ASYMPTOTIC STABILITY OF LATTICE SOLITONS IN THE ENERGY SPACE

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ABSTRACT. Orbital and asymptotic stability for 1-soliton solutions to the Toda lattice equations as well as small solitary waves to the FPU lattice equations are established in the energy space. Unlike analogous Hamiltonian PDEs, the lattice equations do not conserve momentum. Furthermore, the Toda lattice equation is a bidirectional model that does not fit in with existing theory for Hamiltonian system by Grillakis, Shatah and Strauss.

To prove stability of 1-soliton solutions, we split a solution around a 1-soliton into a small solution that moves more slowly than the main solitary wave, and an exponentially localized part. We apply a decay estimate for solutions to a linearized Toda equation which has been recently proved by Mizumachi and Pego to estimate the localized part. We improve the asymptotic stability results for FPU lattices in a weighted space obtained by Friesecke and Pego.

1. INTRODUCTION

In this paper, we study asymptotic stability of solitary waves to a class of Hamiltonian systems of particles connected by nonlinear springs. A typical model of these lattice is Toda lattice

(1) $$\ddot{q}(t,n) = e^{-(q(t,n)-q(t,n-1))} - e^{-(q(t,n+1)-q(t,n))} \quad \text{for } t \in \mathbb{R} \text{ and } n \in \mathbb{Z},$$

where $q(t,n)$ denotes the displacement of the n-th particle at time t and $\dot{}$ denotes differentiation with respect to t. Let $p(t,n) = \dot{q}(t,n)$, $r(t,n) = q(t,n+1) - q(t,n)$, $u(t,n) = {}^t(r(t,n), p(t,n))$ and $V(r) = e^{-r} - 1 + r$. Toda lattice (1) is an integrable system with the Hamiltonian

$$H(u(t)) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2}p(t,n)^2 + V(r(t,n))\right),$$

(see [7]) and it can be rewritten as

(2) $$\frac{du}{dt} = JH'(u),$$

where

$$J = \begin{pmatrix} 0 & e^{\partial} - 1 \\ 1 - e^{-\partial} & 0 \end{pmatrix},$$

and $e^{\pm\partial} = e^{\pm\frac{\partial}{\partial n}}$ are the shift operator defined by $(e^{\pm\partial})f(n) = f(n \pm 1)$ for every sequence $\{f(n)\}_{n\in\mathbb{Z}}$ and H' is the Fréchet derivative of H in $l^2 \times l^2$.

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Toda lattice (2) has a two-parameter family of solitary waves

$$\mathcal{M} = \left\{ u_c(t+\delta) \;\middle|\; c>1,\ \delta\in\mathbb{R} \right\},$$

where $u_c(t,n) = \tilde{u}_c(n-ct)$, $\tilde{u}_c(x) = (\tilde{r}_c(x), \tilde{p}_c(x))$ and

$$\tilde{q}_c(x) = \log\frac{\cosh\{\kappa(x-1)\}}{\cosh\kappa x}, \tag{3}$$

$$\tilde{p}_c(x) = -c\partial_x\tilde{q}_c(x), \quad \tilde{r}_c(x) = \tilde{q}_c(x+1) - \tilde{q}_c(x), \tag{4}$$

and $\kappa = \kappa(c)$ is a unique positive solution of $c = \sinh\kappa/\kappa$.

Friesecke and Pego [9, 10] have proved asymptotic stability of solitary waves to FPU lattice in a weighted space assuming an exponential linear stability property (H1) below.

To state the assumption explicitly, we introduce several notations. Let l^2_a be a Hilbert space of $\mathbb{R}^2$-sequences equipped with the norm

$$\|u\|_{l^2_a} = \left(\sum_{n\in\mathbb{N}} e^{2an}|u(n)|^2\right)^{1/2}.$$

Let $\langle u, v\rangle := \sum_{n\in\mathbb{Z}}(u_1(n)u_2(n) + v_1(n)v_2(n))$ for $\mathbb{R}^2$-sequences $u = (u_1, u_2)$ and $v = (v_1, v_2)$ and $\|u\|_{l^2} = (\langle u, u\rangle)^{1/2}$.

(H1) Let $a > 0$ be a small number. There exist positive numbers K and β such that if

$$\langle v(s), J^{-1}\dot{u}_c(s)\rangle = \langle v(s), J^{-1}\partial_c u_c(s)\rangle = 0, \tag{5}$$

then a solution to

$$\frac{dv}{dt} = JH''(u_c(t))v \tag{6}$$

satisfies

$$\|e^{a(\cdot-ct)}u(t,\cdot)\|_{l^2} \le Ke^{-\beta(t-s)}\|e^{a(\cdot-cs)}u(s)\|_{l^2} \quad \text{for every } t\ge s. \tag{7}$$

Remark 1. Solutions $\dot{u}_c(t)$ and $\partial_c u_c(t)$ to (6) correspond to infinitesimal changes on t and c and they do not decay as $t\to\infty$. Since $J^{-1}\dot{u}_c(t)$ and $J^{-1}\partial_c u_c(t)$ are the corresponding neutral modes to the adjoint equation

$$\frac{dw}{dt} = H''(u_c(t))Jw,$$

the condition (H1) says that a solution to (6) decays exponentially as $t\to\infty$ if it does not include neutral modes $\dot{u}_c(t)$ and $\partial_c u_c(t)$.

Remark 2. In (5), we set

$$J^{-1} = \begin{pmatrix} 0 & \sum_{k=-\infty}^{0} e^{k\partial} \\ \sum_{k=-\infty}^{-1} e^{k\partial} & 0 \end{pmatrix}$$

so that J^{-1} is a bounded operator in l^2_{-a}. (Note that u decays exponentially as $n\to\pm\infty$ if $u\in l^2_{\pm a}$ and $a>0$ and that $\|e^{-\partial}u\|_{l^2_{-a}} = e^{-a}\|u\|_{l^2_{-a}}$.) Since $\dot{u}_c$ and $\partial_c u_c$ decay like $e^{-2\kappa|n-ct|}$ as $n\to\pm\infty$, we have $J^{-1}\dot{u}_c$, $J^{-1}\partial_c u_c\in l^2_{-a}$ for every $a\in(0, 2\kappa(c))$.

Friesecke and Pego prove in [9] that solitary waves to FPU lattice are asymptotically stable in l^2_a if (H1) holds. They have also proved in [10, 11] that small solitary waves of FPU lattice can be approximated by KdV solitons and that they satisfy (H1). In [18], we use the linearized Bäcklund transformation to show that every 1-soliton of Toda lattice satisfies (H1) and prove that it is asymptotically stable in l^2_a without assuming smallness of solitons.

Our goal in the present paper is to prove asymptotic stability of 1-solitons in l^2.

Theorem 1. *Let $c_0 > 1$, $\tau_0 \in \mathbb{R}$ and let $u(t)$ be a solution to (2) with $u(0) = u_{c_0}(\tau_0) + v_0$. For every $\varepsilon > 0$, there exists a positive number $\delta > 0$ satisfying the following: If $\|v_0\|_{l^2} < \delta$, there exist constants $c_+ > 1$ and $\sigma \in (1, c_+)$ and a C^1-function $x(t)$ such that*

$$\begin{aligned}
&\|u(t) - \tilde{u}_{c_0}(\cdot - x(t))\|_{l^2} < \varepsilon,\\
&\lim_{t\to\infty} \left\|u(t) - \tilde{u}_{c_+}(\cdot - x(t))\right\|_{l^2(n\ge\sigma t)} = 0,\\
&\sup_{t\in\mathbb{R}} (|c(t) - c_0| + |\dot{x}(t) - c_0|) = O(\|v_0\|_{l^2}),\\
&\lim_{t\to\infty} c(t) = c_+, \quad \lim_{t\to\infty} \dot{x}(t) = c_+.
\end{aligned}$$

Remark 3. By a simple computation, we see $dH(u_c)/dc > 0$ and $\lim_{c\to 1} H(u_c) = 0$ (see e.g. [24]). So we have arbitrary small 1-solitons in l^2. However, small solitary waves do not belong to an exponentially weighted space if c is close to 1 because $u_c(t)$ decays like $e^{-2\kappa(c)|n-x(t)|}$ as $n \to \infty$ and $\lim_{c\downarrow 1} \kappa(c) = 0$. Thus from Friesecke and Pego [8, 9, 10, 11] and Mizumachi and Pego [18], we cannot see whether a solitary wave can be stable under perturbations which include small solitary waves. Theorem 1 and Theorem 2 below insist that a solitary wave does not collapse by small perturbations including other solitary waves.

Since Benjamin [1] and Bona [2] studied stability of KdV 1-solitons, a lot of results have been obtained on stability of solitary waves to infinite dimensional Hamiltonian systems (see [5] and references therein). In those results, they utilized the fact that the Hamiltonian systems have another conservation law (like momentum for KdV and charge for NLS) and a solitary wave solution is a local minimizer of the Hamiltonian among solutions whose momentum or charge is the same as the solitary wave solution.

However, Toda and FPU lattices are bidirectional models like Boussinesq equations (see [3, 4, 20]) such that a solitary wave solution is a saddle point of the sum of Hamiltonian and the momentum multiplied by the speed of the solitary wave whose second variation has infinite dimensional indefiniteness. Furthermore, a solution to Toda lattice does not conserve momentum in general because Noether's theorem is not applicable to spatial variable $n \in \mathbb{Z}$. Hence stability of solitary waves does not follow from the theory of Hamiltonian system by Grillakis, Shatah and Strauss [13, 14] and Shatah and Strauss [23]. For the same reason, it is not possible to use a Liouville theorem like [15] to prove asymptotic stability of solitary waves.

Luckily, solitary waves for a class of lattice equations including the Toda lattice equation separate from each other as $t \to \infty$. As can be seen from (3) and (4), speed of solitary waves which move to the right is larger than

1 and the larger a solitary wave is the faster it moves, whereas the absolute value of group velocities are less than 1. So a solution to (2) is decoupled into a train of solitary waves and a remainder term as $t \to \infty$.

Friesecke and Pego [8, 9, 10, 11] utilized this fact and prove asymptotic stability of solitary waves to FPU lattice in an exponentially weighted space. They decompose a solitary wave as

$$\begin{aligned} u(t) &= u_{c(t)}(\gamma(t)) + v(t) = \tilde{u}_{c(t)}(\cdot - x(t)) + v(t), \\ x(t) &= c(t)\gamma(t), \end{aligned} \tag{8}$$

where $u_{c(t)}(\gamma(t))$ denotes a main solitary wave, and $c(t)$ and $x(t)$ are modulation parameters of the speed and the phase shift of the main wave, respectively. They prove that a solution which lies in a neighborhood of $\mathcal{M}$ is absorbed into $\mathcal{M}$ exponentially in l^2_a-norm as $t \to \infty$. Their proof basically follows the idea of Pego and Weinstein [21] and impose the symplectical orthogonality condition (5) on v. One of the difficulty to use their method in the energy space is that $J^{-1}\partial_c u_c$ tends to a nonzero constant as $n \to \infty$ and (5) is not well defined for $v \in l^2$.

Our strategy is to decompose $v(t)$ into the sum of a small solution $v_1(t)$ of (2) and $v_2(t)$ that is driven by an interaction of u_c and dispersive part of the solution. Since $v_2(t)$ is exponentially localized in front, we can estimate $v_2(t)$ by using exponential linear stability (7). Since $v_1(t)$ moves more slowly than the main solitary waves, it locally tends to 0 around the solitary wave. To fix the decomposition, we impose the constraint

$$\langle v, J^{-1}\dot{u}_c(\gamma)\rangle = \langle v_2, J^{-1}\partial_c u_c(\gamma)\rangle = 0$$

instead of (5).

Recently, Martel and Merle [16] give a direct proof of the asymptotic stability results in $H^1(\mathbb{R})$ for generalized KdV solitons based on a virial identity (which first appeared in Kato [19]). Because the Toda lattice and KdV equations have a similarity that the dominant solitary wave outruns and is separated from other part of solutions as $t \to \infty$, their idea seems promising. We prove a *virial lemma* [Lemma 9 in Section 3] for $v_1(t)$ and apply local energy decay estimates for other part of the solution instead of proving a virial lemma around solitary waves.This enables us to prove our results without numerics whereas [15, 16] need some numerical computation to prove positivity of a quadratic form. We expect our proof is applicable also for Hamiltonian PDEs like KdV equation by using the renormalization method by Ei [6] and Promislow [22] (see [17] for an application to the generalized KdV equation in a weighted space).

Now, let us consider asymptotic stability of solitary waves to FPU lattice equations. It is interesting to see whether solitary waves to non-integrable lattices are robust to perturbations in the energy class. Let $u(t,n) = {}^t(r(t,n), p(t,n))$ be a solution to

$$\frac{du}{dt} = JH'_F(u) \quad \text{for } t \in \mathbb{R}, \tag{9}$$

where

$$H_F(u(t)) = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2}p(t,n)^2 + V_F(r(t,n))\right),$$

and V_F is a potential satisfying

$$\text{(H2)}\qquad V_F \in C^4(\mathbb{R};\mathbb{R}),\quad V_F(0) = V_F'(0) = 0,\quad V_F''(0) > 0,\quad V_F'''(0) \neq 0.$$

If $c > c_s := \sqrt{V_F''(0)}$ and c is sufficiently close to c_s, Friesecke and Pego [8] show that there exists a unique solution $\tilde{u}_c(x)$

$$-c\partial_x \tilde{u}_c(x) = JH_F'(\tilde{u}_c(x)) \quad \text{for } x \in \mathbb{R} \tag{10}$$

up to translation and its profile is close to that of a KdV soliton. We remark that a solitary wave solution $\tilde{u}_c(n-ct)$ has small amplitude and satisfies $dH(\tilde{u}_c)/dc > 0$ if $c > c_s$ and c is close to c_s. See Friesecke and Wattis [12] for existence of large solitary waves. Friesecke and Pego have proved in [11] that small solitary wave solutions of (9) satisfy (H1) and are asymptotically stable in l^2_a. Assuming (H2), we can prove orbital and asymptotic stability of small solitary waves in l^2 exactly in the same way as Toda lattice.

Theorem 2. *Suppose (H2). Let δ_* be a small positive number and let $c_0 \in (c_s, c_s+\delta_*)$ and $\tau_0 \in \mathbb{R}$. Let $u(t)$ be a solution to (9) with $u(0) = u_{c_0}(\tau_0) + v_0$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ satisfying the following: If $\|v_0\|_{l^2} < \delta$, there exist constants $c_+ > c_s$ and $\sigma \in (c_s, c_+)$ and a C^1-function $x(t)$ such that*

$$\begin{aligned}
&\|u(t) - \tilde{u}_{c_0}(\cdot - x(t))\|_{l^2} < \varepsilon,\\
&\lim_{t\to\infty} \left\|u(t) - \tilde{u}_{c_+}(\cdot - x(t))\right\|_{l^2(n\ge\sigma t)} = 0,\\
&\sup_{t\in\mathbb{R}} \left(|c(t)-c_0| + |\dot{x}(t) - c_0|\right) = O(\|v_0\|_{l^2}),\\
&\lim_{t\to\infty} c(t) = c_+, \quad \lim_{t\to\infty} \dot{x}(t) = c_+.
\end{aligned}$$

Our plan of the present paper is as follows. In Section 2, we introduce a variant of the secular term condition for solutions in the energy class and some estimates that will be used later. In Section 3, we derive modulation equations of $x(t)$ and $c(t)$ and prove

$$\dot{c}(t) = O(\|v_1(t)\|_W^2 + \|v_2(t)\|_X^2) \tag{11}$$

for some weighted space $W \subset l^2_a \cap l^2_{-a}$ and $X \subset l^2_a$. On the other hand, we show that

$$\int_0^\infty (\|v_1(t)\|_W + \|v_2(t)\|_X)^2 dt \lesssim \|v_0\|_{l^2}^2 \tag{12}$$

by using a virial lemma for $v_1(t)$ and a local energy decay estimate (Corollary 6 in Section 2) for $v_2(t)$. Combining (11) and (12) with

$$\|v(t)\|_{l^2}^2 \le C(\|v_0\|_{l^2} + |c(t)-c_0|), \tag{13}$$

which follows from the convexity of the Hamiltonian and the orthogonality condition, we will prove Theorem 1. In Section 4, we give a brief proof of Theorem 2.

Finally, let us introduce some notations. For a Banach space X, we denote by $B(X)$ the space of all linear continuous operators from X to X. We use $a \lesssim b$ and $a = O(b)$ to mean that there exists a positive constant such that $a \le Cb$.

2. Preliminaries

Let $u(t)$ be a solution to (2) which lies in a tubular neighborhood of $\mathcal{M}$. We decompose $u(t)$ as (8). Since $\dot{u}_c = -c\partial_x\tilde{u}_c(\cdot - ct) = JH'(u_c)$, it follows from (3) and (4) that

$$\begin{aligned}\frac{d}{dt}u_{c(t)}(\gamma(t)) =& \dot{c}(t)\partial_c\tilde{u}_{c(t)}(n-x(t)) - \dot{x}(t)\partial_x\tilde{u}_{c(t)}(n-x(t))\\ =& JH'(u_c(t)) + \dot{c}(t)\partial_c u_c(\gamma(t)) + \frac{\dot{x}(t)-c(t)}{c(t)}\dot{u}_{c(t)}(\gamma(t)).\end{aligned}$$

Thus by the definition of v,

$$\frac{dv}{dt} = JH''(u_{c(t)}(\gamma(t)))v(t) + l_1(t) + N_1(t), \tag{14}$$

where

$$\begin{aligned} l_1(t) =& -\dot{c}(t)\partial_c u_{c(t)}(\gamma(t)) - \frac{\dot{x}(t)-c(t)}{c(t)}\dot{u}_{c(t)}(\gamma(t)),\\ N_1(t) =& J\left\{H'(u_{c(t)}(\gamma(t)) + v(t)) - H'(u_{c(t)}(\gamma(t))) - H''(u_{c(t)}(\gamma(t)))v(t)\right\}.\end{aligned}$$

Let $P_c(t)$ be a spectral projection associated with a subspace of neutral modes $\operatorname{span}\{\dot{u}_c(t), \partial_c u_c(t)\}$ and let $Q_c(t) = 1 - P_c(t)$. Then for $v \in l^2_a$ $(0 < a < 2\kappa(c))$,

$$P_c(t)v = \theta(c)\langle v, J^{-1}\dot{u}_c(t)\rangle\partial_c u_c(t) - \theta(c)\langle v, J^{-1}\partial_c u_c(t)\rangle\dot{u}_c(t),$$

where $\theta(c) = (dH(u_c)/dc))^{-1}$. We remark that the projections $P_c(t)$ and $Q_c(t)$ cannot be defined on l^2 because $J^{-1}\partial_c u_c$ does not decay as $n \to \infty$.

Now, we decompose $v(t)$ into the sum of a small solution to (2) and a remainder term that belongs to l^2_a for some $a > 0$. More precisely, we put $v(t) = v_1(t) + v_2(t)$, where

$$\begin{cases}\dfrac{dv_1}{dt} = JH'(v_1),\\ v_1(0) = v_0,\end{cases} \tag{15}$$

and $v_2(t)$ is a solution to

$$\begin{cases}\dfrac{dv_2}{dt} = JH''(u_{c(t)}(\gamma(t)))v_2 + l_1(t) + N_2(t),\\ v_2(0) = \varphi_{c_0}(\tau_0) - \varphi_{c(0)}(\gamma(0)),\end{cases} \tag{16}$$

where $N_2(t) = N_1(t) - JH'(v_1(t)) + JH''(u_{c(t)}(\gamma(t)))v_1$. To fix the decomposition, we will impose the constraint

$$\langle v(t), J^{-1}\dot{u}_{c(t)}(\gamma(t))\rangle = 0, \tag{17}$$

$$\langle v_2(t), J^{-1}\partial_c u_{c(t)}(\gamma(t))\rangle = 0. \tag{18}$$

We remark that $u(t) - v_1(t)$ remains in l^2_a for every $0 \le a < 2\kappa(c_0)$ and $t \in \mathbb{R}$. More precisely, we have the following.

Proposition 3. *Let $c_0 > 1$, $\tau_0 \in \mathbb{R}$ and $v_0 \in l^2$. Let $u(t)$ be a solution to (2) satisfying $u(0) = u_{c_0}(\tau_0) + v_0$ and let $v_1(t)$ be a solution to (15). Then*

$$u(t) \in C^2(\mathbb{R}; l^2) \quad \text{and} \quad u(t) - v_1(t) \in C^2(\mathbb{R}; l^2_a) \quad \text{for } 0 \le a < 2\kappa(c_0).$$

Proof. By [9], we have u, $v_1 \in C^2(\mathbb{R}; l^2)$. Let $v_3(t) = u(t) - v_1(t)$. Then $v_3(0) \in \cap_{0\le a<2\kappa(c_0)} l^2_a$ and

$$\frac{dv_3}{dt} = J(H'(u) - H'(v_1)). \tag{19}$$

Let $u(t) = {}^t(r(t), p(t))$, $v_1(t) = {}^t(r_1(t), p_1(t))$ and let

$$F(u, v_1) = \begin{pmatrix} \frac{V'(r)-V'(r_1)}{r-r_1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have $F(u, v_1) \in C^1(\mathbb{R}; B(l^2_a))$ for every $a \in [0, 2\kappa(c_0))$ and (19) can be rewritten as

$$\frac{dv_3}{dt} = JF(u, v_1)v_3. \tag{20}$$

By [9, Appendix A], we see that there exists a unique solution $v_3 \in C^2(\mathbb{R}; l^2 \cap l^2_a)$ to (20) for every $a \in [0, 2\kappa(c_0))$. Thus we prove $u - v_1 \in C^2(\mathbb{R}; l^2_a)$ for every $a \in [0, 2\kappa(c_0))$. □

If $u(t)$ and $u(t) - v_1(t)$ lie in a tubular neighborhood of a solitary wave in l^2 and l^2_a respectively, we can find modulation parameters $c(t)$ and $\gamma(t)$ satisfying (17) and (18).

Lemma 4. *Let $c_0 > 1$, $\tau_0 \in \mathbb{R}$, $\gamma_0(t) = t + \tau_0$ and $a \in (0, 2\kappa(c_0))$. Let $u(t)$ be a solution to (2) and let $v_1(t)$ be a solution to (15). Then there exist positive numbers δ_0 and δ_1 satisfying the following: If*

$$\sup_{t\in[T_1,T_2]} \left(\|u(t) - u_{c_0}(\gamma_0(t))\|_{l^2} + e^{-ac_0\gamma_0(t)} \|u(t) - u_{c_0}(\gamma_0(t)) - v_1(t)\|_{l^2_a} \right) < \delta_0$$

for some $0 \le T_1 \le T_2 \le \infty$, there exists $(c(t), \gamma(t)) \in C^2([T_1, T_2]; \mathbb{R}^2)$ satisfying (8), (17), (18) and

$$\sup_{t\in[T_1,T_2]} (|\gamma(t) - \gamma_0(t)| + |c(t) - c_0|) < \delta_1.$$

Especially, it holds $|c(0) - c_0| + |\gamma(0) - \tau_0| = O(\|v_0\|_{l^2})$.

Proof. Put

$$F_1(u, \tilde{u}, c, \gamma) := \langle u - u_c(\gamma), J^{-1}\dot{u}_c(\gamma)\rangle, \tag{21}$$

$$F_2(u, \tilde{u}, c, \gamma) := \langle \tilde{u} - u_c(\gamma), J^{-1}\partial_c u_c(\gamma))\rangle. \tag{22}$$

Then

$$\frac{\partial(F_1, F_2)}{\partial(c, \gamma)}(u_{c_0}(\gamma_0), u_{c_0}(\gamma_0), c_0, \gamma_0) = -\left(\frac{d}{dc}H(u_{c_0})\right)^2 \ne 0.$$

Let $U(\delta_0) = \{(u, \tilde{u}) \in l^2 \times l^2_a : \|u - u_c(\gamma_0)\|_{l^2} + e^{-ac\gamma_0}\|\tilde{u} - u_c(\gamma_0)\|_{l^2_a} < \delta_0\}$ and $B(\delta_1) := \{(c, \gamma) \in \mathbb{R}^2 : |c - c_0| + |\gamma - \gamma_0| < \delta_1\}$. Using the implicit function theorem, we see that there exists positive numbers δ_0 and δ_1 and a mapping

$$\Phi : U(\delta_0) \ni (u, \tilde{u}) \mapsto (c, \gamma) \in B(\delta_1)$$

satisfying $F_1(u, \tilde{u}, \Phi(u, \tilde{u})) = F_2(u, \tilde{u}, \Phi(u, \tilde{u})) = 0$. Since F_1 and F_2 are C^2 in $(u, \tilde{u}, \gamma, c) \in U(\delta_0) \times B(\delta_1)$, we have $\Phi \in C^2(U(\delta_0))$.

Let $(c(t),\gamma(t)) = \Phi(u(t), u(t)-v_1(t))$ for $t\in[T_1,T_2]$. Then $c(t)$ and $\gamma(t)$ satisfy (17) and (18) and are of class C^2 because $\Phi\in C^2(U(\delta_0))$ and $(u(t),u(t)-v_1(t))\in C^2(\mathbb{R};U(\delta_0))$. Furthermore, we have

$$\begin{aligned}&|c(t)-c_0|+|\gamma(t)-\gamma_0(t)|\\ \lesssim&\|u(t)-u_{c_0}(\gamma_0(t))\|_{l^2}+e^{-ac_0\gamma(t)}\|u(t)-u_{c_0}(\gamma_0(t))-v_1(t)\|_{l^2_a}.\end{aligned}$$

Especially for $t=0$, we have $|c(0)-c_0|+|\gamma(0)-\tau_0| = O(\|v_0\|_{l^2})$. This completes the proof of Lemma 4. □

To estimate the exponentially decaying part of a solution, we will use the following decay estimate for non-autonomous linearized equations.

Lemma 5 ([10, 18]). *Let $c_0>1$, $a\in(0,2\kappa(c_0))$ and $b(a):=ca-2\sinh(a/2)$. Let $U_0(t,\tau)\varphi$ be a solution to*

$$\begin{cases}\dfrac{dv}{dt}=JH''(u_{c_0})v.\\ v(\tau)=\varphi.\end{cases}\tag{23}$$

Then for every $b\in(0,b(a))$, there exists a positive number K such that for every $\varphi\in l^2_a$ and $t\ge\tau$,

$$e^{-ac_0(t-\tau)}\|U_0(t,\tau)Q_c(\tau)\varphi\|_{l^2_a}\le Ke^{-b(t-\tau)}\|\varphi\|_{l^2_a}.$$

□

Now let $\gamma=\gamma(t)$ be a C^1-function and let $U(t,\tau)v_0$ be a solution to

$$\begin{cases}\dfrac{dv}{dt}=\dot\gamma JH''(u_{c_0}(\gamma))v,\\ v(\tau)=\varphi.\end{cases}\tag{24}$$

If a modulation parameter $\gamma(t)$ is an increasing function and $\dot\gamma(t)$ is bounded away from 0, we have the following.

Corollary 6. *Let c_0, a, b and K be as in Lemma 5 and let $0\le T\le\infty$. Suppose $\inf_{t\in[0,T]}\dot\gamma(t)\ge1/2$, $\varphi\in l^2_a$ and $\langle\varphi,J^{-1}\dot u_{c_0}(\gamma(\tau))\rangle=\langle\varphi,J^{-1}\partial_c u_{c_0}(\gamma(\tau))\rangle=0$. Then*

$$\|U(t,\tau)\varphi\|_{X(t)}\le Ke^{-b(t-\tau)/2}\|\varphi\|_{X(\tau)}\quad\text{for } 0\le\tau\le t\le T,$$

where $\|v\|_{X(t)}:=e^{-ac_0\gamma(t)}\|v\|_{l^2_a}$.

Proof. Let $s=\gamma(t)$, $\tau_1=\gamma(\tau)$ and $\tilde v(s)=v(\gamma^{-1}(s))$. Then for $s\in[0,\gamma(T)]$,

$$\frac{d\tilde v}{ds}=JH''(u_{c_0})\tilde v\quad\text{and}\quad v(s)\in\operatorname{Range}Q_{c_0}(s).$$

Lemma 5 and the fact that $\dot\gamma(t)\ge1/2$ imply

$$\begin{aligned}\|v(t)\|_{X(t)}=e^{-ac_0s}\|\tilde v(s)\|_{l^2_a}\le&Ke^{-b(s-\tau_1)-ac_0\tau_1}\|\varphi\|_{l^2_a}\\ \le&Ke^{-b(t-\tau)/2}e^{-ac_0\gamma(\tau)}\|\varphi\|_{l^2_a}.\end{aligned}$$

This completes the proof of Corollary 6. □

We can estimate $\|v(t)\|_{l^2}$ by applying an argument from [9] that uses the convexity of Hamiltonian and the orthogonality condition (17).

Lemma 7. *Let $u(t)$ be a solution to* (2) *satisfying $u(0) = u_{c_0}(\tau_0) + v_0$. Then there exist positive numbers δ_2 and C satisfying the following: Suppose there exists $T \in [0, \infty]$ such that $v(t)$ satisfies* (8) *and* (17) *for $t \in [0, T]$ and $\sup_{t\in[0,T]} |c(t) - c_0| + \|v_0\|_{l^2} \le \delta_2$. Then*

$$\|v(t)\|_{l^2}^2 \le C(|c(t) - c_0| + \|v_0\|_{l^2}) \quad \text{for } t \in [0, T]. \tag{25}$$

Proof. By (17), we have $\langle H'(u_{c(t)}(\gamma(t))), v(t)\rangle = \langle J^{-1}\dot{u}_{c(t)}(\gamma(t)), v(t)\rangle = 0$. Since $H(u(t))$ does not depend on t, it follows from the convexity of the functional H and the above that

$$\begin{aligned}\delta H :=& H(u_{c_0}(\tau_0) + v_0) - H(u_{c_0})\\ =& H(u_{c(t)}(\gamma(t)) + v(t)) - H(u_{c_0})\\ =& H(u_{c(t)}) + \langle H'(u_{c(t)}(\gamma(t))), v(t)\rangle + \frac{1}{2}\langle H''(u_{c(t)}(\gamma(t)))v(t), v(t)\rangle\\ & - H(u_{c_0}) + O(\|v(t)\|_{l^2}^3)\\ \ge& \frac{1}{2}\|v(t)\|_{l^2}^2 - C'|c(t) - c_0| + O(\|v(t)\|_{l^2}^3),\end{aligned}$$

where C' is a positive constant. Noting that $|\delta H| = O(\|v_0\|_{l^2})$, we have (25) for a $C > 0$. □

Because $l^2 \subset l^r$ for every $r \in [2, \infty]$, Lemma 7 allows us to control every l^r-norm with $r \ge 2$.

3. Proof of Theorem 1

First, we derive from (17) and (18) a system of ordinary differential equations which describe the motion of modulating speed $c(t)$ and phase shift $x(t) = c(t)\gamma(t)$ of the main solitary wave.

Lemma 8. *Let $u(t)$ be a solution to* (2) *and $v_1(t)$ be a solution to* (15). *Suppose that c and γ are C^1-functions satisfying* (17) *and* (18) *on $[0, T]$ and $\inf_{t\in[0,T]} c(t)$*
> 1. Then it holds for $t \in [0, T]$ that

$$\begin{aligned}&\dot{c}(t) = O(\|v_1(t)\|_{W(t)}^2 + \|v_2(t)\|_{X(t)}^2),\\ &\dot{x}(t) - c(t) = O(\|v_1(t)\|_{W(t)} + (\|v(t)\|_{l^2} + \|v_1(t)\|_{l^2})\|v_2(t)\|_{X(t)}),\end{aligned}$$

where $\|u\|_{W(t)} = \left(\sum_{n\in\mathbb{Z}} e^{-\kappa(c(t))|n-x(t)|}|u(n)|^2\right)^{1/2}$, $\|u\|_{X(t)} = e^{-ax(t)}\|u\|_{l^2_a}$ and a is a constant satisfying $0 < a \le \inf_{t\in[0,T]} \kappa(c(t))$.

Proof. Differentiating (17) with respect to t and substituting (14) into the resulting equation, we have

$$\begin{aligned}&\frac{d}{dt}\langle v, J^{-1}\dot{u}_c(\gamma)\rangle\\ =&\langle \dot{v}, J^{-1}\dot{u}_c(\gamma)\rangle + \frac{\dot{x}}{c}\langle v, J^{-1}\ddot{u}_c(\gamma)\rangle + \dot{c}\langle v, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle\\ =&\langle JH''(u_c(\gamma))v, J^{-1}\dot{u}_c(\gamma))\rangle + \langle v, J^{-1}\ddot{u}_c(\gamma)\rangle\\ &+ \langle l_1 + N_1, J^{-1}\dot{u}_c(\gamma)\rangle + \left(\frac{\dot{x}}{c} - 1\right)\langle v, J^{-1}\ddot{u}_c(\gamma)\rangle + \dot{c}\langle v, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle\\ =&0.\end{aligned}$$

Substituting $\ddot{u}_c = JH''(u_c)\dot{u}_c$ and $J^* = -J$ into the above, we have
(26)
$$\dot{c}\left\{\frac{d}{dc}H(u_c) - \langle v, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle\right\} - \left(\frac{\dot{x}}{c} - 1\right)\langle v, J^{-1}\ddot{u}_c(\gamma)\rangle = \langle N_1, J^{-1}\dot{u}_c(\gamma)\rangle.$$

Differentiating (18) with respect to t, we have

$$\begin{aligned}
&\frac{d}{dt}\langle v_2, J^{-1}\partial_c u_c(\gamma)\rangle \\
=&\langle \dot{v}_2, J^{-1}\partial_c u_c(\gamma)\rangle + \frac{\dot{x}}{c}\langle v_2, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle + \dot{c}\langle v_2, J^{-1}\partial_c^2 u_c(\gamma)\rangle \\
=&\langle JH''(u_c(\gamma))v_2, J^{-1}\partial_c u_c(\gamma))\rangle + \langle v_2, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle \\
&+ \langle l_1 + N_2, J^{-1}\partial_c u_c(\gamma)\rangle + \left(\frac{\dot{x}}{c} - 1\right)\langle v_2, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle + \dot{c}\langle v_2, J^{-1}\partial_c^2 u_c(\gamma)\rangle \\
=&0.
\end{aligned}$$

Substituting $\partial_c\dot{u}_c = JH''(u_c)\partial_c u_c$ into the above, we obtain

$$\begin{aligned}
&\left(\frac{\dot{x}}{c} - 1\right)\left\{\frac{d}{dc}H(u_c) + \langle v_2, J^{-1}\partial_c\dot{u}_c(\gamma)\rangle\right\} + \dot{c}\langle v_2, J^{-1}\partial_c^2 u_c(\gamma)\rangle \\
=& -\langle N_2, J^{-1}\partial_c u_c(\gamma)\rangle.
\end{aligned} \tag{27}$$

Since $|N_1(t)| \lesssim |v(t)|^2$ and $|J^{-1}\dot{u}_c(t,n)| \lesssim e^{-2\kappa(c)|n-x(t)|}$ as $n \to \infty$, we have

$$\langle N_1, J^{-1}\dot{u}_c(\gamma)\rangle = O(\|v(t)\|_{W(t)}^2).$$

Let $N_2(t) = \widetilde{N}_1(t) + \widetilde{N}_2(t) + \widetilde{N}_3(t)$, where

$$\begin{aligned}
\widetilde{N}_1(t) =& N_1(t) - JH'(v(t)) + Jv(t), \\
\widetilde{N}_2(t) =& JH'(v(t)) - JH'(v_1(t)) - Jv_2(t), \\
\widetilde{N}_3(t) =& J\left(H''(u_{c(t)}(\gamma(t))) - 1\right)v_1(t).
\end{aligned}$$

We put $G(v) := H'(v) - H'(0) - H''(0)v$ so that $JG(v)$ denotes a part of $\widetilde{N}_1(t)$ that does not interact with the solitary wave $u_c(\gamma)$. Since $|u_c(t,n)| \lesssim e^{-2\kappa(c)|n-x(t)|}$ and $a \le \inf_{t\in[0,T]}\kappa(c(t))$, we have $\|u_{c(t)}v^2\|_{X(t)} \lesssim \|v\|_{W(t)}^2$. Hence by the definition of $\widetilde{N}_1$ and $\widetilde{N}_2$,

$$\|\widetilde{N}_1(t)\|_{X(t)} = \|N_1(t) - JG(v(t))\|_{X(t)} \lesssim \|v(t)\|_{W(t)}^2, \tag{28}$$

$$\begin{aligned}
\|\widetilde{N}_2(t)\|_{X(t)} =& \|JG(v(t)) - JG(v_1(t))\|_{X(t)} \\
\lesssim& (\|v(t)\|_{l^\infty} + \|v_1(t)\|_{l^\infty})\|v_2(t)\|_{X(t)}.
\end{aligned} \tag{29}$$

We see from (3) and (4) that $H''(u_c) - 1$ decays like $e^{-2\kappa|n-x(t)|}$ as $n \to \pm\infty$ and for $a \in (0, \kappa(c(t)))$,

$$\|\widetilde{N}_3(t)\|_{X(t)} \lesssim \|v_1(t)\|_{W(t)}. \tag{30}$$

Let $\|u\|_{X(t)^*} = e^{ax(t)}\|u\|_{l^2_{-a}}$ and $\|u\|_{W(t)^*} = (\sum_{n\in\mathbb{Z}} e^{\kappa(c(t))|n-x(t)|}|u(n)|^2)^{1/2}$. In view of (26), (27) and the fact that

$$\sup_{t\in[0,T]} \left(\|J^{-1}\ddot{u}_{c(t)}(\gamma(t))\|_{W(t)^*} + \|J^{-1}\partial_c\dot{u}_{c(t)}(\gamma(t))\|_{W(t)^*}\right) < \infty,$$
$$\sup_{t\in[0,T]} \left(\|J^{-1}\partial_c\dot{u}_{c(t)}(\gamma(t))\|_{X(t)^*} + \|J^{-1}\partial_c^2 u_{c(t)}(\gamma(t))\|_{X(t)^*}\right) < \infty,$$

we have

$$\mathcal{A}(t)\begin{pmatrix}\dot{c}(t)\\ \dot{x}(t)-c(t)\end{pmatrix} = \begin{pmatrix}O(\|v(t)\|^2_{W(t)})\\ O(\|v_1(t)\|_{W(t)} + (\|v(t)\|_{l^2} + \|v_1(t)\|_{l^2})\|v_2(t)\|_{X(t)})\end{pmatrix},$$

where $\mathcal{A}(t) = \operatorname{diag}(dH(u_c)/dc, dH(u_c)/dc) + O(\|v_1(t)\|_{W(t)} + \|v_2(t)\|_{X(t)})$. We have thus proved Lemma 8. □

Since $v_1(t)$ is smaller than the main wave, it moves more slowly and will be separated from the main wave. The following is an analog of *virial lemma* for small solutions in Martel and Merle [16].

Lemma 9. *Let $v_1(t)$ be a solution to* (15)*.*

(i) *Suppose $v_0 \in l^2$. Then $\sup_{t\in\mathbb{R}}\|v_1(t)\| \le C\|v_0\|_{l^2}$, where C can be chosen as an increasing function of $\|v_0\|_{l^2}$.*

(ii) *Let $c_1 > 1$ and $\tilde{x}(t)$ be a C^1-function satisfying $\inf_{t\in\mathbb{R}}\tilde{x}_t \ge c_1$. Then there exist positive numbers a_0 and δ_3 such that if $a \in (0, a_0)$ and $\|v_0\|_{l^2} \le \delta_3$,*

$$\|\psi_a(t)^{1/2}v_1(t)\|^2_{l^2} + \int_0^t \|\tilde{\psi}_a(t)v_1(s)\|^2_{l^2}ds \lesssim \|\psi_a(0)^{1/2}v_0\|^2_{l^2},$$

where $\psi_a(t,x) = 1 + \tanh a(x-\tilde{x}(t))$ and $\tilde{\psi}_a(t,x) = a^{1/2}\operatorname{sech} a(x - \tilde{x}(t))$.

Corollary 10. *Let $v_1(t)$ be a solution to* (15)*. For every $c_1 > 1$, there exists $\delta_3 > 0$ such that $\lim_{t\to\infty}\|v_1(t)\|_{l^2(n\ge c_1 t)} = 0$ if $\|v_0\|_{l^2} < \delta_3$.*

Proof of Lemma 9. Since $v_1(t) \in C^2(\mathbb{R}; l^2)$ is a solution to (15), we have $H(v_1(t)) = H(v_0)$ for $t \in \mathbb{R}$. Noting that $V(x)$ is coercive and $\inf_{|x|\le R}|x|^{-2}V(x) > 0$ for every $R > 0$, we have

$$\delta'\|v(t)\|^2_{l^2} \le H(v(t)) = H(v_0) \le C(\|v_0\|_{l^2})\|v_0\|^2_{l^2},$$

where C can be chosen as an increasing function of $\|v_0\|_{l^2}$ and δ' is a positive constant depending only on $\|v_0\|_{l^2}$.

Next, we prove (ii). Put

$$v_1(t) = {}^t(r_1(t,n), p_1(t,n)), \quad h_1(t,n) = \frac{1}{2}p_1(t,n)^2 + V(r_1(t,n)).$$

By (2) and the fact that there exists a $C > 0$ such that for every $n \in \mathbb{Z}$,

$$\left|V(r_1(t,n)) - \frac{r_1(t,n)^2}{2}\right| \le C\|v_0\|_{l^2}|r_1(t,n)|^2,$$
$$\left|V'(r_1(t,n)) - r_1(t,n)\right| \le C\|v_0\|_{l^2}|r_1(t,n)|,$$

we have

$$
\begin{aligned}
&\frac{d}{dt}\sum_{n\in\mathbb{Z}}\psi_a(t,n)h_1(t,n)\\
=&\sum_{n\in\mathbb{Z}}p_1(t,n)V'(r_1(t,n-1))\left(\psi_a(t,n-1)-\psi_a(t,n)\right)\ +\sum_{n\in\mathbb{Z}}\partial_t\psi_a(t,n)h_1(t,n)\\
\le&-\frac{\tilde{x}_t(t)}{2}\sum_{n\in\mathbb{Z}}\tilde{\psi}_a(t,n)^2p_1(t,n)^2\\
&+(1+C'\|v_0\|_{l^2})\sum_{n\in\mathbb{Z}}|\psi_a(t,n-1)-\psi_a(t,n)|\,|p_1(t,n)r_1(t,n-1)|\\
&-\frac{\tilde{x}_t(t)}{2}(1-C'\|v_0\|_{l^2})\sum_{n\in\mathbb{Z}}\tilde{\psi}_a(t,n-1)^2r_1(t,n-1)^2,
\end{aligned}
$$

where C' is a positive constant. Let δ_3 and a be sufficiently small numbers. Since $\inf\tilde{x}_t\ge c_1>1$ and

$$
\sup_{n,t}\left|\frac{\psi_a(t,n)-\psi_a(t,n-1)}{\tilde{\psi}_a(t,n)^2}-1\right|=O(a)\quad\text{as } a\downarrow 0,
$$

there exists a $\tilde{\delta}>0$ such that for $t\in[0,T]$,

$$
\frac{d}{dt}\sum_{n\in\mathbb{Z}}\psi_a(t,n)h_1(t,n)\le-\tilde{\delta}\sum_{n\in\mathbb{Z}}\tilde{\psi}_a(t,n)^2(p_1(t,n)^2+r_1(t,n)^2). \tag{31}
$$

Integrating (31) over $[0,t]$, we have

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}}\psi_a(t,n)h_1(t,n)+\tilde{\delta}\sum_{n\in\mathbb{Z}}\int_0^t\tilde{\psi}_a(s,n)^2(p_1(s,n)^2+r_1(s,n)^2)ds\\
\lesssim&\sum_{n\in\mathbb{Z}}\psi_a(0,n)h_1(0,n)\lesssim\|v_0\|_{l^2}^2.
\end{aligned}
$$

We have thus proved Lemma 9. □

Proof of Corollary 10. Let $c_2\in(1,c_1)$ and let $\tilde{x}(t)=c_2t$. Then by Lemma 9, we have $\|v_1(t)\|_{l^2(n\ge c_2t)}\lesssim\|\psi_a(0)^{1/2}v_0\|_{l^2}$. Let $n_0(t)=[(c_1-c_2)t]$, a largest integer which is smaller than $(c_1-c_2)t$. Then we have $n_0(t)\to\infty$ as $t\to\infty$ and

$$
\begin{aligned}
\|v_1(t)\|_{l^2(n\ge c_1t)}\le&\|v_1(t,\cdot+n_0(t))\|_{l^2(n\ge c_2t)}\\
\lesssim&\|\psi_a(0,\cdot)^{1/2}v_0(\cdot+n_0(t))\|_{l^2}.
\end{aligned}
$$

Letting $t\to\infty$, we have $\lim_{t\to\infty}\|v_1(t)\|_{l^2(n\ge c_1t)}=0$. This completes the proof of Corollary 10. □

Next, we will show the decay estimate of v_2.

Lemma 11. *Let $c_0>1$, $a\in(0,\kappa(c_0)/3)$ and δ_4 be a sufficiently small positive number. Suppose that the decomposition* (8), (17) *and* (18) *exists for $t\in[0,T]$ and that $\|v_0\|_{l^2}+\sup_{t\in[0,T]}(|c(t)-c_0|+|\dot{x}(t)-c_0|)\le\delta_4$, where*

$x(t) = c(t)\gamma(t)$. *Then*

(32)
$$\|Q_{c(t)}(\gamma(t))v_2(t)\|_{X(t)} \le C\left(e^{-bt/4}\|v_0\|_{l^2} + \int_0^t e^{-b(t-s)/4}\|v_1(s)\|_{W(s)}ds\right),$$

for $t \in [0,T]$, *and*

$$\int_0^T \|v_2(t)\|^2_{X(t)}dt \le C\|v_0\|^2_{l^2}, \tag{33}$$

where C *is a positive constant independent of* T *and* $\|v\|_{X(t)}$ *and* $\|v\|_{W(t)}$ *are as in Lemma 8.*

Proof. Let $\tilde{v}_2(t) := Q_{c(t)}(\gamma(t))v_2(t)$ and $w(t) = Q_{c_0}(\tilde{\gamma}(t))\tilde{v}_2(t)$, where $\tilde{\gamma}(t) = x(t)/c_0$. Here we choose $\tilde{\gamma}(t)$ so that $u_{c(t)}(\gamma(t))$ and $u_{c_0}(\tilde{\gamma}(t))$ have the same phase shift and for $0 < a < \min(\kappa(c(t)), \kappa(c_0))$,

$$\|Q_{c(t)}(\gamma(t)) - Q_{c_0}(\tilde{\gamma}(t))\|_{B(l^2_a)} = O(|c(t) - c_0|).$$

By (17) and (18),

$$\begin{aligned}\tilde{v}_2(t) =& v_2(t) - \theta(c(t))\langle v_2(t), J^{-1}\dot{u}_c(\gamma(t))\rangle \partial_c u_c(\gamma(t))\\ =& v_2(t) + \theta(c(t))\langle v_1(t), J^{-1}\dot{u}_c(\gamma(t))\rangle \partial_c u_c(\gamma(t)).\end{aligned} \tag{34}$$

Thus we have

$$\frac{d\tilde{v}_2}{dt} = JH''(u_{c(t)})(\gamma(t))\tilde{v}_2 + l_1(t) + l_2(t) + l_3(t) + N_2(t),$$

where

$$\begin{aligned}l_2(t) =& \frac{d}{dt}\left\{\theta(c(t))\langle v_1(t), J^{-1}\dot{u}_{c(t)}(\gamma(t))\rangle \partial_c u_{c(t)}(\gamma(t))\right\},\\ l_3(t) =& -\theta(c(t))\langle v_1(t), J^{-1}\dot{u}_{c(t)}(\gamma(t))\rangle \partial_c \dot{u}_{c(t)}(\gamma(t)).\end{aligned}$$

Since

$$\left[\frac{d}{dt} - \dot{\tilde{\gamma}}JH''(u_{c_0}(\tilde{\gamma})), Q_{c_0}(\tilde{\gamma})\right] = 0,$$

we have

$$\begin{aligned}\dot{w} - \dot{\tilde{\gamma}}JH''(u_{c_0}(\tilde{\gamma}))w =& Q_{c_0}(\tilde{\gamma})\left\{\dot{\tilde{v}}_2 - \dot{\tilde{\gamma}}JH''(u_{c_0}(\tilde{\gamma}))\tilde{v}_2\right\}\\ =& Q_{c_0}(\tilde{\gamma})\left\{\sum_{1\le k\le 4} l_k + \sum_{1\le k\le 3}\widetilde{N}_k\right\},\end{aligned} \tag{35}$$

where

$$\begin{aligned}l_4(t) =& J\left\{H''(u_{c(t)}(\gamma(t))) - H''(u_{c_0}(\tilde{\gamma}(t)))\right\}\tilde{v}_2(t)\\ &- (\dot{\tilde{\gamma}}(t) - 1)JH''(u_{c_0}(\tilde{\gamma}(t)))\tilde{v}_2(t).\end{aligned}$$

In view of Lemma 8, we have for $a \in (0, 2\kappa(c(t)))$,

$$\|l_1\|_{X(t)} \lesssim \|v_1(t)\|_{W(t)} + (\|v(t)\|_{l^2} + \|v_1(t)\|_{l^2} + \|v_2(t)\|_{X(t)})\|v_2(t)\|_{X(t)}. \tag{36}$$

By (15) and the fact that $J^{-1}\dot{u}_c(\gamma)$, $\frac{d}{dt}J^{-1}\dot{u}_c(\gamma)$, $\partial_c u_c(\gamma)$ and $\frac{d}{dt}\partial_c u_c(\gamma)$ decay like $e^{-2\kappa|n-x(t)|}$ as $n \to \pm\infty$, we have

$$\|l_2(t)\|_{X(t)} \lesssim \|v_1(t)\|_{W(t)}. \tag{37}$$

Similarly, we have

$$\|l_3(t)\|_{X(t)} \lesssim \|v_1(t)\|_{W(t)}. \tag{38}$$

Since $x(t) = c_0\tilde{\gamma}(t) = c(t)\gamma(t)$,

$$\begin{aligned} \|l_4(t)\|_{X(t)} \lesssim & (|c(t) - c_0| + |\dot{x}(t) - c_0|)\|\tilde{v}_2(t)\|_{X(t)} \\ \lesssim & \delta_4(\|v_1(t)\|_{W(t)} + \|v_2(t)\|_{X(t)}). \end{aligned} \tag{39}$$

Let $U(t,s)$ be a flow generated by

$$\frac{dw}{dt} = \dot{\tilde{\gamma}}(t)JH''(u_{c_0}(\tilde{\gamma}(t)))w.$$

Applying Corollary 6 to (35) and substituting (28)–(30) and (36)–(39), we have

$$\begin{aligned} & \|w(t)\|_{X(t)} \\ \lesssim & \|U(t,0)w(0)\|_{X(t)} + \sum_{k=1}^{4}\int_0^t \|U(t,s)Q_{c_0}(\tilde{\gamma}(s))l_k(s)\|_{X(t)} \\ & + \sum_{k=1}^{3}\int_0^t \|U(t,s)Q_{c_0}(\tilde{\gamma}(s))\widetilde{N}_k(s)\|_{X(t)} \\ \lesssim & e^{-bt/2}\|w(0)\|_{X(0)} + \int_0^t e^{-b(t-s)/2}\|v_2(s)\|_{X(s)}^2 ds \\ & + \int_0^t e^{-b(t-s)/2}\left\{\|v_1(s)\|_{W(s)} + (\delta_4 + \|v_1(s)\|_{l^2} + \|v_2(s)\|_{l^2})\|v_2(s)\|_{X(s)}\right\}. \end{aligned}$$

Here we use $\|u\|_{W(t)} \lesssim \|u\|_{l^2}$ and $\|u\|_{W(t)} \lesssim \|u\|_{X(t)}$ for $a \in (0, \kappa(c(t))/2)$. By the definition of $\tilde{v}_2$ and w,

$$\|v_2(t)\|_{X(t)} \lesssim \|\tilde{v}_2(t)\|_{X(t)} + \|v_1(t)\|_{W(t)}, \tag{40}$$

$$\begin{aligned} \|\tilde{v}_2(t)\|_{X(t)} \le & \|w(t)\|_{X(t)} + \|(Q_{c(t)}(\gamma(t)) - Q_{c_0}(\tilde{\gamma}(t)))\tilde{v}_2(t)\|_{X(t)} \\ \lesssim & \|w(t)\|_{X(t)} + |c(t) - c_0|\|\tilde{v}_2(t)\|_{X(t)}. \end{aligned} \tag{41}$$

If δ_4 is sufficiently small, Eqs. (40) and (41) imply $\|\tilde{v}_2(t)\|_{X(t)} \lesssim \|w(t)\|_{X(t)}$ and

$$\|v_2(t)\|_{X(t)} \lesssim \|w(t)\|_{X(t)} + \|v_1(t)\|_{W(t)}. \tag{42}$$

It follows from Lemmas 9 (i) and 7 that $\|v_1(t)\|_{l^2} + \|v_2(t)\|_{l^2}^2 \lesssim \|v_0\|_{l^2} + |c(t) - c_0|$. Thus as long as $\sup_{0 \le s \le t}\|w(s)\|_{X(s)} \le \sqrt{\delta_4}$, we have

$$\begin{aligned} \|w(t)\|_{X(t)} \lesssim & e^{-bt/2}\|w(0)\|_{X(0)} \\ & + \int_0^t e^{-b(t-s)/2}\left(\|v_1(s)\|_{W(s)} + \sqrt{\delta_4}\|w(s)\|_{X(s)}\right) ds. \end{aligned}$$

Applying Gronwall's inequality, we have

$$\begin{aligned} \|w(t)\|_{X(t)} \lesssim & e^{-(b/2+O(\sqrt{\delta_4}))t}\|w(0)\|_{X(0)} \\ & + \int_0^t e^{-(b/2+O(\sqrt{\delta_4}))(t-s)}\|v_1(s)\|_{W(s)} ds. \end{aligned} \tag{43}$$

By the definition of w, (16), (34) and Lemma 4,

$$\|w(0)\|_{X(0)} \lesssim \|v_2(0)\|_{X(0)} + \|v_1(0)\|_{l^2} \lesssim \|v_0\|_{l^2}. \tag{44}$$

In view of Lemma 9 (i), (43) and (44), we have $\|w(t)\|_{X(t)} \lesssim \|v_0\|_{l^2} = O(\delta_4)$ and (43) persists for $t \in [0,T]$ if δ_4 is sufficiently small. Thus by (41), we have (32) Combining (32), (34), and Lemma 9 (ii) and using Young's inequality, we have

$$\begin{aligned}\|v_2(t)\|_{L^2(0,T;X(t))} \lesssim & \|v_0\|_{l^2} + \|e^{-bt/4}\|_{L^1(0,T)}\|v_1\|_{L^2(0,T;W(t))} \\ \lesssim & \|v_0\|_{l^2}.\end{aligned}$$

We have thus completed the proof of Lemma 11. □

Now, we are in position to prove the following proposition.

Proposition 12. *Let $c_0 > 1$, $\tau_0 \in \mathbb{R}$ and let $u(t)$ be a solution to* (2) *with $u(0) = u_{c_0}(\tau_0) + v_0$. For every $\varepsilon > 0$, there exists a positive number $\delta > 0$ satisfying the following: If $\|v_0\|_{l^2} < \delta$, there exist a constant $c_+ > 1$ and a C^1-function $x(t)$ such that*

$$\|u(t) - \tilde{u}_{c_0}(\cdot - x(t))\|_{l^2} < \varepsilon, \tag{45}$$

$$\lim_{t\to\infty} \left\|u(t) - \tilde{u}_{c_+}(\cdot - x(t))\right\|_{l^2(n\geq x(t)-R)} = 0 \quad \textit{for every } R > 0, \tag{46}$$

$$\sup_{t\in\mathbb{R}} \left(|c(t) - c_0| + |\dot{x}(t) - c_0|\right) = O(\|v_0\|_{l^2}), \tag{47}$$

$$\lim_{t\to\infty} c(t) = c_+, \quad \lim_{t\to\infty} \dot{x}(t) = c_+. \tag{48}$$

Proof. Let $\delta_5 = \min_{1\leq i\leq 4} \delta_i$ and

$$T_0 := \sup\{t : (8), (17) \text{ and } (18) \text{ hold for } 0 \leq \tau \leq t\},$$

$$T_1 := \sup\left\{t \leq T_0 : \|v_0\|_{l^2} + \sup_{0\leq\tau\leq t} \left(|c(\tau) - c_0| + |\dot{x}(\tau) - c_0|\right) \leq \delta_5\right\}.$$

If δ is sufficiently small, Proposition 3 and Lemma 4 imply that $T_1 > 0$. We will show that $T_0 = T_1$ for small δ. Suppose that $t \in [0, T_1)$. Lemmas 8, 9 and 11 and (40) imply

$$\begin{aligned}|\dot{x}(t) - c(t)| \lesssim & \|v_1(t)\|_{W(t)} + \|v_2(t)\|_{X(t)} \\ \lesssim & \|v_1(t)\|_{W(t)} + \|\tilde{v}_2(t)\|_{X(t)} \lesssim \|v_0\|_{l^2}.\end{aligned} \tag{49}$$

By Lemmas 4 and 8,

$$\begin{aligned}|c(t) - c_0| \leq & |c(0) - c_0| + \int_0^t |\dot{c}(s)|ds \\ \lesssim & \|v_0\|_{l^2} + \int_0^t \left(\|v_1(s)\|_{W(s)}^2 + \|v_2(s)\|_{X(s)}^2\right) ds.\end{aligned}$$

In view of Lemmas 9 (ii) and 11, we have

$$|c(t) - c_0| \lesssim \|v_0\|_{l^2}. \tag{50}$$

It follows from (49) and (50) that $T_0 = T_1$ if δ is sufficiently small.

Next, we will show that $T_0 = \infty$ for small δ. Suppose that for every $\delta > 0$, there exists v_0 such that $\|v_0\|_{l^2} < \delta$ and $T_0 < \infty$. By Lemma 7 and (50),

$$\sup_{t\in[0,T_0]} \|v(t)\|_{l^2}^2 \lesssim \|v_0\|_{l^2}. \tag{51}$$

Using (40), Lemmas 11 and 9 (i), we have

$$\sup_{t\in[0,T_0]}\|v_2(t)\|_{X(t)} \lesssim \sup_{t\in[0,T_0]}\left(\|v_1(t)\|_{W(t)}+\|\tilde v_2(t)\|_{X(t)}\right) \lesssim \|v_0\|_{l^2}. \tag{52}$$

By (51) and (52), we get $\|v(T_0)\|_{l^2}+e^{-ax(T_0)}\|v_2(T_0)\|_{l^2_a} \lesssim \|v_0\|_{l^2}$. Hence it follows from Lemma 4 that the decomposition (8), (17) and (18) can be extended beyond $t=T_0$ if $\|v_0\|_{l^2}$ is small. This is a contradiction. Thus we prove $T_0=\infty$ for small $v_0\in l^2$.

Let δ be a small positive number such that $T_0=T_1=\infty$. Then Lemma 9 (ii) and Lemma 11 imply $\|v_1(t)\|_{W(t)}+\|v_2(t)\|_{X(t)}\in L^2(0,\infty)$. Thus by Lemma 8, we see that $\dot c(t)$ is integrable on $[0,\infty)$ and that there exists c_+ satisfying $\lim_{t\to\infty}c(t)=c_+$.

Next, we will prove (46). As in the proof of Corollary 10, we can prove $\lim_{t\to\infty}\|v_1(t)\|_{W(t)}=0$. Combining this with (54), we have

$$\lim_{t\to\infty}\dot x(t)=\lim_{t\to\infty}c(t)=c_+. \tag{53}$$

By (40), Lemma 11 and the fact that $\|v_1(t)\|_{W(t)}\in L^2(0,\infty)$,

$$\begin{aligned}\|v_2(t)\|_{X(t)} \lesssim&\|v_1(t)\|_{W(t)}+\|\tilde v_2(t)\|_{X(t)}\\ \lesssim&\|v_1(t)\|_{W(t)}+e^{-bt/4}\|v_0\|_{l^2}+\sup_{t/2\le s\le t}\|v_1(s)\|_{W(s)}\\ &+e^{-bt/8}\left(\int_0^{t/2}\|v_1(s)\|^2_{W(s)}ds\right)^{1/2}\to 0\quad\text{as } t\to\infty.\end{aligned} \tag{54}$$

Since $\|v_2(t)\|_{l^2(n\ge x(t)-R)}\lesssim\|v_2(t)\|_{X(t)}$ for every $R>0$, Corollary 10 and (54) imply (46). Combining this (53) and (54), we have

$$\lim_{t\to\infty}\dot x(t)=\lim_{t\to\infty}c(t)=c_+.$$

We have thus completed the proof of Proposition 12. □

Combining Proposition 12 and the *monotonicity* argument given in [16], we obtain Theorem 1.

Proof of Theorem 1. Put

$$\begin{aligned}&{}^t(\tilde r(t,n),\tilde p(t,n)):=v(t,n),\quad h(t,n)=\frac12\tilde p(t,n)^2+V(\tilde r(t,n)),\\ &N_3(t)=J\left\{H'(u_{c(t)}(\gamma(t))+v(t))-H'(u_{c(t)}(\gamma(t)))-H'(v(t))\right\}.\end{aligned}$$

Let $\sigma\in(1,c_+)$, $t_1\ge 0$ and $\tilde x(t)=x(t_1)+\sigma(t-t_1)$. Let $\psi_a(t,n)$ and $\tilde\psi_a(t,n)$ be as in Lemma 9. Then

$$\begin{aligned}&\frac{d}{dt}\sum_{n\in\mathbb Z}\psi_a(t,n)h(t,n)\\ =&\langle H'(v(t)),\psi_a(t)\dot v(t)\rangle+\sum_{n\in\mathbb Z}\partial_t\psi_a(t,n)h(t,n)\\ =&\sum_{n\in\mathbb Z}\tilde p(t,n)V'(\tilde r(t,n-1))(\psi_a(t,n-1)-\psi(t,n))\\ &+\langle\psi_a(t)l_1(t),H'(v(t))\rangle+\langle\psi_a(t)N_3(t),H'(v(t))\rangle+\sum_{n\in\mathbb Z}\partial_t\psi_a(t,n)h(t,n).\end{aligned}$$

Here we use $\frac{dv}{dt} = JH'(v(t)) + l_1(t) + N_3(t)$. Suppose that $a > 0$ and $\|v_0\|_{l^2}$ are sufficiently small. Since $\|v(t)\|_{l^2} \lesssim \|v_0\|_{l^2}$ follows from Proposition 12, we see that there exists a $\delta' > 0$

$$\begin{aligned}\frac{d}{dt}\sum_{n\in\mathbb{Z}} \psi_a(t,n)h(t,n) \le & -\delta' \sum_{n\in\mathbb{Z}} \tilde{\psi}_a(t,n)\left(\tilde{r}(t,n)^2 + \tilde{p}(t,n)^2\right) \\ & + \langle \psi_a(t)l_1(t), H'(v(t))\rangle + \langle \psi_a(t)N_3(t), H'(v(t))\rangle\end{aligned}$$

in exactly the same way as the proof of Lemma 9. By the definitions of $l_1(t)$ and $N_3(t)$ and Lemma 8,

$$\begin{aligned}&|N_3(t)| \lesssim |u_{c(t)}(\gamma(t))v(t)|, \\ &|\langle l_1(t), H'(v(t))\rangle| \lesssim (\|v_1(t)\|_{W(t)} + \|v_2(t)\|_{X(t)})^2.\end{aligned}$$

Combining the above, we have

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} \psi_a(t,n)\left(\tilde{r}(t,n)^2 + \tilde{p}(t,n)^2\right) \\ \lesssim & \sum_{n\in\mathbb{Z}} \psi_a(t_1,n)h(t_1,n) + \int_{t_1}^{t} (\|v_1(s)\|_{W(s)} + \|v_2(s)\|_{X(s)})^2 ds \\ \lesssim & \sum_{n\in\mathbb{Z}} \psi_a(t_1,n)\left(|v_1(t_1,n)|^2 + |v_2(t_1,n)|^2\right) + \int_{t_1}^{t} (\|v_1(s)\|_{W(s)} + \|v_2(s)\|_{X(s)})^2 ds.\end{aligned}$$

As in the proof of Corollary 10, we have

$$\lim_{t_1\to\infty} \sum_{n\in\mathbb{Z}} \psi_a(t_1,n)|v_1(t_1,n)|^2 = 0.$$

On the other hand, Lemma 11 implies

$$\sum_{n\in\mathbb{Z}} \psi_a(t_1,n)|v_2(t_1,n)|^2 \lesssim \|v_2(t_1)\|_{X(t_1)}^2 \to 0 \quad \text{as } t_1 \to \infty.$$

Furthermore, Lemmas 9 and 11 and Proposition 12 imply

$$\lim_{t_1\to\infty} \int_{t_1}^{\infty} (\|v_1(s)\|_{W(s)}^2 + \|v_2(s)\|_{X(s)}^2) ds = 0.$$

Combining the above, we obtain

$$\lim_{t_1\to\infty} \sup_{t\ge t_1} \|v(t)\|_{l^2(n\ge\sigma t)} = 0.$$

Thus we complete the proof of Theorem 1. □

4. Proof of Theorem 2

In this section, we will prove orbital and asymptotic stability of solitary waves to FPU lattice (9). For a two-parameter family of solitary wave solutions $\{u_c(t+\delta) : c \in [c_1, c_2],\ \delta \in \mathbb{R}\}$ that satisfies the condition (P1)–(P4) below, we can prove the orbital and asymptotic stability of solitary wave solutions in exactly the same way as Theorem 1.

(P1) There exists an open interval I such that $V''(r) > 0$ for every $r \in I$ and that $\overline{\{r_c(x) : x \in \mathbb{R}\}} \subset I$ for every $c \in [c_1, c_2]$.

(P2) There exists $a > 0$ such that the map $\mathbb{R} \times [c_1, c_2] \ni (t, c) \mapsto u_c(t) \in l^2_a \cap l^2_{-a}$ is C^2.

(P3) The solitary wave energy $H_F(u_c)$ satisfies $dH_F(u_c)/dc \neq 0$ for $c \in [c_1, c_2]$.

(P4) Let $c_0 \in [c_1, c_2]$ and $a \in (0, 2\kappa(c_0/c_s))$. Let $U_0(t,\tau)\varphi$ be a solution to

$$\begin{cases} \dfrac{dv}{dt} = JH_F''(u_{c_0})v. \\ v(\tau) = \varphi. \end{cases} \tag{55}$$

Then there exist positive numbers b and K such that for every $\varphi \in l_a^2$ and $t \geq \tau$,

$$e^{-ac_0(t-\tau)}\|U_0(t,\tau)Q_c(\tau)\varphi\|_{l_a^2} \leq Ke^{-b(t-\tau)}\|\varphi\|_{l_a^2}.$$

Proof of Theorem 2. If $c > c_s$ and c is sufficiently close to c_s, there exists a unique solitary wave solution to (10) up to translation ([8, Theorem 1.1]). By [8, Theorem 1.1], we see that a solitary wave solution satisfies (P1) and (P3) if c is close to c_s. Slightly modifying the proof of [8, Proposition 6.1] and [9, Proposition A.3], we obtain (P2). Since (P4) holds for small solitary waves (see [11]), Theorem 2 can be proved in exactly the same way as Theorem 1. □

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