Switching effect upon the quantum Brownian motion near a reflecting boundary

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The quantum Brownian motion of a charged particle in the electromagnetic vacuum fluctuations is investigated near a perfectly reflecting flat boundary, taking into account the smooth switching process in the measurement.

Constructing a smooth switching function by gluing together a plateau and the Lorentzian switching tails, it is shown that the switching tails have a great influence on the measurement of the Brownian motion in the quantum vacuum. Indeed, it turns out that the result with a smooth switching function and the one with a sudden switching function are qualitatively quite different. It is also shown that anti-correlations between the switching tails and the main measuring part plays an essential role in this switching effect.

The switching function can also be interpreted as a prototype of an non-equilibrium process in a realistic measurement, so that the switching effect found here is expected to be significant in actual applications in vacuum physics.

PACS numbers: 05.40.Jc, 03.70.+k, 12.20.Ds

I. INTRODUCTION

It is well known that a non-trivial spectral profile of the vacuum fluctuations produces observable effects. One important example in this category is the quantum vacuum near reflecting boundaries, which is directly related to various applications; Casimir effect, quantum effects in the early universe, quantum noise in a gravitational-wave detector, and so on. One way of probing such non-trivial vacuum fluctuations is to study the Brownian motion of a test particle released in the vacuum in question [1, 2]. Another approach is, for instance, to investigate the interaction between a mirror and the nearby vacuum fluctuations [3].

In the present paper, we study the velocity dispersions of the Brownian motion of a charged test particle in the quantized electromagnetic vacuum near a perfectly reflecting, flat boundary. (Let us assume that the boundary coincides with the x-y plane ($z = 0$) for later convenience.) This analysis is along the line of some preceding studies on the quantum Brownian motion, among which the case of an uncharged, polarizable test particle [1] and the case of a charged test particle [2] are closely related to the present one. We here, however, would like to pay special attention to the influence of the switching process in measuring the velocity dispersions.

The reason why we focus on the switching process is as follows. In Ref. [2], Yu and Ford calculated the velocity dispersions of a classical charged particle in the electromagnetic vacuum near a reflecting boundary, assuming a *sudden switching* process. Here the "sudden switching" process indicates the measurement process in which the detector is abruptly turned on and turned off at the time $t = 0$ and $t = \tau$, say, respectively. They reported that the z-component of the velocity dispersion of the test particle, $\langle \Delta v_z^2 \rangle$, does not vanish in late time, but shows an asymptotic late-time behavior $\langle \Delta v_z^2 \rangle \sim C/z^2$ (C is some constant). They interpreted this behavior as a transient effect due to a sudden switching process. However, a sudden switching may not be very realistic in view of the uncertainty principle between time and energy since the sudden switching implies that a coupling between the field and the particle is switched on/off instantaneously with some finite energy exchange. Furthermore, the fact that the late-time behavior of $\langle \Delta v_z^2 \rangle$ does not depend on the main measuring time τ suggests that there should be cancellation during the time τ. If so, the switching tails at the edge of the main measuring process might have significant influence on the result.

It is desirable, thus, to reanalyze the same system under a more realistic measuring process with smooth switching tails and to see how the late-time behavior of the measured $\langle \Delta v_z^2 \rangle$ depends on the switching process. The aim of the present paper is to undertake this analysis.

We will see that, contrary to the macroscopic measurements, the measurement of the quantum vacuum fluctuations is considerably influenced by the switching tails in a highly non-trivial manner. In particular the anti-correlation between the main measuring part and the switching tails plays an essential role.

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‡One of the author (M.S.) has been supported by the Japan Ministry of Education, Culture, Sports, Science and Technology with the grant #14740162.

There are three time-scales characterizing the present system as is discussed below. We also study how the results depend on the time scales to get basic ideas about when switching is regarded as "smooth" or "sudden".

The quantum switching effect analyzed here is expected to find various applications related to non-stationary aspects of the vacuum fluctuations.

In Sec. II, we first review some basic results of the case of sudden switching discussed in [2], and then we show that a smooth switching process do lead to a totally different result. In Sec. III, after introducing a reasonable switching function, the velocity dispersions of the probe particle are explicitly computed, paying special attention to the singular integrals caused by the mirror-reflections of the light signal. We find that a particular anti-correlation between the main measuring part and the switching tails plays an essential role in the measuring process. Sec. IV is to clarify the origin of the anti-correlation effect found in the preceding section and we confirm that it indeed comes from the interplay between the measuring part and the switching tails. In Sec. V, the case in which the measuring time is shorter than $2z$ is considered. Section VI is devoted for summary and several discussions.

II. SUDDEN SWITCHING AND SMOOTH SWITCHING

A. The case of sudden switching

Let us first recall the analysis of the sudden switching case discussed in Ref.[2]. Throughout the paper, the analysis is done in the Minkowski spacetime with a standard coordinate system (t, x, y, z).

A flat, infinitely spreading mirror of perfect reflectivity is installed at $z = 0$ and the quantum vacuum of the electromagnetic field is considered inside the half space $z > 0$. Then we consider the measurement of the quantum fluctuations of the vacuum by using a classical charged particle with mass m and charge e as a probe. When the velocity of the particle is much smaller than the light-velocity, one can assume that the particle couples solely with the electric field $\vec{E}(\vec{x}, t)$. Then the equation of motion for the particle is given by

$$m\frac{d\vec{v}}{dt} = e\vec{E}(\vec{x}, t) \quad . \tag{1}$$

Furthermore, when the position of the particle does not change so much within the time-scale in question, Eq. (1) is approximately solved to

$$\vec{v}(\tau) \simeq \frac{e}{m}\int_0^\tau \vec{E}(\vec{x}, t)dt \quad . \tag{2}$$

Based on Eq.(2) along with $\langle E_i(\vec{x}, t)\rangle_R = 0$, the velocity dispersions of the particle, $\langle \Delta {v_i}^2\rangle$ $(i = x, y, z)$, are given by

$$\langle \Delta {v_i}^2\rangle = \frac{e^2}{m^2}\int_0^\tau dt' \int_0^\tau dt'' \langle E_i(\vec{x}, t')E_i(\vec{x}, t'')\rangle_R \tag{3}$$

with[4]

$$\langle E_z(\vec{x}, t')E_z(\vec{x}, t'')\rangle_R = \frac{1}{\pi^2}\frac{1}{(T^2 - (2z)^2)^2} \tag{4}$$

$$\langle E_x(\vec{x}, t')E_x(\vec{x}, t'')\rangle_R = \langle E_y(\vec{x}, t')E_y(\vec{x}, t'')\rangle_R = -\frac{1}{\pi^2}\frac{T^2 + 4z^2}{(T^2 - (2z)^2)^3} \quad , \tag{5}$$

where $T := t' - t''$ and the suffix "R" is for "renormalized". (We set $c = \hbar = 1$ hereafter throughout the paper.)

Now the explicit computation of Eq.(3) along with Eqs.(4) and (5) results in [2]

$$\langle \Delta {v_z}^2\rangle \doteq \frac{e^2}{32\pi^2 m^2}\frac{\tau}{z^3}\ln\left(\frac{2z+\tau}{2z-\tau}\right)^2 \quad , \tag{6}$$

$$\langle \Delta {v_x}^2\rangle = \langle \Delta {v_y}^2\rangle \doteq \frac{e^2}{\pi^2 m^2}\left\{\frac{\tau}{64z^3}\ln\left(\frac{2z+\tau}{2z-\tau}\right)^2 - \frac{\tau^2}{8z^2(\tau^2 - 4z^2)}\right\} \quad , \tag{7}$$

irrespective of whether $\tau > 2z$ or $\tau < 2z$. Here we note that a regularization using the generalized principal value [5] would lead to the same result in Ref.[2] to get these results when $\tau > 2z$. We introduce a special equality symbol "$\doteq$" (e.g. in Eqs.(6) and (7)) and an estimation symbol "$\approx$" (e.g. in Eqs.(8) and (9) below) to remind us that a regularization should be employed to get the result when the integral is a multi-pole integral. Indeed the kernel $\langle E_i(\vec{x}, t')E_i(\vec{x}, t'')\rangle_R$ possesses a double pole and a triple pole for $i = z$ and $i = x, y$, respectively, at $T = 2z$. Thus a regularization should be employed when $\tau > 2z$.

Let us note at this stage that there are two time-scales characterizing the present situation. One is the measuring time τ and the other is the traveling time z of the light-signal from the test particle to the plate.

Now the results Eqs.(6) and (7) yield the asymptotic late-time behavior

$$\langle \Delta {v_z}^2\rangle \approx \frac{e^2}{4\pi^2 m^2 z^2} + O\left((z/\tau)^2\right) \quad , \tag{8}$$

$$\langle \Delta {v_x}^2\rangle = \langle \Delta {v_y}^2\rangle \approx -\frac{e^2}{3\pi^2 m^2 \tau^2} + O\left((z/\tau)^2\right) \; . \tag{9}$$

Eq.(8) indicates that $\langle \Delta {v_z}^2\rangle$ *remains finite even in the late-time limit,* $\tau/z \longrightarrow \infty$. It would mean that an energy of the order of $\frac{1}{2}m\langle \Delta {v_z}^2\rangle$ is gained during this process.

Ref.[2] interpreted this asymptotic behavior of $\langle \Delta {v_z}^2\rangle$ in Eq.(8) as a transient effect caused by some energy change due to the "sudden-switching". Indeed, the formula Eq.(3) corresponds to the measuring process with the sudden-switching in which the measuring device is abruptly switched on and switched off at the time 0 and τ, respectively. From the viewpoint of the switching function, this measuring process is represented by a step-function

$$\begin{aligned} \Theta(t) &= 1 \quad (\text{for } \ 0 < t < \tau) \\ &= 0 \quad (\text{otherwise}) \ . \end{aligned} \tag{10}$$

It consists of the measuring part of the duration τ and infinitely steep switching tails. If the behavior could be interpreted as the transient effect during the switching process, then it is expected to see more or less similar behavior even when a different switching process is chosen other than the sudden switching. Let us study this point next.

B. The case of smooth switching

We now replace $\Theta(t)$ (Eq.(10)) with the Lorentzian function as a typical smooth switching function. The Lorentzian function with the characteristic time-scale τ is

$$f_\tau(t) = \frac{1}{\pi}\frac{\tau^2}{t^2+\tau^2} \quad , \tag{11}$$

normalized as

$$\int_{-\infty}^{\infty} f_\tau(t)dt = \tau \quad .$$

Instead of Eq.(3), the velocity dispersions shall be given by

$$\begin{aligned} \langle \Delta {v_i}^2 \rangle &= \frac{e^2}{m^2}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \\ & \qquad f_\tau(t')f_\tau(t'')\langle E_i(\vec{x},t')E_i(\vec{x},t'')\rangle_R \ . \end{aligned} \tag{12}$$

The function $f_\tau(t)$ represents solely smooth switching tails without any flat measuring part. In this case, the model is characterized by two time-scales, i.e. the switching-duration time τ and the traveling time z of the light-signal from the test particle to the plate.

If the asymptotic behavior (Eq.(8)) is due to the transient effect caused by energy input during the switching process, then, a similar kind of behavior is expected for Eq.(12). It turns out, however, these integrals are shown to be

$$\langle \Delta {v_z}^2 \rangle \doteq \frac{e^2}{16\pi^2 m^2\tau^2}\frac{1}{(1+\frac{z^2}{\tau^2})^2} \quad , \tag{13}$$

$$\langle \Delta {v_x}^2 \rangle = \langle \Delta {v_y}^2 \rangle \doteq -\frac{e^2}{16\pi^2 m^2\tau^2}\frac{1-\frac{z^2}{\tau^2}}{(1+\frac{z^2}{\tau^2})^3} \quad . \tag{14}$$

Thus we see that

(i) The short time behavior of $\langle \Delta {v_i}^2 \rangle$ ($\tau \ll 2z$) is same;

$$\begin{aligned} \langle \Delta {v_z}^2 \rangle &\sim \big(\langle \Delta {v_x}^2 \rangle = \langle \Delta {v_y}^2 \rangle\big) \\ &\sim \frac{e^2}{16\pi^2 m^2 z^2}\frac{\tau^2}{z^2} + O((\tau/z)^4) \quad , \end{aligned} \tag{15}$$

for both the step-function case and the Lorentzian switching case.

(ii) However, the long time behavior ($\tau \gg 2z$) of the z component is quite different. For the Lorentzian switching case, it turns out that

$$\langle \Delta {v_z}^2 \rangle \approx \frac{e^2}{16\pi^2 m^2\tau^2} + O((z/\tau)^4) \tag{16}$$

$$\langle \Delta {v_x}^2 \rangle = \langle \Delta {v_y}^2 \rangle \approx -\frac{e^2}{16\pi^2 m^2\tau^2} + O((z/\tau)^4) \tag{17}$$

Thus the late-time behavior of $\langle \Delta {v_z}^2 \rangle$ in the Lorentzian switching case is quite different from the step-function case (Eq.(8)); as $\tau/z \to \infty$, the former goes away while the latter remains finite independently of τ. The qualitatively different late-time behavior of $\langle \Delta {v_z}^2 \rangle$ shown in Eq. (8) and Eq. (16) is quite puzzling. The former depends on z, an intrinsic scale of the system, and remains even in the late time, while the latter does not depend on z and goes away in the late time.

It can be said that both the switching functions studied so far are not realistic enough. On the one hand, a sudden switching could have virtually picked up the contribution from the highly fluctuating vacuum, which might have been forbidden by the uncertainty principle. On the other hand, the pure Lorentzian switching model we have just investigated lacks a plateau of the measuring part, which is not realistic either: In a proper measurement, the measuring time scale τ should be large enough compared to z, the intrinsic scale of the system, so that the measuring function is regarded as nearly flat except for the switching ends.

We shall introduce a switching function which blends smoothly the step-function and the Lorentzian-tails of arbitrary duration in the arbitrary ratio. In the next few sections, we shall construct such a generalized model and investigate it in detail.

III. ANALYSIS OF THE VELOCITY FLUCTUATIONS OF A PROBE PARTICLE WITH A LORENTZ-PLATEAU SWITCHING FUNCTION

Here we undertake the reanalysis of the model introduced above with a more realistic switching function. We start with constructing a reasonable switching function.

A. Lorentz-plateau switching function

We here construct a switching function which is characterized by a stable measuring-part (of a time-scale

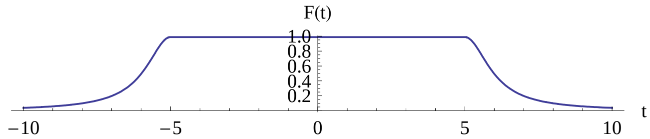

FIG. 1: Typical example of the Lorentz-plateau function.

τ_1) and two switching-tails describing the turn-on and the turn-off processes (of a total time-scale τ_2). This "Lorentz-plateau" function $F_{\tau\mu}(t)$ is a blend of a plateau part and the Lorentz function, characterized by two parameters τ and μ and defined as

$$\begin{aligned} F_{\tau\mu}(t) &= 1 \qquad\qquad\qquad\qquad (\text{for} \quad |t| \le \tau/2) \\ &= \frac{\mu^2}{(|t|/\tau - 1/2)^2 + \mu^2} \quad (\text{for} \quad |t| > \tau/2). \end{aligned} \tag{18}$$

Its flat part (corresponding to the main measuring period) is matched to two tail-parts (corresponding to switching tails) each of which is half of the Lorentzian function. The smoothness of the matching (C^1-class) is enough for our analysis since partial-integrals are not included in our analysis scheme. (If one wishes, however, one may anytime modify $F_{\tau\mu}$ to a smoother one.) The time-scale characterizing the measuring-part is $\tau := \tau_1$, while $\tau_2 := \pi\mu\tau$ characterizes the time-scale of the switching-tails:

$$\begin{aligned} &\int_{-\tau/2}^{\tau/2} F_{\tau\mu}(t)dt = \tau = \tau_1 \quad , \\ &2\int_{\tau/2}^{\infty} F_{\tau\mu}(t)dt = \pi\mu\tau =: \tau_2 \quad . \end{aligned} \tag{19}$$

Thus the parameter

$$\mu = \frac{\tau_2}{\pi\tau_1} \tag{20}$$

is the switching-duration parameter which characterizes the switching duration relative to the main measuring time-scale. In this general setting, the situation described by Eq.(3) (as considered in Ref.[2]) corresponds to the limit $\mu \longrightarrow 0$ with a fixed τ_1. Let us call this limiting situation the *sudden-switching* limit, for brevity. On the other hand, the situation described by Eq.(12) corresponds to the limit $\mu \longrightarrow \infty$ with a fixed τ_2 (i.e. $\tau_1 \to 0$ with a fixed τ_2), which shall be called the *Lorentzian* limit. In most part of the analysis below, it suffices to assume μ to be less than 1. (However, the case $\mu \gg 1$ is also considered when necessary.)

In view of Eqs.(3) and (12) along with Eqs.(4) and (5), what we need to estimate is of the form

$$\mathcal{I} = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \; F_{\tau\mu}(t') F_{\tau\mu}(t'') \mathcal{K}(t' - t'') \quad , \tag{21}$$

where $\mathcal{K}$ is an even function of $T := t' - t''$ with an appropriate asymptotic behavior as $|T| \to \infty$. General properties of the integral Eq.(21) are analyzed in *Appendix* A.

Introducing dimension-free variables $x := (t' - t'')/\tau$ and $y := (t' + t'')/\tau$, Eq.(21) becomes

$$\begin{aligned} \mathcal{I} \;=\; &\frac{\tau^2}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \\ &F_{\tau\mu}\Big(\frac{\tau}{2}(x+y)\Big) F_{\tau\mu}\Big(\frac{\tau}{2}(y-x)\Big) \; \mathcal{K}(\tau x) \; . \end{aligned} \tag{22}$$

Now the kernel $\mathcal{K}$ is essentially a two-point time-correlation function, so that the integral region for Eq.(22) is naturally divided into 4 classes of sub-regions, M, S_1, S_2 and MS ("M" and "S" are for "measuring" and "switching", respectively). Namely the class M comes from the two-point correlation solely within the measuring part ($|t| < \tau/2$), the class S_1 from the one within the same switching tail (either $t > \tau/2$ or $t < -\tau/2$), the class S_2 from the one between different switching tails ($t > \tau/2$ and $t < -\tau/2$) and finally the class MS from the two-point correlation between the measuring part ($|t| < \tau/2$) and the switching tails ($|t| > \tau/2$). (See *Appendix* A for computational details.)

B. Estimation of velocity dispersions using the Lorentz-plateau switching function

Having prepared a reasonable switching function, we now estimate the velocity dispersions of a probe-particle near a perfectly reflecting plate.

First let us note that, the model to be analyzed is now characterized by three time-scales rather than two, i.e. the measuring time $\tau = \tau_1$, the switching-duration time τ_2 and the traveling time z of the light-signal from the test particle to the plate.

Let us focus on $\langle \Delta {v_z}^2 \rangle$. We can make use of general formulas given in *Appendix* A. Comparing Eq.(4) with Eq.(21), we can set

$$\begin{aligned} \mathcal{K}(T) :&= \frac{e^2}{\pi^2 m^2} \frac{1}{(T^2 - (2z)^2)^2} \\ &= \frac{e^2}{\pi^2 m^2} \frac{1}{\tau_1^4} \frac{1}{(x^2 - \sigma_1^2)^2} = \frac{e^2}{\pi^2 m^2} \frac{1}{\mu^4 \tau_1^4} \frac{1}{(\chi^2 - \sigma_2^2)^2} \quad . \end{aligned}$$

Here several variables and parameters are introduced for simplicity,

$$\begin{aligned} &T := t' - t'' \; , \; x := T/\tau \; , \; \chi := x/\mu \; , \\ &\sigma_1 := 2z/\tau \; , \; \sigma_2 := 2z/(\mu\tau) = \sigma_1/\mu \quad , \end{aligned} \tag{23}$$

while $\tau = \tau_1$ and τ_2 are given in Eq.(19), and μ is given in Eq.(20).

Then Eq.(A5) in *Appendix* A gives the formula for

$\langle \Delta {v_z}^2 \rangle$,

$$\langle \Delta {v_z}^2 \rangle = \frac{2\ e^2}{\pi^2 m^2 \tau^2} \int_0^1 dx\ \frac{1-x}{(x^2-{\sigma_1}^2)^2}$$
$$+\frac{4\ e^2}{\mu^2 \pi m^2 \tau^2} \int_0^\infty d\chi\ \frac{1}{(\chi^2+4)(\chi^2-{\sigma_2}^2)^2}$$
$$+\frac{4\ e^2}{\mu^2 \pi^2 m^2 \tau^2} \int_0^\infty d\chi\ \times$$
$$\times \left\{ \frac{1}{(\chi^2-{\sigma_2}^2)^2} - \frac{1}{\{(\chi+1/\mu)^2-{\sigma_2}^2\}^2} \right\} \mathcal{F}(\chi)$$
$$=: \langle \Delta {v_z}^2 \rangle_M + \langle \Delta {v_z}^2 \rangle_S + \langle \Delta {v_z}^2 \rangle_{MS}\ , \qquad (24)$$

where $\mathcal{F}(\chi)$ is given by Eq.(A6).

Let us investigate three terms $\langle \Delta {v_z}^2 \rangle_M$, $\langle \Delta {v_z}^2 \rangle_S$ and $\langle \Delta {v_z}^2 \rangle_{MS}$ in Eq.(24) in more detail.

We now focus on the case $\tau > 2z$ since our main interest is in the late-time behavior of the vacuum fluctuations. (The case $\tau < 2z$ shall be treated separately in Sec.V.) In this case, all of the three integrals in Eq.(24) are singular integrals since $0 < \sigma_1 < 1$ and $\sigma_1 < \sigma_2 < 1/\mu \leq \infty$.

Let us first estimate the integral $\langle \Delta {v_z}^2 \rangle_M$, coming from the M-region. With the help of Eq.(C2) in *Appendix* C, we get

$$\langle \Delta {v_z}^2 \rangle_M :\ \doteq\ \frac{e^2}{2\pi^2 m^2 \tau^2} \frac{1}{\sigma_1^3} \ln\left(\frac{1+\sigma_1}{1-\sigma_1}\right)$$
$$\sim \frac{e^2}{\pi^2 m^2 \tau^2 \sigma_1^2}\ , \qquad (25)$$

where in the last line $\sigma_1 \ll 1$ ($\tau \gg 2z$) has been assumed. We note that $\langle \Delta {v_z}^2 \rangle_M$ is the contribution purely from the M-region, which corresponds to the velocity dispersion in the case of sudden switching. The above expression exactly coincides with the result given in Ref.[2].

By a similar prescription for the singular integral along with Eq.(B1) in *Appendix* B, we can estimate the term $\langle \Delta {v_z}^2 \rangle_S$, coming from the S_1- and S_2-regions, as

$$\langle \Delta {v_z}^2 \rangle_S\ \doteq\ \frac{\mu^2 e^2}{m^2 \tau^2} \frac{1}{(\sigma_1^2+4\mu^2)^2}$$
$$\sim O\left(\frac{\mu^2}{\sigma_1^2}\right) \cdot \langle \Delta {v_z}^2 \rangle_M \quad (\text{for } \mu < \sigma_1)$$
$$\sim O\left(\frac{\sigma_1^2}{\mu^2}\right) \cdot \langle \Delta {v_z}^2 \rangle_M \quad (\text{for } \mu > \sigma_1) \qquad (26)$$

The term $\langle \Delta {v_z}^2 \rangle_{MS}$, coming from the MS-regions along with the S_1- and S_2-regions, is estimated as follows. Noting that $0 \leq \mathcal{F}(\chi) < \frac{\pi}{2}$, it follows

$$\langle \Delta {v_z}^2 \rangle_{MS} = O(1) \cdot \frac{2e^2}{\mu^2 \pi m^2 \tau^2} \times \int_0^{1/\mu} \frac{1}{(\chi^2-{\sigma_2}^2)^2} d\chi\ . \qquad (27)$$

We note that the integral above is a singular one since $\sigma_2 < 1/\mu$, which can be treated with the help of Eq.(B1). It is notable that, in the above computation for $\langle \Delta {v_z}^2 \rangle_{MS}$, the cancellation has occurred as is shown the upper-bound of the integral region. Tracing back the origin of this cancellation, we can see from the general argument in *Appnedix* A, that it comes from the S_2-region, which describes the correlation between the pre- and the post-measuring switching tails. Thus, this cancellation phenomenon caused by the correlation between the pre- and the post-measuring switching tails seems to be quite universal and is probably worth while pursuing further.

The integral can be estimated as

$$\langle \Delta {v_z}^2 \rangle_{MS} = O(1) \cdot \frac{2e^2}{\mu^2 \pi m^2 \tau^2} \times \qquad (28)$$
$$\times \left\{ \frac{\mu^2}{2\sigma_1^2 \rho} + O(\rho) - \frac{\mu^3}{2\sigma_1^2} \left(\frac{1}{1-\sigma_1^2} - \frac{1}{2\sigma_1} \ln \frac{1+\sigma_1}{1-\sigma_1} \right) \right\}.$$

When $\sigma_1 \ll 1$, it is further modified as

$$\langle \Delta {v_z}^2 \rangle_{MS} = O(1) \cdot \frac{e^2}{\pi m^2 \sigma_1^2 \tau^2} \left(\frac{1}{\rho} - \frac{2\mu\sigma_1^2}{3} \right)\ . \qquad (29)$$

As mentioned at the end of the previous section, the regularization procedure removes the first term on the R.H.S., yielding

$$\langle \Delta {v_z}^2 \rangle_{MS}\ \approx\ -O(1) \cdot \frac{2\mu e^2}{3\pi m^2 \tau^2}$$
$$\sim -O(\mu\, \sigma_1^2) \langle \Delta {v_z}^2 \rangle_M\ . \qquad (30)$$

Gathering Eqs.(25), (26) and (30) together, and changing back to the variables τ_1,τ_2 and z, we get an estimation for the total velocity dispersion in z-direction

$$\langle \Delta {v_z}^2 \rangle$$
$$=\ \langle \Delta {v_z}^2 \rangle_M + \langle \Delta {v_z}^2 \rangle_S + \langle \Delta {v_z}^2 \rangle_{MS}$$
$$\approx\ \left\{ 1 + \frac{\pi^4 z^2 \tau_2^2}{4(\pi^2 z^2 + \tau_2^2)^2} - O(1) \cdot \frac{8}{3} \left(\frac{z}{\tau_1}\right)^2 \frac{\tau_2}{\tau_1} \right\} \langle \Delta {v_z}^2 \rangle_M \qquad (31)$$

Thus, under the condition $\tau_1 \gg 2z$, we derive the behavior of the velocity dispersion $\langle \Delta {v_z}^2 \rangle$ as a function of the three parameters τ_1, τ_2 and z:

(i) When $\tau_2 \ll 2z \ll \tau_1$, $\langle \Delta {v_z}^2 \rangle \approx \langle \Delta {v_z}^2 \rangle_M$.

(ii) When $\tau_2 \approx 2z \ll \tau_1$, $\langle \Delta {v_z}^2 \rangle \approx \frac{3}{2} \langle \Delta {v_z}^2 \rangle_M$.

(iii) When $2z \ll \tau_1 \ll \tau_2$ and $\frac{\tau_2}{\tau_1} = O\left(\left(\frac{\tau_1}{2z}\right)^2\right)$,

$\langle \Delta {v_z}^2 \rangle \approx \langle \Delta {v_z}^2 \rangle_S$.

(iv) When $2z \ll \tau_1 \ll \tau_2$ and $\frac{\tau_2}{\tau_1} \gg \frac{\tau_1^2}{(2z)^2}$,

$\langle \Delta {v_z}^2 \rangle \approx -O\left(\frac{\tau_2}{\tau_1}\left(\frac{2z}{\tau_1}\right)^2\right) \cdot \langle \Delta {v_z}^2 \rangle_M$

$\sim -O(1) \cdot \frac{2\, e^2}{3m^2 \pi^2} \frac{\tau_2}{\tau_1^3}$.

When the time-scale τ_2 of the switching tails is much shorter than the time-scale $2z$, the velocity dispersion $\langle \Delta v_z{}^2 \rangle$ reduces to the result of the sudden switching case given in Ref.[2] (the case (i)). As the time-scale τ_2 increases up to around the time scale $2z$, however, $\langle \Delta v_z{}^2 \rangle$ becomes around 3/2 times of $\langle \Delta v_z{}^2 \rangle_M$ (the case (ii)). It means that the contribution from the switching tails, $\langle \Delta v_z{}^2 \rangle_S$, is almost of the same order as the contribution from the measuring part, $\langle \Delta v_z{}^2 \rangle_M$. Hence the condition for the switching to be regarded as the "sudden switching" is $\tau_2 \ll 2z$, i.e. the switching time-scale is much smaller than the scale characterizing the system configuration.

Next, as the switching time τ_2 increases the velocity dispersion decreases, reducing to the Lorentzian switching case (Eq. (11)) at around $\tau_2 \sim O\left(\left(\frac{\tau_1}{2z}\right)^2\right)\tau_1$ (the case (iii)). This occurs mainly due to the cancellation of the M-term by the negative contribution from the MS-term, which is actually the correlation between the switching part and the main measuring part.

Finally, the case (iv) shows the possible total negative dispersion when the switching time is really large. However, we should also note that the time-scales cannot be arbitrarily large on account of the assumption that the position of the particle does not change so much during the whole process of probing the vacuum (see below Eq.(1)). The latter condition can be characterized by

$$\sqrt{|\langle \Delta v_z{}^2 \rangle|}\ \Delta T < z \quad , \tag{32}$$

where ΔT is the time-scale of the whole probing process. For the case (iv), we set $\Delta T = \tau_2$ to get

$$\frac{\tau_2}{\tau_1} < \left(\frac{3\pi^2 m^2 z^2}{2e^2}\right)^{1/3} \quad . \tag{33}$$

To get an idea, let us set m to be the electron mass. Then $\frac{\tau_2}{\tau_1} < 12.7\left(\frac{z}{\lambda_e}\right)^{2/3}$ where λ_e is the Compton length of the electron ($\sim 10^{-10}$cm). This inequality is likely to be satisfied when the system configuration is so arranged. Here we point out that this anti-correlation effect can possibly be used to control the total quantum fluctuations in applications.

IV. ANTI-CORRELATION DUE TO SWITCHING PROCESSES

It has been found in the preceding section that $\langle \Delta v_z{}^2 \rangle_{MS}$ becomes negative after the regularization, which plays a key role in the whole process of vacuum measurement. Since the quantity $\langle \Delta v_z{}^2 \rangle_{MS}$ is the combination of contributions from the MS-, S_1- and S_2-regions (see *Appnedix* A), it is desirable to pin down where the negative correlation comes from.

Here the switching function shall be modified in three ways to find out where the negative correlation comes in.

A. Measuring part with one switching tail

We choose as a switching function,

$$\begin{aligned} F^{(A)}_{\tau\mu}(t) &= 1 && (\text{for } \ |t| \le \tau/2) \\ &= 0 && (\text{for } \ t > \tau/2) \\ &= \frac{\mu^2}{(t/\tau + 1/2)^2 + \mu^2} && (\text{for } \ t < -\tau/2). \end{aligned} \tag{34}$$

The above switching function is not an even function in t; it consists of the pre-measurement tail, the main measurement part and a sudden switching-off.

It is easy to see that only the M-region and 2 MS-regions (among four) contribute to the integral Eq.(21). Thus

$$\begin{aligned} &\mathcal{I}^{(A)} = \mathcal{I}^{(M)} + 2\mathcal{I}^{(MS)} \\ &= 2\tau^2 \int_0^1 dx\ (1-x)\mathcal{K}(\tau x) \\ &\quad +2\mu^2\tau^2 \int_0^\infty d\chi\ \{\mathcal{K}(\mu\tau\chi) - \mathcal{K}(\mu\tau(\chi + 1/\mu)\} \tan^{-1}\chi \\ &\quad =: \mathcal{I}^{(A)}_M + \mathcal{I}^{(A)}_{MS} \quad . \end{aligned} \tag{35}$$

Comparing with Eq.(A5), it is notable that the behavior of $\tan^{-1}\chi$ in $\mathcal{I}^{(A)}_{MS}$ is very similar to $\mathcal{F}(\chi)$ given in Eq.(A6). Indeed both are monotonically increasing functions which approach 0 and $\frac{\pi}{2}$ as χ goes to 0 and ∞, respectively. Thus we notice that $\mathcal{I}^{(A)}_{MS}$ is qualitatively same as $\mathcal{I}_{MS}$ up to the numerical factor about $\frac{1}{2}$. (The factor around $\frac{1}{2}$ comes because the number of the MS-regions is now 2 rather than 4). Afterwards the computations go almost the same as done in Sec.III. Thus we get

$$\begin{aligned} &\langle \Delta v_z{}^2 \rangle^{(A)} \quad (\sim \langle \Delta v_z{}^2 \rangle_M + \frac{1}{2}\langle \Delta v_z{}^2 \rangle_{MS}) \\ &\approx \frac{e^2}{\pi^2 m^2 \tau^2 \sigma_1^2} - O(1)\cdot\frac{2\mu e^2}{3\pi m^2 \tau^2} \\ &\sim \left\{1 - O\left(\frac{\mu}{\sigma_1^2}\right)\right\}\langle \Delta v_z{}^2 \rangle^{(A)}_M \quad . \end{aligned} \tag{36}$$

It has turned out that, thus, *the negative correlation comes from the time-correlation between the measuring part and the switching tail*. The result does not change even when we choose $F^{(A')}_{\tau\mu}(t) := F^{(A)}_{\tau\mu}(-t)$ as a switching function. It is expected that the switching function $F^a_{\tau\mu}(t)$ itself is also useful in some applications.

For confirmation, we also consider 2 more modified switching functions below.

B. Single switching tail

We choose as a switching function,

$$\begin{aligned} F^{(B)}_{\tau\mu}(t) &= \frac{\mu^2}{(t/\tau + 1/2)^2 + \mu^2} && (\text{for } \ t < -\tau/2) \\ &= 0 && (\text{otherwise}) \quad . \end{aligned} \tag{37}$$

The above switching function consists only of the half of the Lorentzian function with a sudden switching-off.

Only one S_1-region (among the two) contributes to the integral Eq.(21). Then we get

$$\begin{aligned}\langle \Delta v_z{}^2 \rangle^{(B)} &\approx \frac{\mu^2 e^2}{2m^2\tau^2}\frac{1}{(\sigma_1^2+4\mu^2)^2} \quad , \\ &\sim \frac{1}{2}\langle \Delta v_z{}^2\rangle_S \quad . \end{aligned} \tag{38}$$

Note that the above result is half of the result given in Eq.(26). There is no change even when we choose $F_{\tau\mu}^{(C')}(t) := F_{\tau\mu}^{(C)}(-t)$ as a switching function. Thus, the correlations within the same switching tail do nothing with the negative correlation effect.

C. Switching tails without the measuring part

We choose as a switching function,

$$\begin{aligned}F_{\tau\mu}^{(C)}(t) &= 0 \qquad\qquad\qquad (\text{for} \ \ |t| \le \tau/2) \\ &= \frac{\mu^2}{(|t|/\tau - 1/2)^2 + \mu^2} \quad (\text{for} \ \ |t| > \tau/2). \end{aligned} \tag{39}$$

The above switching function consists only of the switching tails. Only the S_1- and S_2-regions contribute to the integral Eq.(21).

Following the estimations in Sec.III, we easily get

$$\langle \Delta v_z{}^2 \rangle^{(C)} \approx \langle \Delta v_z{}^2\rangle_S + \frac{4\mathcal{Z}\mu e^2}{3\pi^2 m^2\tau^2} \quad , \tag{40}$$

where the factor $\mathcal{Z}$ is some numerical factor much smaller than 1 and $\sigma_1 \ll 1$ has been assumed. Thus, the correlations between the two switching tails and those within the same switching tail do not yield negative contributions. Noting the inequality $\langle \Delta v_z{}^2\rangle^{(C)} > 2\langle \Delta v_z{}^2\rangle^{(B)}$, it is seen that *the correlations between the two switching tails cause small enhancement of* $\langle \Delta v_z{}^2\rangle$.

D. Origin of the negative correlation effect

From the results of the subsections IV A-IV C, it is now clear that the origin of the negative correlation effect resides in the correlation between the measuring part and the switching tail. Furthermore, one switching tail along with the measuring part is enough to cause this effect. In this way, it is seen that the interplay between the measuring part and the switching tail is a key to understand the measurement process of quantum vacuum.

V. ANALYSIS FOR THE CASE $\tau < 2z$

We have mainly studied the case $\tau_1 > 2z$ so far. In this section, let us investigate the case $\tau_1 < 2z$ in some detail.

The analysis goes in the similar manner up to Eq.(24). We note that $\sigma_1 > 1$ and

$$\sigma_2 := \sigma_1/\mu > \text{Max}(\sigma_1, 1/\mu) > \min(\sigma_1, 1/\mu) > 1$$

in this case. The situation now is that the measuring time-scale τ_1 is shorter than the intrinsic time-scale $2z$, and the switching time-scale τ_2 is even shorter than τ_1. Contrary to the case $\sigma_1 < 1$ ($\tau_1 > 2z$), the first term $\langle \Delta v_z{}^2\rangle_M$ in Eq.(24), which comes purely from the M-region, is now a regular integral due to $\sigma_1 > 1$ and it exactly matches the original result of the step-function case shown in Ref.[2] (Eq.(15)). Physically, it corresponds to the situation in which the measuring time τ_1 is so short that the information exchange between the mirror and the particle has not yet taken place. Therefore it is expected that the presence of the reflecting boundary does not play a vital role in this case. This is why the integral for $\langle \Delta v_z{}^2\rangle_M$ is regular when $\tau_1 < 2z$.

On the other hand, $\langle \Delta v_z{}^2\rangle_S$, which comes from the S_1- and S_2-regions, is given by a singular integral. The appearance of the singular integral is understood as the long-tail nature of the Lorentzian function. Though the measuring-part is too short to cause correlations due to the reflecting boundary, the long tails of the switching-part still pick up correlations. By a regularization, $\langle \Delta v_z{}^2\rangle_S$ is given by

$$\begin{aligned}\langle \Delta v_z{}^2\rangle_S &\doteq \frac{\mu^2\ e^2}{m^2\sigma_1^4\tau^2}\frac{1}{\left(1+\frac{4\mu^2}{\sigma_1^2}\right)^2} \\ &\sim \frac{\mu^2 e^2}{m^2\sigma_1^4\tau^2} \quad . \end{aligned}$$

Finally $\langle \Delta v_z{}^2\rangle_{MS}$ in Eq.(24), coming from the MS-, S_1- and S_2-regions, turns out to be finite. By shifting the variable $\chi' := \chi + 1/\mu$ in the second term, it can be estimated as

$$\langle \Delta v_z{}^2\rangle_{MS} \sim O(1)\cdot\frac{2e^2}{\pi m^2\mu^2\tau^2}\int_0^{1/\mu}\frac{1}{(\chi^2-\sigma_2^2)^2}d\chi \quad ,$$

which is a regular integral since $\sigma_2 > 1/\mu$. Performing the integral, we get

$$\begin{aligned}\langle \Delta v_z{}^2\rangle_{MS} &\sim O(1)\cdot\frac{\mu\ e^2}{2\pi m^2\sigma_1^3\tau^2}\ln\left(\frac{\sigma_1+1}{\sigma_1-1}\right) \\ &\quad + O(1)\cdot\frac{\mu\ e^2}{\pi m^2\sigma_1^4\tau^2}\frac{1}{1-\frac{1}{\sigma_1^2}} \\ &\sim O(1)\cdot\frac{2\mu\ e^2}{\pi m^2\sigma_1^4\tau^2} \quad . \end{aligned}$$

It is curious that $\langle \Delta v_z{}^2\rangle_{MS}$ does not contain any singular integral when $\tau < 2z$ although the Lorentzian switching tails take part in $\langle \Delta v_z{}^2\rangle_{MS}$. The reason may be that this sector contains the description of cancellations between the pre- and post-measurement tails (the S_2 region).

Leaving only the most dominant terms, we get the estimation

$$\langle \Delta {v_z}^2 \rangle \approx \frac{e^2}{\pi^2 m^2 \sigma_1^4 \tau^2} \quad (\text{for } \mu\sigma_1 < 1) \;, \qquad (41)$$

$$\approx O(1) \cdot \frac{2\mu\, e^2}{\pi m^2 \sigma_1^4 \tau^2} \quad (\text{for } \mu\sigma_1 > 1) \;, \qquad (42)$$

where Eq.(41) comes from the M-region and coincides with Eq.(15), while Eq.(42) comes from the S_1-, S_2- and MS-regions.

VI. SUMMARY AND DISCUSSIONS

In the present paper, the effect of a switching process upon the measurement of the Brownian motion of a charged test particle near a perfect reflecting boundary has been investigated.

We have started with the fact that the late-time asymptotic behavior of $\langle \Delta {v_z}^2 \rangle$ does not depend on the measuring time τ_1 but only on the distance to the boundary, z, when the sudden-switching is employed (Eq.(8)) [2]. The τ_1-independence suggests that effective cancellations should be taking place during the measuring process and, if so, the result should be sensitive to the switching-tails at the edge of the measuring part. This is a natural reasoning considering the highly fluctuating nature of the vacuum. In the measurement of a normal system, the switching effect is likely to be ignored if the measuring time-scale is much larger than the switching time-scale. When dealing with the quantum vacuum like the present case, however, the highly fluctuating vacuum might cause the cancellations during the main measuring process so that the switching effect can also be an important ingredient.

Next we have proceeded to calculate the velocity dispersion with a Lorentzian function, which represents a pure smooth switching-process without a flat measuring part. We have then shown that the result is very different from the sudden-switching case (Eq.(16)).

Finally constructing the Lorentz-plateau switching function, we have shown that the result is very sensitive to the switching tails in probing the quantum vacuum. We have also derived a reasonable criteria for the switching to be regarded as "sudden" or "smooth". Only the condition that the switching time-scale τ_2 is much smaller than the measuring time-scale τ_1 is not enough for the switching to be qualified as "sudden". As clarified in III (the case (i) there), the criteria for the validity of the sudden-switching approximation should be $\tau_2 \ll z$ as well as $\tau_2 \ll \tau_1$, where z is interpreted as the traveling time of a signal from the particle to the reflecting boundary.

The above criteria, however, may not be easy to be satisfied so that the sudden-switching approximation should be taken care more carefully when we consider an actual procedure of measurement. We can imagine an example for measuring $\langle \Delta {v_z}^2 \rangle$ as follows. Assume a wide conducting plate of a square shape (the edge size L) is fixed in the vacuum. For clarity of the argument, let us introduce the standard (x, y, z)-coordinates in such a way as the plate is contained in the x-y plane with the original point O being at the center of the plate. From a distant point $P(-A, 0, z)$ ($A \gg L$ and $z \ll L$), a charged particle is shot parallel to the plate with an incident velocity $\vec{v}_0 = (v_0, 0, 0)$. The particle initially goes in the empty space far away from the plate and then passes near the edge of the plate, and finally enters into the region bounded by the plate. The smooth switching function discussed in the paper would be interpreted as a mathematical description of this process of shooting a probe from a far distance. In this situation, the switching time scale is around z/v_0 and the intrinsic time-scale determined by the system configuration is z. It is obvious that the switching time-scale can not be smaller than the intrinsic time scale z in this case, since $v_0 < 1$. This is just one example, but it at least shows that the sudden-switching approximation is not valid all the time and that we should be more careful about the switching effect in dealing with the quantum vacuum.

In view of the above example, it might also be possible to look at the switching function from a different angle, i.e. as a mathematical description of what the test particle would experience when the vacuum shifts from the Minkowski vacuum to a Casimir-like vacuum. Based on this interpretation, it is not surprising to see $\langle \Delta {v_z}^2 \rangle$ remain constant in the late-time, depending only on z for the sudden switching case. For, it is interpreted as the sudden energy shift due to the sudden change of vacuum state. In the case of the pure Lorentzian switching-function, on the other hand, the corresponding interpretation is that the vacuum changes smoothly from the asymptotic Minkowski vacuum to the Casimir-like vacuum, going back to the asymptotic Minkowski vacuum again. Then the test particle is never stabilized in this varying vacuum so that the result is naturally so different from the sudden switching case. Then the setup using Lorentz-Plateau function in this connection would be interpreted as describing a smooth transient process from the Minkowski vacuum to the Casimir-like vacuum. Thus it is expected that the Lorentz-Plateau function constructed in the present paper might be very useful to analyze the situations such as a smooth transient from one vacuum to another.

Finally it is appropriate to make some comments on the singular integrals and their regularization procedure. Tracing back the origin of the singular integral, it comes from the singularity at $T = 2z$ in the integral kernel (Eq.(23) or Eq.(4)). This singularity is understood as produced by the reflecting boundary. Due to the mirror-reflections of signals with the light-velocity, the values of the electric field at the two world-points (t', x, y, z) and (t'', x, y, z) are expected to be strongly correlated when $T = t' - t'' = 2z$. These correlations accumulate in the velocity fluctuations of the particle at z when the measurement time τ is longer than the travel time $2z$ for the signals. It is natural, thus, to expect that the resulting singular terms of the form A/ρ ($A > 0$ and $\rho \longrightarrow 0$) contain the information on the reflecting boundary. However the standard regularization procedure[5] corresponds to discarding these singular terms in effect. It should be clarified when this type of regularization is valid and when not.

With the above physical interpretation of the singular integrals, another natural way of regularization should be possible. It has been assumed that the probe particle and the reflecting boundary or the mirror are treated as classical objects. However in reality they also cannot escape quantum fluctuations. Taking into account their quantum fluctuations, the effective path-lengths of signals are expected to be vague. It is estimated that the quantum fluctuations of the probe particle are more significant than those of the mirror. It is natural to assume the effective size of the particle to be of the order of its Compton length $\lambda_c = 1/m$, corresponding to setting the infinitesimal parameter ρ to be $\rho = \lambda_c/\tau = \frac{1}{m\tau}$. Just for an illustration, let us consider the case of an electron ($\lambda_c \sim 10^{-10}$ cm) with $\tau = 1$ μsec. Then $\rho \sim 10^{-15}$. It turns out that, thus, the singular terms would all the time dominate in the velocity fluctuations. Since the results could be drastically influenced, it should also be clarified whether the cut-off type of regularization is valid.

Akcnowledgement

We would like to thank L. H. Ford for various helpful comments on many aspects on the present research.

APPENDIX A: GENERAL FEATURES OF THE INTEGRAL WITH A LORENTZ-PLATEAU SWITCHING FUNCTION

We here analyze general properties of the integral given in Eq.(21) or Eq.(22).

We note that the x-y plane is divided into 9 integral regions by 4 border lines, $x+y = \pm 1$ and $y-x = \pm 1$, and the 9 regions are further classified into 4 classes, M, S_1, S_2 and MS, as discussed after Eq.(22). In each integral region, the y-integral can be done independently of the kernel $\mathcal{K}$, leaving the x-integral. Now we shall investigate each of 4 types of integral regions one by one.

(i) M-Region : The region defined by $|x + y| \leq 1$ and $|x - y| \leq 1$.
It coincides with the sudden-switching case considered in Ref.[2]. The integral $\mathcal{I}^{(M)}$, coming from this region, is computed as

$$\mathcal{I}^{(M)} = \frac{\tau^2}{2}\left(\int_{-1}^{0} dx \int_{-x-1}^{x+1} dy + \int_{0}^{1} dx \int_{x-1}^{-x+1} dy\right)\mathcal{K}(\tau x)$$
$$= 2\tau^2 \int_0^1 dx\ (1-x)\mathcal{K}(\tau x) \quad , \tag{A1}$$

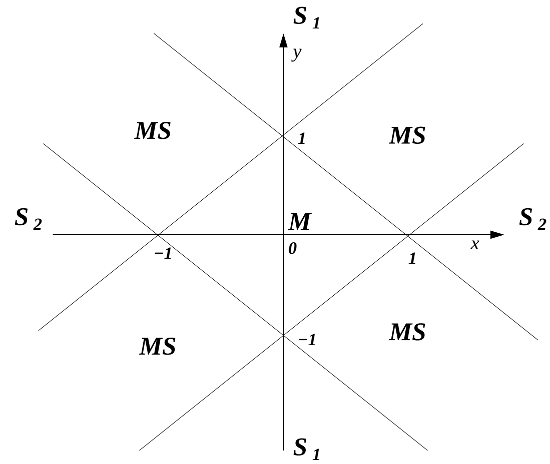

FIG. 2: Illustraion of four types of integral regions.

where the last line follows using the even function property of $\mathcal{K}$.

(ii) MS-Regions : The 4 regions defined by '$x + y \geq 1$ and $|y - x| \leq 1$', '$|x + y| \leq 1$ and $y - x \geq 1$, '$x + y \leq -1$ and $|y - x| \leq 1$, and '$|x + y| \leq 1$ and $y - x \leq -1$'.

For illustration, let us focus on the region '$x+y \geq 1$ and $|y - x| \leq 1$'. The integral $\mathcal{I}^{(MS)}$, coming from this region, is computed as

$$\mathcal{I}^{(MS)} = \frac{\tau^2}{2}\left(\int_0^1 dx \int_{-x+1}^{x+1} dy + \int_1^{\infty} dx \int_{x-1}^{x+1} dy\right)\cdot$$
$$\cdot \frac{\mu^2}{\left(\frac{x+y}{2} - \frac{1}{2}\right)^2 + \mu^2}\mathcal{K}(\tau x)$$
$$= 4\tau^2\mu \int_0^{\infty} dx\ \left(\mathcal{K}(\tau x) - \mathcal{K}(\tau(x+1)\right) \times$$
$$\times \tan^{-1}\frac{x}{\mu} \quad , \tag{A2}$$

where the even function property of $\mathcal{K}$ has been employed to get the result. It turns out that each of the 4 regions yield exactly the same contribution $\mathcal{I}^{(MS)}$ given by Eq.(A2).

(iii) S_1-Regions : The 2 regions defined by '$x + y \geq 1$ and $y - x \geq 1$' and '$x + y \leq -1$ and $y - x \leq -1$'.
By performing the y-integral and using the even function property of $\mathcal{K}$, it turns out that these 2 regions yield the same contribution,

$$\mathcal{I}^{(S_1)} = 2\tau^2\mu^3 \int_0^{\infty} dx\ \frac{\mathcal{K}(\tau x)}{x^2 + 4\mu^2} \times$$
$$\times \left\{\pi - \tan^{-1}\frac{x}{\mu} - \frac{\mu}{x}\ln\left(1 + \frac{x^2}{\mu^2}\right)\right\}. \tag{A3}$$

(iv) S_2-*Regions : The two regions defined by '$x+y\geq 1$ and $y-x\leq -1$' and '$x+y\leq -1$ and $y-x\geq 1$'.* By performing the y-integral and using the even function property of $\mathcal{K}$, it turns out that these 2 regions yield exactly the same contribution,

$$\mathcal{I}^{(S_2)} = 2\tau^2\mu^3\int_0^\infty dx\,\frac{\mathcal{K}(\tau(x+1))}{x^2+4\mu^2}\times \times\left\{\tan^{-1}\frac{x}{\mu}+\frac{\mu}{x}\ln\left(1+\frac{x^2}{\mu^2}\right)\right\}. \quad \text{(A4)}$$

Gathering the results Eqs.(A1)-(A4), we get

$$\begin{aligned}\mathcal{I} &= \mathcal{I}^{(M)}+4\mathcal{I}^{(MS)}+2\mathcal{I}^{(S_1)}+2\mathcal{I}^{(S_2)}\\ &= 2\tau^2\int_0^1 dx\,(1-x)\mathcal{K}(\tau x)+4\pi\mu^2\tau^2\int_0^\infty d\chi\,\frac{\mathcal{K}(\mu\tau\chi)}{\chi^2+4}\\ &\quad +4\mu^2\tau^2\int_0^\infty d\chi\,\left\{\mathcal{K}(\mu\tau\chi)-\mathcal{K}(\mu\tau(\chi+1/\mu)\right\}\mathcal{F}(\chi)\\ &=: \mathcal{I}_M+\mathcal{I}_S+\mathcal{I}_{MS} \quad , \qquad \text{(A5)}\end{aligned}$$

with

$$\mathcal{F}(\chi) := \left(1-\frac{1}{\chi^2+4}\right)\tan^{-1}\chi-\frac{1}{\chi(\chi^2+4)}\ln(1+\chi^2) \quad . \qquad \text{(A6)}$$

Here a variable $\chi := x/\mu$ has been introduced in $\mathcal{I}_S$ and $\mathcal{I}_{MS}$ (see Eq.(23) for definitions of variables and parameters).

The expression Eq.(A5) reveals several general properties of the integral representation Eq.(21).

First of all, the two limiting cases of $\mathcal{I}$, $\mu\to 0$ and $\mu\to\infty$, can be easily obtained (let us once again recall μ is the switching-duration parameter defined in Eq.(20)): On the one hand, we see

$$\mathcal{I}\longrightarrow \quad \mathcal{I}_M = 2\tau^2\int_0^1 dx\,(1-x)\mathcal{K}(\tau x)$$

as $\mu\to 0$ with a fixed τ (or equivalently $\tau_2\to 0$ with a fixed τ). This limiting expression is a general formula corresponding to Eq.(3), so that the sudden-switching limit ($\mu\to 0$ with a fixed τ) is well-defined in general.

On the other hand, expressing Eq.(A5) in terms of τ_2 instead of τ, it follows

$$\mathcal{I}\longrightarrow \quad \mathcal{I}_S = \frac{2\tau_2^2}{\pi^2}\int_{-\infty}^\infty d\xi\frac{\mathcal{K}(\tau_2\xi)}{\xi^2+4/\pi^2}$$

as $\tau\to 0$ with a fixed τ_2 (or equivalently $\tau\to 0$ with a fixed $\mu\tau$). This limiting expression is equivalent to the result obtained from Eq.(21) with $F_{\tau\mu}(t)$ being replaced by the following Lorentzian function

$$f(t) = \frac{1}{\pi^2}\frac{\tau_2^2}{t^2+(\tau_2/\pi)^2} \quad ,$$

which is nothing but the limiting function of $F_{\tau\mu}(t)$ as $\tau\to 0$ with a fixed μ (or equivalently $\tau\to 0$ with a fixed τ_2). Thus the Lorentzian limit is also well-defined in general.

In this manner, the switching function $F_{\mu\tau}(t)$ smoothly bridges the gap between the step-function and the Lorentz-function and is expected to be quite useful for investigating various switching effects in quantum vacuum.

Next, let us assume $\mathcal{K}(T)>0$ for the sake of later application. It is easy to see that $\mathcal{F}(\chi)$ is a monotonically increasing function with $\mathcal{F}(0)=0$ and $\lim_{\chi\to\infty}\mathcal{F}(\chi)=\frac{\pi}{2}$. Then one can estimate

$$\mathcal{I}_{MS}\sim O(1)\cdot 2\pi\mu^2\tau^2\int_0^{1/\mu}d\chi\,\mathcal{K}(\mu\tau\chi) \quad .$$

Since $\mathcal{I}_M$ and $\mathcal{I}_{MS}$ are estimated by integrals with a compact integral region, they may or may not be singular integrals depending on whether the pole of $\mathcal{K}(T)$ is included inside their integral regions. In our model, the case $\tau>2z$ makes them singular while the case $\tau<2z$ non-singular. On the other hand, $\mathcal{I}_S$ is always a singular integral due to the long-tail nature of the Lorentzian function. When integrals are singular due to the pole of the kernel $\mathcal{K}(T)$, some regularization procedure can enter the analysis. The regularization employed in the present context[5] (following Ref.[2]) is in effect to throw away an infinitely large positive terms. (See arguments after Eq.(7).) Thus the apparent positive quantity can become negative after regularization. We encounter this situation in Sec. III B. In the case discussed in Sec. III B, the integrals $\mathcal{I}_M$ and $\mathcal{I}_S$ remain positive, while $\mathcal{I}_{MS}$ becomes negative after regularization. What happens then is that, as the switching time-scale becomes longer, either $\mathcal{I}_S$ or $\mathcal{I}_{MS}$ dominates depending on the choice of the time-scale parameters. When $\mathcal{I}_S$ dominates, the situation is close to the Lorentzian smearing case (Eqs.(11)-(16)). When the negative term $\mathcal{I}_S$ dominates, on the other hand, the velocity dispersion in z-direction becomes negative.

APPENDIX B: FORMULAS FOR SINGULAR INTEGRALS

Here we present the formulas for particular singular integrals needed in our analysis:

$$\begin{aligned}\mathcal{I}(\sigma,\xi:2): &= \Re\int_{\curvearrowleft_\rho(\sigma)}\frac{1-\xi z}{(z^2-\sigma^2)^2}dz\\ &= \frac{1-\xi\sigma}{2\sigma^2\rho}+O(\rho) \qquad (\sigma,\xi\in\mathbf{R}) \quad , \quad \text{(B1)}\\ \mathcal{I}(\sigma,\xi:3): &= \Re\int_{\curvearrowleft_\rho(\sigma)}\frac{1-\xi z}{(z^2-\sigma^2)^3}dz\\ &= -\frac{3-\xi\sigma}{8\sigma^4\rho}+O(\rho) \qquad (\sigma,\xi\in\mathbf{R}) \ . \quad \text{(B2)}\end{aligned}$$

Here $\curvearrowleft_\rho(\sigma)$ indicates a semicircle (with an anti-clockwise direction) in the upper-plane of z with its radius being ρ (>0) and its center located at $z=\sigma$. More precisely,

$\curvearrowleft_\rho(\sigma) := \{z \in \mathbf{C} | z = \sigma + \rho\, e^{i\theta},\ \theta \in [0,\pi]\}$ with its direction matched with increasing θ.

We here derive only the formula Eq.(B1). One can derive Eq.(B2) in the same manner.

Now setting $z = \sigma + \rho\, e^{i\theta}$, it is straightforward to see that

$$\mathcal{I}(\sigma, \xi : 2) = -\frac{1}{\rho} \int_0^\pi \frac{\Im \mathcal{A}(\theta)}{\mathcal{D}^2(\theta)}\, d\theta \quad , \tag{B3}$$

where $\mathcal{A}(\theta) := (2\sigma + \rho e^{-i\theta})^2((1-\xi\sigma)e^{-i\theta} - \xi\rho)$ and $\mathcal{D}(\theta) := 4\sigma^2 + \rho^2 + 4\sigma\rho\cos\theta$.

Due to the relation, $\cos\theta = \{\mathcal{D}(\theta) - (4\sigma^2 + \rho^2)\}/4\sigma\rho$, the imaginary part of the function $\mathcal{A}(\theta)$ can be expressed in powers of $\mathcal{D}$:

$$\Im \mathcal{A}(\theta) = -(p\, \mathcal{D}(\theta)^2 - q\, \mathcal{D}(\theta) + r)\sin\theta \quad ,$$

where

$$p := \frac{\beta}{4\sigma^2} \quad , \quad q := \frac{\rho^2}{2\sigma^2} \quad ,$$
$$r := \beta\left(\frac{\rho^4}{4\sigma^2} - \rho^2\right) + \xi\left(\frac{\rho^4}{2\sigma} - 2\sigma\rho^2\right) \quad ,$$

with $\beta := 1 - \xi\sigma$. Thus

$$\begin{aligned} \rho\, \mathcal{I}(\sigma, \xi : 2) &= \int_0^\pi \left(p - \frac{q}{\mathcal{D}(\theta)} + \frac{r}{\mathcal{D}(\theta)^2}\right)\sin\theta d\theta \\ &= 2p - \frac{q}{4\sigma\rho}\ln\left(\frac{1+\frac{\rho}{2\sigma}}{1-\frac{\rho}{2\sigma}}\right)^2 + \frac{r}{8\sigma^4}\left(1 - \frac{\rho^2}{4\sigma^2}\right)^{-2} \\ &= \frac{\beta}{2\sigma^2} + O(\rho^2) \quad , \end{aligned} \tag{B4}$$

yielding the formula Eq.(B1).

We note that the results remain the same even when $\curvearrowleft_\rho(\sigma)$ is replaced by $\{\curvearrowleft_\rho(\sigma)\}^*$ in Eqs.(B1) and (B2). Here $\{\curvearrowleft_\rho(\sigma)\}^*$ denotes the complex conjugate of the curve $\curvearrowleft_\rho(\sigma)$, i.e. $\{\curvearrowleft_\rho(\sigma)\}^* := \{z \in \mathbf{C} | z = \sigma + \rho\, e^{-i\theta},\ \theta \in [0,\pi]\}$ with its direction matched with increasing θ. This is obvious since integrals in Eqs.(B1) and (B2) are computed from the real part of the integrands.

One can also check the above claim by the following consideration. Let $f(z)$ be any function which has an isolated pole at $z = \sigma$ with its residue, $\mathrm{Res}(f, \sigma)$, being real. Let $C_\rho(\sigma)$ be a circle (with the anti-clockwise direction) of radius ρ with its center at $z = \sigma$. When ρ is chosen to be sufficiently small, then,

$$\int_{C_\rho(\sigma)} f = \int_{\curvearrowleft_\rho(\sigma)} - \int_{\{\curvearrowleft_\rho(\sigma)\}^*} = 2\pi i\, \mathrm{Res}(f, \sigma) \quad .$$

Considering the real part of this equation, we get

$$\Re \int_{\curvearrowleft_\rho(\sigma)} = \Re \int_{\{\curvearrowleft_\rho(\sigma)\}^*} \quad .$$

Thus the claim is confirmed once again by choosing $f(z) = \frac{1-\xi z}{(z^2-\sigma^2)^2}$ or $f(z) = \frac{1-\xi z}{(z^2-\sigma^2)^3}$.

APPENDIX C: ASYMPTOTIC PRINCIPAL VALUES OF SINGULAR INTEGRALS

Here we introduce a special treatment for a singular integral, represented by a symbol $\wp_{(\rho)}$. Let $f(x)$ be a real function which is possibly singular at $x = \sigma$. Now, for a sufficiently small $\rho\ (>0)$, we define

$$\wp_{(\rho)} \int_A^B f(x)dx := \left(\int_A^{\sigma-\rho} + \int_{\sigma+\rho}^B\right) f(x)dx \quad , \tag{C1}$$

where $A < \sigma < B$. Conventionally, if the R.H.S. of Eq.(C1) converges as $\rho \to 0$, the value of convergence is called the principal value of the integral $\int f(x)dx$, denoted by $p.v. \int f(x)dx$. We here have generalized the concept of the principal value and have left the positive small quantity ρ as a free parameter. Let us call the above $\wp_{(\rho)} \int f(x)dx$ *an asymptotic principal value of order* ρ. Note that $\wp_{(\rho)} \int f(x)dx$ need not necessarily to converge as $\rho \to 0$. For neatness, we shall write it just $\wp \int f(x)dx$ from now on.

We now show the following integral formula, needed in our analysis: For $0 < \sigma < 1$,

$$\wp \int_0^1 dx\, \frac{1-x}{(x^2-\sigma^2)^2} = \frac{1}{8\sigma^3}\ln\left(\frac{1+\sigma}{1-\sigma}\right)^2 + \frac{1-\sigma}{2\sigma^2\rho} + O(\rho) \quad . \tag{C2}$$

To show this formula, set $f(z) := \frac{1-z}{(z^2-\sigma^2)^2}$. Circumventing $z = \sigma$ from above, we enclose the contour in the complex plane to get

$$\wp \int_0^1 + \int_{\curvearrowleft_\rho(\sigma)} + \int_1^{1+iR} + \int_{1+iR}^{iR} + \int_{iR}^0 = 0 \quad .$$

Here the path $\curvearrowleft_\rho(\sigma)$ is the same as in *Appendix* B; the paths for $\int_1^{1+iR}$ and $\int_{iR}^0$ run parallel to the imaginary axis, while the path for $\int_{1+iR}^{iR}$ runs parallel to the real axis. Taking the limit $R \longrightarrow \infty$, then, the real part of the above equation yields

$$\wp \int_0^1 = \Re \int_0^{i\infty} - \Re \int_1^{1+i\infty} + \mathcal{I}(\sigma, 1 : 2) \quad ,$$

where the last term is given by Eq.(B1) with $\xi = 1$. It is now straightforward to compute the first two integrals on the R.H.S., yielding the formula Eq.(C2). By the similar arguments as in *Appendix* B, the same formula Eq.(C2) is obtained even though we choose a contour circumventing $z = \sigma$ from below.

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