

Weak-Pseudo-Hermiticity of Non-Hermitian Hamiltonians with Position-Dependent Mass

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Abstract

We extend the definition of η -weak-pseudo-Hermiticity to the class of potentials endowed with position-dependent mass. The construction of non-Hermitian Hamiltonians through some generating function are obtained. Special cases of potentials are thus deduced.

Keywords : η -weak-pseudo-Hermiticity; Non-Hermitian Hamiltonians;
 \mathcal{PT} -symmetry; Effective mass.

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1 Introduction

The Hamiltonians are called \mathcal{PT} -invariant if they are invariant under a joint transformation of parity \mathcal{P} and time-reversal \mathcal{T} [1-8]. A conjecture due to Bender and Boettcher [1] has relaxed \mathcal{PT} -symmetry as a necessary condition for the reality of the spectrum. Here, the Hermiticity assumption $\mathcal{H} = \mathcal{H}^\dagger$ is replaced by the \mathcal{PT} -symmetric one; i.e. $[\mathcal{PT}, \mathcal{H}] = 0$, where \mathcal{P} denotes the parity operator (space reflection) and has as effects : $x \rightarrow -x$, $p \rightarrow -p$ and \mathcal{T} mimics the time-reversal and has as effects : $x \rightarrow x$, $p \rightarrow -p$, and $i \rightarrow -i$. Note that \mathcal{T} changes the sign of i because it preserves the fundamental commutation relation of the quantum mechanics known as the Heisenberg algebra, i.e. $[x, p] = i\hbar$ [1-3].

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According to Mostafazadeh [9-12], the basic mathematical structure underlying the properties of \mathcal{PT} -symmetry is explored and can now be found to be connected to the concept of a pseudo-Hermiticity. The pseudo-Hermiticity has been found to be a more general concept than those of Hermiticity and \mathcal{PT} -symmetry. As a consequence of this, the reality of the bound-state eigenvalues can be associated with it.

In terms of these settings, a Hamiltonian \mathcal{H} is called pseudo-Hermitian if it obeys to [9,11]

$$\mathcal{H}^\dagger = \eta \mathcal{H} \eta^{-1}, \quad (1)$$

where η is a Hermitian invertible linear operator and a dagger (\dagger) stands for the adjoint of the corresponding operator. A non-Hermitian Hamiltonian has a real spectrum if and only if it is pseudo-Hermitian with respect to a linear Hermitian automorphism [10], and may be factored as

$$\eta = \mathcal{D}^\dagger \mathcal{D}, \quad (2)$$

where $\mathcal{D} : \mathfrak{H} \rightarrow \mathfrak{H}$ is a linear automorphism (\mathfrak{H} is the Hilbert space). Note that choosing $\eta = 1$ reduces the assumption (1) to the Hermiticity of the Hamiltonian.

On the other hand, Bagchi and Quesne [13] have established that the twin concepts of pseudo-Hermiticity and weak-pseudo-Hermiticity are complementary to one another. In the pseudo-Hermiticity case, η can be written as a first-order differential operator and may be anti-Hermitian, while in the weak-pseudo-Hermitian case, η is a second-order differential operator and must be necessarily Hermitian.

The quantum mechanical systems with position-dependent mass have attracted, in recent years, much attention on behalf of physicists [15-20]. The effective mass Schrödinger equation was first introduced by BenDaniel and Duke in order to explain the behaviors of electrons in semi-conductors [15]. It also have many applications in the fields of materials science and condensed matter physics [20,21].

In the present paper, a class of non-Hermitian Hamiltonians, known in the literature, as well as their accompanying ground-state wavefunctions are generated as a by-product of the generalized η -weak-pseudo-Hermiticity endowed with position-dependent mass. Here our primary

concern is to point out that, being different from the realization of Ref.[13] considering therein $A(x)$ as a pure imaginary function, there is no inconsistency if a shift on the momentum p of the type $p \rightarrow p - \frac{A(x)}{U(x)}$ is used, where $A(x)$ and $U(x) (\neq 0)$ are, respectively, complex- and real-valued functions. It opens a way towards the construction of non-Hermitian Hamiltonians (not necessarily \mathcal{PT} -symmetric). On these settings, Eq.(2) becomes $\eta \rightarrow \tilde{\eta} = \tilde{\mathcal{D}}^\dagger \tilde{\mathcal{D}}$. Such operator, i.e. $\tilde{\mathcal{D}}$, may be looked upon as a gauge-transformed version of \mathcal{D} , depending essentially on the function $A(x)$. Consequently, it is found that the wavefunction is also subjected to a gauge transformation of the type $\psi(x) \rightarrow \xi(x) = \Lambda(x) \psi(x)$ where $\Lambda(x) = \exp \left[i \int^x dy \frac{A(y)}{U(y)} \right]$.

2 Generalized pseudo-Hermitian Hamiltonians

The general form of the Hamiltonian introduced by von Roos [16] for the spatially varying mass $M(x) = m_0 m(x)$ reads

$$\mathcal{H} = \frac{1}{4} \left[m^\alpha(x) p m^\beta(x) p m^\gamma(x) + m^\gamma(x) p m^\beta(x) p m^\alpha(x) \right] + V(x), \quad (3)$$

where the constraint $\alpha + \beta + \gamma = -1$ holds and $V(x) = V_{\text{Re}}(x) + iV_{\text{Im}}(x)$ is a complex-valued potential. Here, $p (= -i\frac{d}{dx})$ is a momentum with $\hbar = m_0 = 1$, and $m(x)$ is dimensionless real-valued mass function.

Using the restricted Hamiltonian from the $\alpha = \gamma = 0$ and $\beta = -1$ constraints, the Hamiltonian (3) becomes

$$\mathcal{H} = pU^2(x)p + V(x), \quad (4)$$

with $U^2(x) = \frac{1}{2m(x)}$. The shift on the momentum p in the manner

$$p \rightarrow p - \frac{A(x)}{U(x)}, \quad (5)$$

where $A : \mathbb{R} \rightarrow \mathbb{C}$ is a complex-valued function, allows to bring the Hamiltonian of Eq.(4) in the form

$$\mathcal{H} \rightarrow \mathcal{H}' = \left[p - \frac{A(x)}{U(x)} \right] U^2(x) \left[p - \frac{A(x)}{U(x)} \right] + V(x). \quad (6)$$

In Ref.[11], it was showed that for every anti-pseudo-Hermitian Hamiltonian \mathcal{H} , there is an antilinear operator τ fulfilling the condition

$$\mathcal{H}^\dagger = \tau \mathcal{H} \tau^{-1}. \quad (7)$$

Let us extend the proof of Ref.[12] to our Hamiltonian (6). To this end, τ should be constructed suitably. According to Mostafazadeh [12], $\tau = \mathcal{T} e^{i\alpha(x)}$ is the product of linear and antilinear operators, and $\alpha : \mathbb{R} \rightarrow \mathbb{C}$ is a complex-valued function. Therefore, the Hermiticity of τ is established straightforwardly

$$\tau^\dagger = e^{-i\alpha^*(x)} \mathcal{T}^\dagger = e^{-i\alpha^*(x)} \mathcal{T} = \mathcal{T} e^{i\alpha(x)} = \tau, \quad (8)$$

where the identities $\mathcal{T}^\dagger = \mathcal{T}$ and $\mathcal{T} f(x) \mathcal{T} = f^*(x)$ are used and $f : \mathbb{R} \rightarrow \mathbb{C}$.

According to Mostafazadeh in Ref.[12], the function $\alpha(x)$ can be gen-

eralized to $\alpha(x) = -2 \int^x dy \frac{A(y)}{U(y)}$, therefore

$$\begin{aligned}
\tau \mathcal{H}' \tau^{-1} &= \mathcal{T} e^{i\alpha(x)} \left[p - \frac{A(x)}{U(x)} \right] U^2(x) \left[p - \frac{A(x)}{U(x)} \right] e^{-i\alpha(x)} \mathcal{T} \\
&\quad + \mathcal{T} e^{i\alpha(x)} V(x) e^{-i\alpha(x)} \mathcal{T} \\
&= \mathcal{T} \left[p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] e^{i\alpha(x)} U^2(x) e^{-i\alpha(x)} \left[p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] \mathcal{T} \\
&\quad + V^*(x) \\
&= \mathcal{T} \left[p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] U^2(x) \left[p - \frac{A(x)}{U(x)} - \partial_x \alpha \right] \mathcal{T} + V^*(x) \\
&= \mathcal{T} \left[p + \frac{A(x)}{U(x)} \right] U^2(x) \left[p + \frac{A(x)}{U(x)} \right] \mathcal{T} + V^*(x) \\
&= \left[-p + \frac{A^*(x)}{U(x)} \right] U^2(x) \left[-p + \frac{A^*(x)}{U(x)} \right] + V^*(x) \\
&= \left[p - \frac{A^*(x)}{U(x)} \right] U^2(x) \left[p - \frac{A^*(x)}{U(x)} \right] + V^*(x) \\
&= \mathcal{H}'^\dagger, \tag{9}
\end{aligned}$$

where for every differential function $\alpha(x)$, the following identity holds $e^{-i\alpha(x)} p e^{i\alpha(x)} = p + \partial_x \alpha(x)$ while the position x commutes with $e^{i\alpha(x)}$ and remains unaffected under a last transformation; i.e. $e^{-i\alpha(x)} x e^{i\alpha(x)} = x$. Here we note that for every function $f : \mathbb{R} \rightarrow \mathbb{C}$, the identity $\mathcal{T} f(x, p) \mathcal{T} = f^*(x, -p)$ is used.

In the other hand, and according to Ref.[11], it was checked that \mathcal{PT} -symmetry ($[\mathcal{PT}, \mathcal{H}] = 0$) and anti-pseudo-Hermiticity operator τ imply pseudo-Hermiticity of \mathcal{H} with the respect of a linear Hermitian automorphism $\eta : \mathfrak{H} \rightarrow \mathfrak{H}$ according to

$$\eta = \tau \mathcal{PT}, \tag{10}$$

and it turns out that the choice of η is not unique. As was made for τ , let us generalize η according to

$$\eta = \exp \left[2i \int^x dy \frac{A^*(y)}{U(y)} \right] \mathcal{P}, \tag{11}$$

then the Hermiticity of η is established straightforwardly

$$\begin{aligned}
\eta^\dagger &= \mathcal{P} \exp \left[-2i \int^x dy \frac{A(y)}{U(y)} \right] = \exp \left[-2i \int^{-x} dy \frac{A(y)}{U(y)} \right] \mathcal{P} \\
&= \exp \left[2i \int^{-x} d(-y) \frac{A(y)}{U(y)} \right] \mathcal{P} = \exp \left[2i \int^x dy \frac{A(-y)}{U(-y)} \right] \mathcal{P} \\
&= \exp \left[2 \int^x dy \frac{i \operatorname{Re} A(-y) - \operatorname{Im} A(-y)}{U(-y)} \right] \mathcal{P} \\
&= \exp \left[2 \int^x dy \frac{i \operatorname{Re} A(y) + \operatorname{Im} A(y)}{U(y)} \right] \mathcal{P} \\
&= \exp \left[2i \int^x dy \frac{\operatorname{Re} A(y) - i \operatorname{Im} A(y)}{U(y)} \right] \mathcal{P} \\
&= \exp \left[2i \int^x dy \frac{A^*(y)}{U(y)} \right] \mathcal{P} \\
&= \eta,
\end{aligned} \tag{12}$$

where we use $\mathcal{P}^\dagger = \mathcal{P}$ and, for every function $f : \mathbb{R} \rightarrow \mathbb{C}$, the following identity holds $\mathcal{P}f(x)\mathcal{P} = f(-x)$. In Eq.(12), the real and imaginary parts of $A(x)$ are, respectively, even and odd functions; i.e. $\operatorname{Re} A(-x) = \operatorname{Re} A(x)$, $\operatorname{Im} A(-x) = -\operatorname{Im} A(x)$ and $U(x)$ must be an even function, i.e. $U(x) = U(-x)$.

In summary, the \mathcal{PT} -symmetry and anti-pseudo-Hermiticity with respect to τ imply pseudo-Hermiticity with respect to $\tau\mathcal{PT}$ and which coincides with the η operator [11]. Therefore, it is obvious that the (weak-) pseudo-Hermiticity as defined in Eq.(10) adapts very well to the problems relating with position-dependent effective mass.

3 The generalized weak-pseudo-Hermiticity generators

As η is weak-pseudo-Hermitian, then the operators \mathcal{D} and \mathcal{D}^\dagger are connected to the first-order differential operator through [14]

$$\begin{aligned}\mathcal{D} &= U(x) \partial_x + \phi(x), \\ &= iU(x) p + \phi(x),\end{aligned}\tag{13.a}$$

$$\begin{aligned}\mathcal{D}^\dagger &= -\partial_x U(x) + \phi^*(x), \\ &= -ipU(x) + \phi^*(x),\end{aligned}\tag{13.b}$$

where we have used the abbreviation $\partial_x = \frac{d}{dx}$. Here $\phi : \mathbb{R} \rightarrow \mathbb{C}$ is a complex-valued function. It is obvious that the operator \mathcal{D} becomes, under transformation (5),

$$\begin{aligned}\tilde{\mathcal{D}} &= iU(x) \left[p - \frac{A(x)}{U(x)} \right] + \phi(x), \\ &= iU(x) p - iA(x) + \phi(x).\end{aligned}\tag{14}$$

Therefore, the operator $\tilde{\mathcal{D}}$ may be looked upon as a gauge-transformed version of \mathcal{D} , depending on $A(x)$ such that $\tilde{\mathcal{D}} = \mathcal{D} - iA(x)$. In terms of these, $\tilde{\eta}$ becomes

$$\begin{aligned}\tilde{\eta} &= \tilde{\mathcal{D}}^\dagger \tilde{\mathcal{D}} \\ &= [\mathcal{D}^\dagger + iA^*(x)] [\mathcal{D} - iA(x)] \\ &= \mathcal{D}^\dagger \mathcal{D} - i\mathcal{D}^\dagger A(x) + iA^*(x) \mathcal{D} + A^*(x) A(x),\end{aligned}\tag{15}$$

and taking into account that $\phi(x) = f(x) + ig(x)$ and $A(x) = a(x) + ib(x)$, (15) can be recast as

$$\begin{aligned}\tilde{\eta} &= \mathcal{D}^\dagger \mathcal{D} + 2iU(x) a(x) \partial_x + i[U(x) A(x)]' - i\phi^*(x) A(x) \\ &\quad + i\phi(x) A^*(x) + |A(x)|^2,\end{aligned}\tag{16}$$

where prime denotes derivative with respect to x . At this point, let us now

evaluate η appearing in Eq.(16) using Eq.(13), we obtain

$$\begin{aligned}\mathcal{D}^\dagger \mathcal{D} &= [-\partial_x U(x) + \phi(x)] [U(x) \partial_x + \phi(x)] \\ &= -U^2(x) \partial_x^2 - 2U(x) [U'(x) + ig(x)] \partial_x + |\phi(x)|^2 \\ &\quad - [U(x) \phi(x)]',\end{aligned}\tag{17}$$

Combining Eq.(17) with Eq.(16), we obtain a second-order differential operator of $\tilde{\eta}$

$$\tilde{\eta} = -U^2(x) \partial_x^2 - 2\mathcal{K}(x) \partial_x + \mathcal{L}(x),\tag{18}$$

where $\mathcal{K}(x)$ and $\mathcal{L}(x)$ are defined as

$$\mathcal{K}(x) = U(x) U'(x) + iU(x) g(x) - iU(x) a(x),\tag{19.a}$$

$$\begin{aligned}\mathcal{L}(x) &= |\phi(x)|^2 + |A(x)|^2 - [U(x) \phi(x)]' + i[U(x) A(x)]' \\ &\quad - i\phi^*(x) A(x) + i\phi(x) A^*(x).\end{aligned}\tag{19.b}$$

One can easily check that $\tilde{\eta}$ given in Eq.(18) is, indeed, Hermitian since it is written in the form $\tilde{\eta} = \tilde{\mathcal{D}}^\dagger \tilde{\mathcal{D}}$. On the other hand, taking into account $p = -i\partial_x$, the Hamiltonian of Eq.(6) may be expressed as

$$\mathcal{H}' = -U^2(x) \partial_x^2 - 2\mathcal{M}_1(x) \partial_x + \mathcal{N}_1(x) + V(x),\tag{20}$$

where, by definition

$$\mathcal{M}_1(x) = U(x) U'(x) - iU(x) A(x),\tag{21.a}$$

$$\mathcal{N}_1(x) = i[U(x) A(x)]' + A^2(x).\tag{21.b}$$

The adjoint of the Hamiltonian (20) reads as

$$\mathcal{H}'^\dagger = -U^2(x) \partial_x^2 - 2\mathcal{M}_2(x) \partial_x + \mathcal{N}_2(x) + V^*(x),\tag{22}$$

with

$$\mathcal{M}_2(x) = U(x) U'(x) - iU(x) A^*(x),\tag{23.a}$$

$$\mathcal{N}_2(x) = i[U(x) A^*(x)]' + A^{*2}(x).\tag{23.b}$$

It should be noted that \mathcal{D} and \mathcal{D}^\dagger are two intertwining operators, therefore, the defining condition (1) may be expressed as $\eta\mathcal{H} = \mathcal{H}^\dagger\eta$. Thereupon, a generalization beyond the pair $\tilde{\eta}$ and \mathcal{H}' is straightforward, given

$$\tilde{\eta}\mathcal{H}' = \mathcal{H}'^\dagger\tilde{\eta}. \quad (24)$$

Letting both sides of (24) act on every function, e.g. on a wavefunction. Using Eqs.(18), (20), (22) and comparing between their varying differential coefficients, we can easily recognized from the coefficients corresponding to the third derivative that $A(x)$ must be real function, i.e. $b(x) = 0$.

By comparing both coefficients corresponding to the second derivative, one deduces the expression connecting the potential to its conjugate through

$$V(x) = V^*(x) - 4iU(x)g'(x). \quad (25)$$

On the other hand, the coefficients corresponding to the first derivative give the shape of the potential

$$V^{*'}(x) = 2f(x)f'(x) - 2g(x)g'(x) - [U(x)f(x)]'' + 2i[U(x)g'(x)]', \quad (26)$$

and by integrating Eq.(26) taking into account its conjugate, we get

$$\begin{aligned} V(x) &\equiv V_{\text{Re}}(x) + iV_{\text{Im}}(x) \\ &= f^2(x) - g^2(x) - [U(x)f(x)]' - 2iU(x)g'(x) + \delta, \end{aligned} \quad (27)$$

with δ is a constant of integration. It is obvious that both imaginary parts of Eqs.(25) and (27) coincide.

The last remaining coefficients correspond to the null derivative and give the following pure-imaginary expression

$$\begin{aligned} &-4U(x)f(x)f'(x)g'(x) - 4U(x)f^2(x)g'(x) + 4U^2(x)f'(x)g'(x) \\ &+ 4U(x)U'(x)f'(x)g(x) + 4U(x)U'(x)f(x)g'(x) + 2U^2(x)f''(x)g(x) \\ &+ 3U^2(x)U'(x)g''(x) + 2U(x)U''(x)f(x)g(x) - U^2(x)U''(x)g'(x) \\ &- 2U(x)U'(x)U''(x)g(x) + U^3(x)g'''(x) - U^2(x)U'''(x)g(x) = 0. \end{aligned} \quad (28)$$

Using Eq.(24) together with the eigenvalues of the Schrödinger equation for the Hamiltonian and its adjoint, namely $\mathcal{H}' |\xi_i\rangle = \mathcal{E}'_i |\xi_i\rangle$ and $\langle \xi_j | \mathcal{H}'^\dagger = \langle \xi_j | \mathcal{E}'_j^*$, where $|\xi_q\rangle \in \mathfrak{H}$ ($q = i, j$), and then multiplying them by $\tilde{\eta}$ on the left- and right-hand sides, respectively, we can easily obtain due to Eq.(24), on subtracting, that any two eigenvectors $|\xi_i\rangle$ and $|\xi_j\rangle$ satisfy

$$\begin{aligned}
\langle \xi_j | (\mathcal{H}'^\dagger \tilde{\eta} - \tilde{\eta} \mathcal{H}') |\xi_i\rangle &= \langle \xi_j | (\mathcal{E}'_j^* \tilde{\eta} - \mathcal{E}'_i \tilde{\eta}) |\xi_i\rangle \\
&= (\mathcal{E}'_j^* - \mathcal{E}'_i) \langle \xi_j | \tilde{\eta} |\xi_i\rangle \\
&= (\mathcal{E}'_j^* - \mathcal{E}'_i) \langle \xi_j \| \xi_i \rangle_{\tilde{\eta}} \\
&\equiv 0,
\end{aligned} \tag{29}$$

where $\langle \xi_j \| \xi_i \rangle_{\tilde{\eta}} \equiv \langle \xi_j | \tilde{\eta} |\xi_i\rangle$ is the Hermitian indefinite inner product of the Hilbert space \mathfrak{H} defined by $\tilde{\eta}$ [9,11]. According to the proposition 2 in Ref.[9], a direct implication of Eq.(29) has the following properties

- (i) The eigenvectors with non-real eigenvalues have a vanishing η -norm, i.e. $\mathcal{E}'_i \notin \mathbb{R}$ implies that $\| |\xi_i\rangle \|_{\tilde{\eta}}^2 = \langle \xi_i \| \xi_i \rangle_{\tilde{\eta}} = 0$.
- (ii) Any two eigenvectors are η -orthogonal *unless* their eigenvalues are complex conjugates, i.e. $\mathcal{E}'_i \neq \mathcal{E}'_j^*$ implies that $\langle \xi_i \| \xi_j \rangle_{\tilde{\eta}} = 0$.

The inner product $\langle \cdot \| \cdot \rangle_{\tilde{\eta}}$ is generally positive-definite, i.e. $\langle \cdot \| \cdot \rangle_{\tilde{\eta}} > 0$. Thus, the Hilbert space equipped with this inner product may be identified as the physical Hilbert space $\mathfrak{H}_{\text{phys}}$ [1-3]. Therefore, according to Eq.(29), it is obvious that $\mathcal{E}' = \mathcal{E}'^*$. Hence, the eigenvalue \mathcal{E}' is real, i.e. $\mathcal{E}'_{\text{Im}} = 0$. In terms of these, η -orthogonality suggests that the eigenvector (wavefunction), here $\xi(x)$, is related to \mathcal{H}' through the identity $\tilde{\eta}\xi(x) = 0$ [14], i.e.

$$\tilde{\mathcal{D}}\xi(x) = 0, \tag{30}$$

and keeping in mind Eq.(14), and after integration, we obtain the ground-

state wavefunction (not necessarily normalizable)

$$\begin{aligned}
\xi(x) &= \Lambda(x) \psi(x) \\
&= \exp \left[i \int^x dy \frac{A(y)}{U(y)} \right] \psi(x) \\
&\propto \exp \left[- \int^x dy \frac{f(y)}{U(y)} - i \int^x dy \frac{g(y) - a(y)}{U(y)} \right], \quad (31)
\end{aligned}$$

where $\psi(x)$ is the ground-state wavefunction when the restriction $A(x) = 0$ holds. Then $\xi(x)$, as for $\tilde{\mathcal{D}}$, is also subjected to a gauge transformation in the manner of $\psi(x) \rightarrow \xi(x) = \Lambda(x) \psi(x)$.

In these settings, letting $\tilde{\mathcal{D}}$ acts on both sides of (31), we obtain

$$\begin{aligned}
\tilde{\mathcal{D}}\xi(x) &\equiv [U(x) \partial_x - iA(x) + \phi(x)] \Lambda(x) \psi(x) \\
&= U(x) \Lambda'(x) \psi(x) + U(x) \Lambda(x) \psi'(x) - iA(x) \Lambda(x) \psi(x) \\
&\quad + \phi(x) \Lambda(x) \psi(x) \\
&= \Lambda(x) [U(x) \partial_x + \phi(x)] \psi(x) \\
\implies \mathcal{D}\psi(x) &= 0, \quad (32)
\end{aligned}$$

where $\Lambda'(x) = i \frac{A(x)}{U(x)} \Lambda(x)$. That means that the wavefunctions thus obtained can be deduced either by $\tilde{\mathcal{D}}\xi(x) = 0$ or by $\mathcal{D}\psi(x) = 0$.

In the remainder of the article, we write \mathcal{E} instead of \mathcal{E}' . Now, using the Schrödinger equation $\mathcal{H}'\xi(x) = \mathcal{E}\xi(x)$, with \mathcal{H}' given in Eq.(20), $\xi(x)$ in Eq.(31) and $\mathcal{E} = \mathcal{E}_{\text{Re}} + i\mathcal{E}_{\text{Im}}$, we end up by relating $f(x)$ to $g(x)$ and $U(x)$ through

$$f(x) = \frac{U'(x)g(x) - U(x)g'(x)}{2g(x)}, \quad (33)$$

where for the sake of simplicity we consider $\delta \equiv \mathcal{E}_{\text{Re}}$. Hence, it becomes clear that $g(x)$ is our generating function leading to identify the function $f(x)$, and then the potential $V(x)$.

This in turn leads to the following question. Is (33) the equation connecting $f(x)$ to the generating function $g(x)$? The answer to this question amounts to check for the satisfaction of Eq.(28). It is then straightforward,

after a long calculation, to be convinced that $f(x)$, as defined in (33), is a farfetched function (solution).

In order to deal with position-dependent mass, we introduce the auxiliary function defined by the mapping $\mu(x) \equiv \int^x \frac{dy}{U(y)}$, where $\mu(x)$ is a dimensionless mass integral which will appear frequently in subsequent developments. The function $f(x)$ can be written as

$$f(x) = -\frac{g'(x)}{2\mu'(x)g(x)} - \frac{\mu''(x)}{2\mu'^2(x)}. \quad (34)$$

and the potential $V(x)$ acquires the form

$$\begin{aligned} V_{\text{eff}}(x) - \mathcal{E}_{\text{Re}} = & -g^2(x) - \frac{g'^2(x)}{4g^2(x)\mu'^2(x)} + \frac{g''(x)}{2g(x)\mu'^2(x)} - \frac{g'(x)\mu''(x)}{2g(x)\mu'^3(x)} \\ & - 2i\frac{g'(x)}{\mu'(x)}, \end{aligned} \quad (35)$$

where $V_{\text{eff}}(x)$ is called the effective potential and is related to $V(x)$ by

$$V(x) = V_{\text{eff}}(x) - \mathcal{V}_\mu(x), \quad (36)$$

with

$$\mathcal{V}_\mu(x) = \frac{\mu'''(x)}{\mu'^3(x)} - \frac{5\mu''^2(x)}{4\mu'^4(x)}. \quad (37)$$

4 Effective potentials and corresponding wavefunctions

The strategy to determine both effective potentials and ground-state wavefunctions is as follows. As $g(x)$ is a generating function, all expressions depend on it. We may choose various generating functions $g(x)$ and obtain all others expressions such as $f(x)$, $V_{\text{eff}}(x)$ and $\tilde{\eta}$. Knowing $f(x)$ and $g(x)$, the proper ground-state wavefunctions can be found from Eq.(32), i.e. without the gauge-term. Without giving the details of our calculation which are straightforward, we present the results of various expressions in standard form.

4.1 $3D$ –Harmonic oscillator potential

$$g(x) = \alpha\mu(x), \quad (38.a)$$

$$f(x) = -\frac{1}{2\mu(x)} - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (38.b)$$

$$V_{\text{HO}}(x) = -\alpha^2\mu^2(x) - \frac{1}{4\mu^2(x)} - 2i\alpha, \quad (38.c)$$

$$\psi_{\text{HO}}^{(0)}(x) \propto \frac{\sqrt{\mu(x)}}{U(x)} \exp\left[-\frac{i\alpha}{2}\mu^2(x)\right]. \quad (38.d)$$

4.2 Morse potential

$$g(x) = \exp[-\alpha\mu(x)], \quad (39.a)$$

$$f(x) = \frac{\alpha}{2} - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (39.b)$$

$$V_{\text{M}}(x) = -\exp[-2\alpha\mu(x)] + 2i\alpha \exp[-\alpha\mu(x)] + \frac{\alpha^2}{4}, \quad (39.c)$$

$$\psi_{\text{M}}^{(0)}(x) \propto \frac{1}{U(x)} e^{-\frac{\alpha}{2}\mu(x)} \exp\left[\frac{2i}{\alpha} e^{-\alpha\mu(x)}(x)\right] \quad (39.d)$$

4.3 Scarf II potential

$$g(x) = \operatorname{sech}[\alpha\mu(x)], \quad (40.a)$$

$$f(x) = \frac{\alpha}{2} \tanh[\alpha\mu(x)] - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (40.b)$$

$$V_{\text{Sc}}(x) = -\left(1 + \frac{3\alpha^2}{4}\right) \operatorname{sech}^2[\alpha\mu(x)] \\ + 2i\alpha \operatorname{sech}[\alpha\mu(x)] \tanh[\alpha\mu(x)] + \frac{\alpha^2}{4}, \quad (40.c)$$

$$\psi_{\text{Sc}}^{(0)}(x) \propto \frac{1}{U(x) \sqrt{\cosh[\alpha\mu(x)]}} \exp\left[-\frac{i}{\alpha} \arctan \tanh \frac{\alpha}{2}\mu(x)\right] \quad (40.d)$$

4.4 Generalized Pöschl-Teller potential

$$g(x) = \operatorname{cosech} [\alpha\mu(x)], \quad (41.a)$$

$$f(x) = \frac{\alpha}{2} \coth [\alpha\mu(x)] - \frac{\mu''(x)}{\mu'^2(x)}, \quad (41.b)$$

$$V_{\text{GPT}}(x) = - \left(1 - \frac{3\alpha^2}{4} \right) \operatorname{cosech}^2 [\alpha\mu(x)] \\ + 2i\alpha \operatorname{cosech} [\alpha\mu(x)] \coth [\alpha\mu(x)] + \frac{\alpha^2}{4}, \quad (41.c)$$

$$\psi_{\text{GPT}}^{(0)}(x) \propto \frac{1}{U(x) \sqrt{\sinh [\alpha\mu(x)]}} \tanh^{-\frac{2i}{\alpha}} \left[\frac{\alpha\mu(x)}{2} \right]. \quad (41.d)$$

4.5 Pöschl-Teller potential

$$g(x) = \operatorname{sech} [\alpha\mu(x)] \operatorname{cosech} [\alpha\mu(x)], \quad (42.a)$$

$$f(x) = \alpha \coth [2\alpha\mu(x)] - \frac{\mu''(x)}{2\mu'^2(x)}, \quad (42.b)$$

$$V_{\text{PT}}(x) = \left(\frac{3\alpha^2}{4} - 1 + 2i\alpha \right) \operatorname{cosech}^2 [\alpha\mu(x)] \\ - \left(\frac{3\alpha^2}{4} - 1 - 2i\alpha \right) \operatorname{sech}^2 [\alpha\mu(x)] + \alpha^2, \quad (42.c)$$

$$\psi_{\text{PT}}^{(0)}(x) \propto \frac{1}{U(x) \sqrt{\sinh [2\alpha\mu(x)]}} \tanh^{-\frac{2i}{\alpha}} [\alpha\mu(x)]. \quad (42.d)$$

The above models are displayed in their usual forms and give quite well-known exact solvable non-Hermitian effective potentials as well as their accompanying ground-state wavefunctions. The first one represents a generalized η -weak-pseudo-Hermitian $3D$ -harmonic oscillator. The second model corresponds to the non- \mathcal{PT} -symmetric Morse potential and is already obtained by [22,23], where the $\gamma = b_R$ constraint is considered therein, using $\mathfrak{sl}(2, \mathbb{C})$ potential algebra as a complex Lie algebra by a simple complexification of the coordinates in a group theoretical point of view

and also in [24], labelled LIII according to Lévai [25], once a substitution $b \rightarrow ib$ is made therein. The remainder models belong to so called PI class [25] which contains five individual potentials. The third model represents a generalized η -weak-pseudo-Hermitian \mathcal{PT} -symmetric Scarf II Potential, labelled PI_1 , which is established in [22,23,24] with the same constraints quoted above. Finally, the two last models represent, respectively, a generalized η -weak-pseudo-Hermitian generalized Pöschl-Teller (PI_2) and a generalized η -weak-pseudo-Hermitian Pöschl-Teller (PI_5) potentials and are already established, respectively, in [22,23,24] and [24].

5 Conclusion

A well-known class of non-Hermitian Hamiltonians endowed with position-dependent mass are generated as a by-product of a generalized η -weak-pseudo-Hermiticity thanks to a shift on the momentum p of the type $p \rightarrow p - \frac{A(x)}{U(x)}$, and which allows to avoid the Hermitian invertible linear operator η for the benefit of $\tilde{\eta}$. We show that, being different from the realization of Ref.[13], there is no inconsistency to generate a well-known class of non-Hermitian Hamiltonians if the last shift is used, leading then to consider that $\tilde{\mathcal{D}}$ may be looked upon as a gauge-transformed version of \mathcal{D} and depending essentially on the function $A(x)$, i.e. $\delta\mathcal{D} \equiv \tilde{\mathcal{D}} - \mathcal{D} = -iA(x)$. As a consequence of this, the wavefunction $\xi(x)$ is also subjected to a gauge transformation in the manner $\psi(x) \rightarrow \xi(x) = \Lambda(x)\psi(x)$, with $\Lambda(x) = \exp\left[i\int^x dy \frac{A(y)}{U(y)}\right]$ and where $\psi(x)$ is the ground-state wavefunction when the $A(x) = 0$ constraint holds.

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