part of the wave function $\psi(a)$ satisfies

$$-\psi''(a) + (144ka^2 - 24Ba^{1-3w})\psi(a) = 24Ea\,\psi(a), \tag{30}$$

and the prime means derivative with respect to a.

Now, we consider late time Universe when $Sp_\Phi^{1+\alpha} \ll Aa^{3(1+\alpha)}$. Using the expression

$$\left(Sp_\Phi^{1+\alpha} + Aa^{3(1+\alpha)}\right)^{\frac{1}{1+\alpha}} \approx A^{\frac{1}{1+\alpha}} a^3 \left[1 + \frac{1}{1+\alpha}\frac{Sp_\Phi^{1+\alpha}}{Aa^{3(\alpha+1)}} + \frac{1}{2}\frac{1}{1+\alpha}\left(\frac{1}{1+\alpha} - 1\right)\frac{S^2 p_\Phi^{2(1+\alpha)}}{A^2 a^{6(\alpha+1)}} + \dots\right], \tag{31}$$

up to the first order, the super-Hamiltonian (16) takes the form

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6ka + a^3 \rho_f + A^{\frac{1}{1+\alpha}} a^3 + \frac{A^{\frac{\alpha}{1+\alpha}}}{1+\alpha} a^{-3\alpha} Sp_\Phi^{1+\alpha}, \tag{32}$$

The following additional canonical transformations

$$T = -(1+\alpha)A^{-\frac{\alpha}{1+\alpha}} p_\Phi^{-(1+\alpha)} p_S, \quad p_T = \frac{A^{\frac{\alpha}{1+\alpha}}}{1+\alpha} Sp_\Phi^{1+\alpha}, \tag{33}$$

simplify the super-Hamiltonian to

$$\mathcal{H} = -\frac{p_a^2}{24a} - 6ka + Ba^{-3w} + A^{\frac{1}{1+\alpha}} a^3 + a^{-3\alpha} p_T. \tag{34}$$

The classical dynamics is governed by the Hamilton equations, derived from Eq. (20) and Poisson brackets as

$$
\begin{cases}
\dot{a} = & \{a, N\mathcal{H}\} = -\dfrac{Np_a}{12a}, \\[2mm]
\dot{p}_a = & \{p_a, N\mathcal{H}\} = -\dfrac{N}{24a^2}p_a^2 + 6Nk + 3wNBa^{-3w-1} \\[2mm]
& \qquad -3NA^{\frac{1}{1+\alpha}}a^2 + 3\alpha N\, a^{-3\alpha-1}p_T, \\[2mm]
\dot{T} = & \{T, N\mathcal{H}\} = Na^{-3\alpha}, \\[2mm]
\dot{p}_T = & \{p_T, N\mathcal{H}\} = 0.
\end{cases}
\tag{35}
$$

We also have the constraint equation $\mathcal{H} = 0$. Choosing the gauge $N = a^{3\alpha}$, we have the following solutions for the system

$$
T = t, \tag{36}
$$

$$
p_T = \text{const.}, \tag{37}
$$

$$
\ddot{a} = (3\alpha - \frac{1}{2})\frac{\dot{a}^2}{a} - \frac{k}{2}a^{6\alpha-1} - \frac{1}{4}wBa^{6\alpha-3w-2} + \frac{1}{4}A^{\frac{1}{1+\alpha}}a^{6\alpha+1} - \frac{1}{4}\alpha p_T a^{3\alpha-2}, \tag{38}
$$

$$
0 = -6a^{-6\alpha+1}\dot{a}^2 - 6ka + Ba^{-3w} + A^{\frac{1}{1+\alpha}}a^3 + a^{-3\alpha}\, p_T. \tag{39}
$$

It is important to note that these equations predict an accelerating Universe for late times. For large values of the scale factor we can simplify the above equations and find the acceleration parameter

$$
q = \frac{a\ddot{a}}{\dot{a}^2} = 3\alpha - 1, \tag{40}
$$

which is positive for $\alpha > 1/3$. Now, imposing the standard quantization conditions on the canonical momenta and demanding that the super-Hamiltonian operator annihilates the wave function, we are led to SWD equation in minisuperspace ($\hbar = 1$)

$$
\frac{\partial^2 \Psi}{\partial a^2} - (144ka^2 - 24Ba^{1-3w} - 24A^{\frac{1}{1+\alpha}}a^4)\Psi - i24a^{1-3\alpha}\frac{\partial \Psi}{\partial t} = 0. \tag{41}
$$

Here, according to (36), $T = t$ corresponds to the time coordinate. Demanding that the Hamiltonian operator $\hat{H}$ to be self-adjoint, the inner product of any two wave functions Φ and Ψ must take the form [46, 47]

$$
(\Phi, \Psi) = \int_0^\infty a^{1-3\alpha}\, \Phi^*\Psi da. \tag{42}
$$

The SWD equation (41) can be solved by separation of variables as follows

$$
\psi(a,t) = e^{iEt}\psi(a), \tag{43}
$$

8

where the a dependent part of the wave function $\psi(a)$ satisfies

$$-\psi''(a) + \left(144ka^2 - 24Ba^{1-3w} - 24A^{\frac{1}{1+\alpha}}a^4\right)\psi(a) = 24Ea^{1-3\alpha}\,\psi(a), \tag{44}$$

and the prime means derivative with respect to a. Note that effective Chaplygin gas term $(24A^{\frac{1}{1+\alpha}})$ plays the role of a positive cosmological constant. In particular, when $\alpha = 1/3$ and $w = 1/3$, this equation reduces to the FRW model with positive cosmological constant and radiation which has been studied in Ref. [48].

3   Results

In this Section we first study the issue of singularity avoidance in quantum cosmology in the early Universe and then present some analytical solutions in both early and late Universes.

For $k = 0$ the time-independent Wheeler-DeWitt equation (30), in the dust dominated Universe ($w = 0$), reduces to

$$\psi'' + 24(E + B)a\psi = 0. \tag{45}$$

The above equation has the following general time-dependent solutions under the form of Bessel functions

$$\Psi'_E = e^{iEt}\sqrt{a}\left[c_1 J_{\frac{1}{3}}\left(\frac{\sqrt{96E'}}{3}a^{\frac{3}{2}}\right) + c_2 Y_{\frac{1}{3}}\left(\frac{\sqrt{96E'}}{3}a^{\frac{3}{2}}\right)\right], \tag{46}$$

where $E' = E + B$. Now, the wave packets can be constructed by superimposing these solutions to obtain physically allowed wave functions. The general structure of these wave packets are

$$\Psi(a,t) = \int_0^\infty A(E')\Psi'_E(a,t)dE'. \tag{47}$$

We choose $c_2 = 0$ for satisfying the first boundary condition (28). Defining $r = \frac{\sqrt{96E'}}{3}$, simple analytical expressions for the wave packet can be found by choosing $A(E')$ to be a quasi-gaussian function

$$\Psi(a,t) = \sqrt{a}e^{-iBt}\int_0^\infty r^{\nu+1}e^{-\gamma r^2 + i\frac{3}{32}r^2 t}J_\nu(ra^{\frac{3}{2}})dr, \tag{48}$$

where $\nu = \frac{1}{3}$ and γ is an arbitrary positive constant. The above integral is known [49], and the wave packet takes the form

$$\Psi(a,t) = a\frac{e^{-\frac{a^3}{4Z} - iBt}}{(-2Z)^{\frac{4}{3}}}, \tag{49}$$
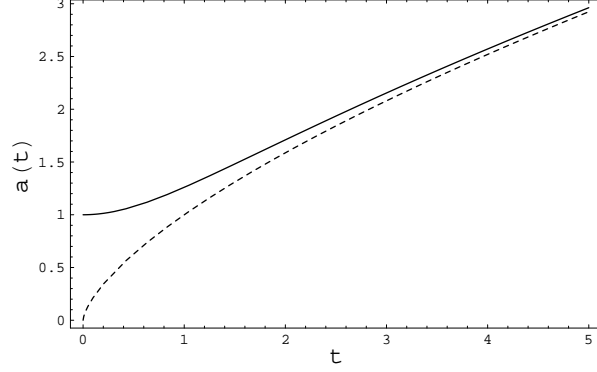
9

Figure 1: The time behavior of the expected value for the scale factor $\langle a \rangle(t)$ (solid line) and the classical scale factor $a(t)$ (dashed line) for dust dominated Universe ($w = 0$) and flat space time ($k = 0$).

where $Z = \gamma - i\frac{3}{32}t$. Now, we can verify what these quantum models predict for the behavior of the scale factor of the Universe. By adopting the many-worlds interpretation [50, 51], and with regards to the inner product relation (27), the expectation value of the scale factor

$$< a > (t) = \frac{\int_0^\infty a\Psi(a,t)^* a\Psi(a,t)da}{\int_0^\infty a\Psi(a,t)^* \Psi(a,t)da}, \tag{50}$$

is easily computed, leading to

$$< a > (t) \propto \left[\frac{9}{(32)^2\gamma^2}t^2 + 1\right]^{\frac{1}{3}}. \tag{51}$$

These solutions represent a no singular Universe which goes asymptotically over to the corresponding flat classical model for dust ($w = 0$) dominated epoch (22-25)(Fig. 1)

$$a(t) \propto t^{2/3}. \tag{52}$$

In the case $k = 1$ and $w = 0$ the time-independent Wheeler-DeWitt equation (30) reduces to

$$-\psi''(a) + \left(-24E'a + 144a^2\right)\psi(a) = 0. \tag{53}$$

Defining new variable $x = 12a - E'$ we find

$$-\frac{d^2\psi}{dx^2} + \left[-\frac{E'^2}{144} + \frac{x^2}{144}\right]\psi(a) = 0. \tag{54}$$

Equation (54) is similar to the time-independent Schrödinger equation for a simple harmonic oscillator with unit mass and energy λ

$$-\frac{d^2\psi}{dx^2} + \left[-2\lambda + w^2x^2\right]\psi(x) = 0, \tag{55}$$

10

where $2\lambda = E'^2/144$ and $w = 1/12$. Therefore, the allowed values of λ are $w(n+1/2)$ and the possible values of E' are

$$E'_n = \sqrt{12(2n+1)}\,, \qquad n = 0, 1, 2, \ldots \quad . \tag{56}$$

therefore, the stationary solutions are

$$\Psi_n(a,t) = e^{iE_n t}\varphi_n\left(12a - E'_n\right), \tag{57}$$

where

$$\varphi_n(x) = H_n\left(\frac{x}{\sqrt{12}}\right)e^{-x^2/24}\,, \tag{58}$$

and H_n are Hermite polynomials. The wave functions (57) are similar to the stationary quantum wormholes as defined in [52]. However, neither of the boundary conditions (28) can be satisfied by the these wave functions.

In $k = -1$ and $w = 0$ case, equation (30) reduces to

$$\psi''(a) + \left(24E'a + 144a^2\right)\psi(a) = 0, \tag{59}$$

where the solutions are

$$\Psi(a,t) = e^{iEt}(12a + E')^{-1/2}\left\{C_1 M_{\frac{iE^2}{48},\frac{1}{4}}\left(\frac{i(12a + E')^2}{12}\right) + C_2 W_{\frac{iE^2}{48},\frac{1}{4}}\left(\frac{i(12a + E')^2}{12}\right)\right\}, \tag{60}$$

where $M_{\kappa,\lambda}$ and $W_{\kappa,\lambda}$ are Whittaker functions. The Whittaker functions do not automatically vanish at $a = 0$. Therefore, we need to take both $C_1 \neq 0$ and $C_2 \neq 0$ to satisfy $\Psi(0,t) = 0$.

For $w = -1/3$, the SWD equation (30) can be written as

$$-\psi''(a) + 24(6k - B)a^2\psi(a) = 24Ea\,\psi(a), \tag{61}$$

which as before, has the solutions in the form of Simple Harmonic Oscillator (57) with discrete spectrum or Whittaker function (60) for positive or negative value of $(6k - B)$, respectively.

For $k = 0$ and $w = 1/3$ (radiation), the WD equation (30) reduces to

$$-\psi''(a) - 24B\psi(a) = 24Ea\psi(a), \tag{62}$$

which can be rewritten as

$$\psi''(a) + 24E\left(a + \frac{B}{E}\right)\psi(a) = 0, \tag{63}$$

11

by taking $x = a + \frac{B}{E}$ we have

$$\frac{d^2}{dx^2}\psi(x) + 24Ex\psi(x) = 0, \tag{64}$$

which is the Airy's differential equation. We solve this equation for $E > 0$ and $E < 0$, separately.

For $E > 0$, this equation has two solutions as $\text{Ai}\left[-(24E)^{1/3}x\right]$ and $\text{Bi}\left[-(24E))^{1/3}x\right]$. First one is exponentially decreasing function of x and the second one grows exponentially and is physically unacceptable. Therefore, the solution is

$$\psi(a) = \text{Ai}\left[-(24E))^{1/3}\left(a + \frac{B}{E}\right)\right]. \tag{65}$$

We choose the first boundary condition (28), which leads to

$$\text{Ai}\left[-(24E)^{1/3}\frac{B}{E}\right] = 0. \tag{66}$$

Airy's function $\text{Ai}(x)$ has infinitely many negative zeros $z_n = -a_n$, where $a_n > 0$, therefore, the energy levels quantize and take the values

$$E_n = \left(\frac{24^{1/3}B}{a_n}\right)^{3/2}. \tag{67}$$

The time-dependent eigenfunctions take the form

$$\Psi_n(a,t) = e^{iE_n t}\text{Ai}\left[-(24E_n)^{1/3}\left(a + \frac{B}{E_n}\right)\right]. \tag{68}$$

For $E < 0$, this equation has also two solutions as $\text{Ai}\left[(24|E|)^{1/3}x\right]$ and $\text{Bi}\left[(24|E|))^{1/3}x\right]$. Since the second one grows exponentially and is physically unacceptable, the solution is

$$\psi(a) = \text{Ai}\left[(24|E|))^{1/3}\left(a - \frac{B}{|E|}\right)\right]. \tag{69}$$

We choose the first boundary condition (28), which leads to

$$\text{Ai}\left[-(24|E|)^{1/3}\frac{B}{|E|}\right] = 0, \tag{70}$$

therefore, the energy levels quantize and take the values

$$E_n = -\left(\frac{24^{1/3}B}{a_n}\right)^{3/2}. \tag{71}$$

The time-dependent eigenfunctions take the form

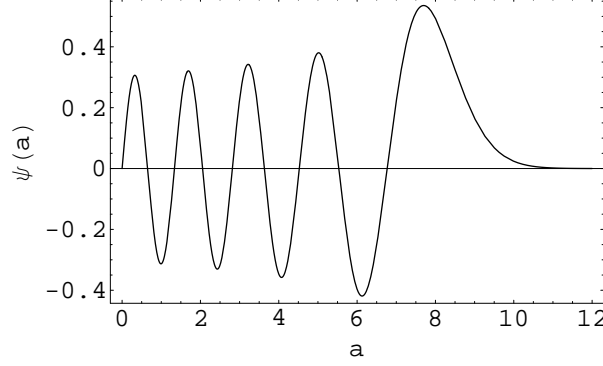$$\Psi_n(a,t) = e^{iE_n t}\text{Ai}\left[(24|E_n|)^{1/3}\left(a - \frac{B}{|E_n|}\right)\right]. \tag{72}$$

Figure 2: Plot of the wave function ($\psi(a)$) for $B = 1$ and $n = 8$, showing the oscillatory behavior for the small values of the scale factor and exponential damping for the large values of the scale factor.

It is important to note that Airy's function Ai(x) has an oscillatory behavior for $x < 0$ ($a < \frac{B}{|E_n|}$) whiles for $x > 0$ ($a > \frac{B}{|E_n|}$) decreases monotonically and is an exponentially damped function for large x (Fig. 2). Therefore, the solutions (72) show a classical behavior for small a and a quantum behavior for large a. This is contrary to usually expected results for previous case. In fact detecting quantum gravitational effects in large Universes is noticeable which has been also observed in FRW, Stephani, and Kaluza-Klein models [53, 35, 54].

In $k = 1$ and $w = 1/3$ (radiation) case, the WD equation (30) reduces to

$$-\psi''(a) + (144a^2 - 24B)\psi(a) = 24Ea\psi(a). \tag{73}$$

The above equation can be written as

$$-\psi''(a) + 144\left[\left(a - \frac{E}{12}\right)^2 - \left(\frac{E}{12}\right)^2\right]\psi = 0, \tag{74}$$

by taking $x = a - \frac{E}{12}$ we have

$$-\frac{d^2}{dx^2}\psi(x) + 144x^2\psi(x) = (E^2 + 24B)\psi(x). \tag{75}$$

This equation is identical to the time-independent Schrödinger equation for a simple harmonic oscillator with unit mass and energy λ

$$-\frac{d^2\psi(x)}{dx^2} + \omega^2 x^2\psi(x) = 2\lambda\psi(x), \tag{76}$$

where $2\lambda = (E^2 + 24B)$ and $\omega^2 = 144$. Therefore, the allowed values of λ are $\omega(n + 1/2)$ and the possible

13

values of E are

$$E_n = \sqrt{6(n + 1/2) - 24B} \ , \qquad n = 0, 1, 2, ... \quad , \tag{77}$$

therefore, the stationary solutions are

$$\Psi_n(a, t) = e^{iE_n t} \varphi_n \left(a - \frac{E_n}{12} \right), \tag{78}$$

$$\varphi_n(x) = H_n \left((12)^{\frac{1}{2}} x \right) e^{-3 \, x^2}, \tag{79}$$

where H_n are Hermite polynomials. However, neither of the boundary conditions (28) can be satisfied by these wave functions.

Now, we present some analytical solutions for the late time Universe. For flat space time ($k = 0$), dust epoch ($w = 0$), and standard Chaplygin gas ($\alpha = 1$), equation (44) reduces to

$$-\psi''(a) + \left(-24Ba - 24A^{\frac{1}{1+\alpha}} a^4 \right) \psi(a) = 24Ea^{-2} \, \psi(a), \tag{80}$$

where the solutions are

$$
\begin{aligned}
\Psi(a, t) \ = \ & e^{iEt} e^{2i\sqrt{\frac{2}{3}} A^{\frac{1}{2(1+\alpha)}} a^3} a^{\frac{1}{2} - \frac{1}{2}\sqrt{1-96E}} \\
\times \ & \left\{ C_1 \ U \left(\frac{1}{6} \left(-2i\sqrt{6}BA^{\frac{-1}{2(1+\alpha)}} - \sqrt{1-96E} + 3 \right), 1 - \frac{1}{3}\sqrt{1-96E}, -4i\sqrt{\frac{2}{3}} A^{\frac{1}{2(1+\alpha)}} a^3 \right) \right. \\
+ \ & \left. C_2 \ L^{-\frac{1}{3}\sqrt{1-96E}}_{\frac{1}{6}(2i\sqrt{6}BA^{\frac{-1}{2(1+\alpha)}} + \sqrt{1-96E} - 3)} \left(-4i\sqrt{\frac{2}{3}} A^{\frac{1}{2(1+\alpha)}} a^3 \right) \right\}.
\end{aligned}
\tag{81}
$$

Here $U(a, b, c)$ is the confluent hypergeometric function and $L_n^a(x)$ is the generalized Laguerre polynomial. We need to take both $C_1 \neq 0$ and $C_2 \neq 0$ to satisfy $\Psi(0, t) = 0$.

For flat space time ($k = 0$), stiff matter ($w = 1$), and standard Chaplygin gas ($\alpha = 1$), equation (44) reduces to

$$-\psi''(a) + \left(-24Ba^{-2} - 24A^{\frac{1}{1+\alpha}} a^4 \right) \psi(a) = 24Ea^{-2} \, \psi(a), \tag{82}$$

with the solutions as

$$
\begin{aligned}
\Psi(a, t) \ = \ & e^{iEt} \left\{ C_1 \ \sqrt{a} \, J_{-\frac{1}{6}\sqrt{-96(B+E)+1}} \left(2\sqrt{\frac{2}{3}} A^{\frac{1}{2(1+\alpha)}} a^3 \right) \right. \\
+ \ & \left. C_2 \ \sqrt{a} \, J_{\frac{1}{6}\sqrt{-96(B+E)+1}} \left(2\sqrt{\frac{2}{3}} A^{\frac{1}{2(1+\alpha)}} a^3 \right) \right\}.
\end{aligned}
\tag{83}
$$

Here again, we have $C_1 \neq 0$ and $C_2 \neq 0$ in order to satisfy the first boundary condition (28).

4 Conclusions

In this work we have investigated minisuperspace FRW quantum cosmological models with Chaplygin gas and perfect fluid as the matter content in early and late times. The use of Schutz's formalism for the Chaplygin gas allowed us to obtain SWD equations with the perfect fluid's effective potential. We have obtained eigenfunctions and therefore acceptable wave packets were constructed by appropriate linear combination of these eigenfunctions. The time evolution of the expectation value of the scale factor has been determined in the spirit of the many-worlds interpretation of quantum cosmology. We have showed that contrary to the classical case, the expectation value of the scale factor avoids singularity at the quantum level. Moreover, this model predicts an accelerated Universe for late times.

References

[1] A. G. Riess *et al.* [Supernova Search Team Collaboration], Astron. J. **116**, 1009 (1998) [arXiv:astro-ph/9805201]; S. Perlmutter *et al.* [Supernova Cosmology Project Collaboration], Astrophys. J. **517**, 565 (1999) [arXiv:astro-ph/9812133]; J. L. Tonry *et al.*, Astrophys. J. **594**, 1 (2003) [arXiv:astro-ph/0305008].

[2] D. N. Spergel *et al.*, Astrophys. J. Suppl. **148**, 175 (2003) [arXiv:astro-ph/0302209]; C. L. Bennett *et al.*, Astrophys. J. Suppl. **148**, 1 (2003) [arXiv:astro-ph/0302207].

[3] M. Tegmark *et al.* [SDSS Collaboration], [arXiv:astro-ph/0310723].

[4] V. Sahni, Class. Quant. Grav. **19**, 3435 (2002) [arXiv:astro-ph/0202076].

[5] S. Weinberg, Rev. Mod. Phys. **61**, 1 (1989).

[6] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003) [arXiv:astro-ph/0207347].

[7] C. Wetterich, Nucl. Phys. B **302**, 668 (1988); B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988); R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998) [arXiv:astro-ph/9708069]; P. F. González-Díaz, Phys. Rev. D **62**, 023513 (2000) [arXiv:astro-ph/0004125]; Y. Fujii, Phys. Rev. D **62**, 064004 (2000) [arXiv:gr-qc/9908021].

[8] P. Brax and J. Martin, Phys. Rev. D **61**, 103502 (2000) [arXiv:astro-ph/9912046].

[9] A. Y. Kamenshchik, U. Moschella and V. Pasquier, Phys. Lett. B **511**, 265 (2001) [arXiv:gr-qc/0103004].

[10] M. C. Bento, O. Bertolami and A. A. Sen, Phys. Rev. D **66**, 043507 (2002) [arXiv:gr-qc/0202064].

[11] M. C. Bento, O. Bertolami and A. A. Sen, Phys. Rev. D **67**, 063003 (2003) [arXiv:astro-ph/0210468].

[12] M. C. Bento, O. Bertolami and A. A. Sen, Phys. Lett. B **575**, 172 (2003) [arXiv:astro-ph/0303538]; L. Amendola, F. Finelli, C. Burigana and D. Carturan, JCAP **0307**, 005 (2003) [arXiv:astro-ph/0304325].

[13] R. Bean and O. Dore, Phys. Rev. D **68**, 023515 (2003) [arXiv:astro-ph/0301308]; A. Dev, D. Jain and J.S. Alcaniz, astro-ph/0311056; M. Biesiada, W. Godlowski and M. Szydlowski, astro-ph/0403305.

[14] J. C. Fabris, S. V. Goncalves and P. E. de Souza, Gen. Rel. Grav. **34**, 53 (2002) [arXiv:gr-qc/0103083]; Gen. Rel. Grav. **34**, 2111 (2002) [arXiv:astro-ph/0203441]; T. Multamaki, M. Manera and E. Gaztanaga, Phys. Rev. D **69**, 023004 (2004) [arXiv:astro-ph/0307533].

[15] V. Gorini, A. Kamenshchik and U. Moschella, Phys. Rev. D **67**, 063509 (2003) [arXiv:astro-ph/0209395]; R. Colistete, J. C. Fabris, S. V. Goncalves and P. E. de Souza, [arXiv:gr-qc/0210079], H. Sandvik, M. Tegmark, M. Zaldarriaga and I. Waga, [arXiv:astro-ph/0212114]; L. M. Beca, P. P. Avelino, J. P. de Carvalho and C. J. Martins, Phys. Rev. D **67**, 101301 (2003) [arXiv:astro-ph/0303564].

[16] N. Ogawa, Phys. Rev. D **62**, 085023 (2000) [arXiv:hep-th/0003288].

[17] M. Bordemann and J. Hoppe, Phys. Lett. B **317**, 315 (1993) [arXiv:hep-th/9307036].

[18] M. Hassaine and P. A. Horvathy, Lett. Math. Phys. **57**, 33 (2001) [arXiv:hep-th/0101044].

[19] G. W. Gibbons, Grav. Cosmol. **8**, 2 (2002) [arXiv:hep-th/0104015].

[20] M. Hassaine, Phys. Lett. A **290**, 157 (2001) [arXiv:hep-th/0106252].

[21] G. M. Kremer, Gen. Rel. Grav. **35**, 1459 (2003) [arXiv:gr-qc/0303103].

[22] H. B. Benaoum, [arXiv:hep-th/0205140].

[23] V. Gorini, A. Kamenshchik, U. Moschella and V. Pasquier, [arXiv:gr-qc/0403062].

[24] O. Bertolami, [arXiv:astro-ph/0403310].

[25] M. Szydlowski and W. Czja, Phys. Rev. D **69**, 023506 (2004) [arXiv:astro-ph/0306579].

[26] R. Jackiw, [arXiv:physics/0010042].

[27] M. R. Setare, Phys.Lett. B 644, 99 (2007).

[28] M. R. Setare, Phys. Lett. B 648, 329 (2007).

[29] M. Roos, [arXiv:0704.0882].

[30] M. Bouhmadi-López, P. V. Moniz, Phys. Rev. D **71**, 063521 (2005).

[31] P. Pedram, S. Jalalzadeh and S. S. Gousheh, Int. J. Theor. Phys. DOI: 10.1007/s10773-007-9436-9 [arXiv:0705.3587].

[32] B. F. Schutz, Phys. Rev. D **2**, 2762 (1970).

[33] B. F. Schutz, Phys. Rev. D **4**, 3559 (1971).

[34] P. Pedram, S. Jalalzadeh and S. S. Gousheh, Phys. Lett. B. In press, doi:10.1016/j.physletb.2007.08.077, [arXiv:0708.4143].

[35] P. Pedram, S. Jalalzadeh and S. S. Gousheh, Class. Quantum Grav. In press, [arXiv:0709.1620].

[36] F. G. Alvarenga, J. C. Fabris, N. A. Lemos, and G. A. Monerat, Gen. Rel. Grav. **34** 651 (2002).

[37] G. A. Monerat, G. Oliveira-Neto, E. V. Corrêa Silva, L. G. Ferreira Filho, P. Romildo, Jr., J. C. Fabris, R. Fracalossi, S. V. B. Gonçalves, and F. G. Alvarenga, Phys. Rev. D **76**, 024017 (2007)

[38] O. Bertolami and J. Mourao, Class Quantum Grav. **8**, 1271 (1991).

[39] O. Bertolami and V. Duvvuri, Phys. Lett. B **640**, 121 (2006).

[40] T. Barreiro, A.A. Sen, Phys. Rev. D **70**, 124013 (2004).

[41] M. Heydari-Fard and H. R. Sepangi, to appear in Phys. Rev. D, [arXiv: 0710.2666].

[42] S. Jalalzadeh and H. R. Sepangi, Class. Quant Grav. **22**, 2035 (2005).

[43] G. F. R. Ellis and S. W. Hawking, Large Scale Structure of Space Time, (Cambridge University Press, 1973); R. Mansouri and F. Nasseri, Phys. Rev. D **60**, 123512 (1999).

[44] R. Arnowitt, S. Deser and C. W. Misner, *Gravitation: An Introduction to Current Research*, edited by L. Witten, Wiley, New York (1962).

[45] V. G. Lapchinskii and V. A. Rubakov, Theor. Math. Phys. **33**, 1076 (1977).

[46] N. A. Lemos, J. Math. Phys. **37**, 1449 (1996).

[47] F. G. Alvarenga and N. A. Lemos, Gen. Rel. Grav. **30**, 681 (1998).

[48] J. Acacio de Barros, E. V. Corrêa Silva, G. A. Monerat, G. Oliveira-Neto, L. G. Ferreira Filho, and P. Romildo, Phys. Rev. D **75**, 104004 (2007).

[49] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980), formula 6.631-4.

[50] F. J. Tipler, Phys. Rep. **137**, 231 (1986).

[51] H. Everett, III, Rev. Mod. Phys. **29**, 454 (1957).

[52] S. W. Hawking and D. B. Page, Phys. Rev. D **42**, 2655 (1990).

[53] N. A. Lemos, F. G. Alvarenga, Gen. Rel. Grav. **31**, 1743 (1999), [arXiv:gr-qc/9906061].

[54] R. Coliteste, Jr., J. C. Fabris and N. Pinto-Neto, Phys. Rev. D **57**, 4707 (1998).