Antisymmetric Tensor Fields, 4-Vector Fields, Indefinite Metrics and Normalization*

Valeri V. Dvoeglazov
Universidad de Zacatecas, Apartado Postal 636, Suc. UAZ
Zacatecas 98062, Zac., México
E-mail: valeri@planck.reduaz.mx,
URL: http://planck.reduaz.mx/˜ valeri/

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Abstract

On the basis of our recent modifications of the Dirac formalism we generalize the Bargmann-Wigner formalism for higher spins to be compatible with other formalisms for bosons. Relations with dual electrodynamics, with the Ogievetskii-Polubarinov notoph and the Weinberg 2(2J+1) theory are found. Next, we introduce the dual analogues of the Riemann tensor and derive corresponding dynamical equations in the Minkowski space. Relations with the Marques-Spehler chiral gravity theory are discussed. The problem of indefinite metrics, particularly, in quantization of 4-vector fields is clarified.

1 Introduction

The general scheme for derivation of higher-spin equations was given in [1]. A field of rest mass m and spin $j \geq \frac{1}{2}$ is represented by a completely symmetric multispinor of rank $2j$. The particular cases $j = 1$ and $j = \frac{3}{2}$ were given in the textbooks, e. g., ref. [2]. Generalized equations for higher spins can be derived from the first principles on using some modifications of the Bargmann-Wigner formalism. The generalizations of the equations in the

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$(1/2,0)\oplus(0,1/2)$ representation are well known. The Tokuoka-SenGupta-Fushchich formalism and the Barut formalism are based on the equations presented in refs. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

2 Generalized Spin-1 Case

We begin with

$$\left[i\gamma_\mu\partial_\mu + a - b\partial^2 + \gamma_5(c - d\partial^2)\right]_{\alpha\beta}\Psi_{\beta\gamma} = 0\,, \tag{1}$$

$$\left[i\gamma_\mu\partial_\mu + a - b\partial^2 - \gamma_5(c - d\partial^2)\right]_{\alpha\beta}\Psi_{\gamma\beta} = 0\,, \tag{2}$$

∂^2 is the d'Alembertian. Thus, we obtain the Proca-like equations:

$$\partial_\nu A_\lambda - \partial_\lambda A_\nu - 2(a + b\partial_\mu\partial_\mu)F_{\nu\lambda} = 0\,, \tag{3}$$

$$\partial_\mu F_{\mu\lambda} = \frac{1}{2}(a + b\partial_\mu\partial_\mu)A_\lambda + \frac{1}{2}(c + d\partial_\mu\partial_\mu)\tilde{A}_\lambda\,, \tag{4}$$

$\tilde{A}_\lambda$ is the axial-vector potential (analogous to that used in the Duffin-Kemmer set of equations for $J=0$). Additional constraints are:

$$i\partial_\lambda A_\lambda + (c + d\partial_\mu\partial_\mu)\tilde{\phi} = 0\,, \tag{5}$$

$$\epsilon_{\mu\lambda\kappa\tau}\partial_\mu F_{\lambda\kappa} = 0\,, (c + d\partial_\mu\partial_\mu)\phi = 0\,. \tag{6}$$

The spin-0 Duffin-Kemmer equations are:

$$(a + b\partial_\mu\partial_\mu)\phi = 0\,, i\partial_\mu\tilde{A}_\mu - (a + b\partial_\mu\partial_\mu)\tilde{\phi} = 0\,, \tag{7}$$

$$(a + b\,\partial_\mu\partial_\mu)\tilde{A}_\nu + (c + d\,\partial_\mu\partial_\mu)A_\nu + i(\partial_\nu\tilde{\phi}) = 0\,. \tag{8}$$

The additional constraints are:

$$\partial_\mu\phi = 0\,, \partial_\nu\tilde{A}_\lambda - \partial_\lambda\tilde{A}_\nu + 2(c + d\partial_\mu\partial_\mu)F_{\nu\lambda} = 0\,. \tag{9}$$

In such a way the spin states are *mixed* through the 4-vector potentials. After elimination of the 4-vector potentials we obtain the equation for the AST field of the second rank:

$$[\partial_\mu\partial_\nu F_{\nu\lambda} - \partial_\lambda\partial_\nu F_{\nu\mu}] + \\ + \left[(c^2 - a^2) - 2(ab - cd)\partial_\mu\partial_\mu + (d^2 - b^2)(\partial_\mu\partial_\mu)^2\right]F_{\mu\lambda} = 0\,, \tag{10}$$

which should be compared with our previous equations which follow from the Weinberg-like formulation [13, 14, 15]. Just put:

$$c^2 - a^2 \Rightarrow \frac{-Bm^2}{2}\,, \qquad c^2 - a^2 \Rightarrow +\frac{Bm^2}{2}\,, \tag{11}$$

$$-2(ab - cd) \Rightarrow \frac{A-1}{2}\,, \qquad +2(ab - cd) \Rightarrow \frac{A+1}{2}\,, \tag{12}$$

$$b = \pm d\,. \tag{13}$$

Of course, these sets of algebraic equations have solutions in terms A and B. We found them and restored the equations. The parity violation and the spin mixing are *intrinsic* possibilities of the Proca-like theories.

In fact, there are several modifications of the BW formalism. One can propose the following set:

$$\left[i\gamma_\mu \partial_\mu + \epsilon_1 m_1 + \epsilon_2 m_2 \gamma_5\right]_{\alpha\beta} \Psi_{\beta\gamma} = 0\,, \tag{14}$$

$$\left[i\gamma_\mu \partial_\mu + \epsilon_3 m_1 + \epsilon_4 m_2 \gamma_5\right]_{\alpha\beta} \Psi_{\gamma\beta} = 0\,, \tag{15}$$

where $\epsilon_i = i\partial_t/E$ are the sign operators. So, at first sight, we have 16 possible combinations for the AST fields. We first come to

$$\begin{aligned} &\left[i\gamma_\mu \partial_\mu + m_1 A_1 + m_2 A_2 \gamma_5\right]_{\alpha\beta} \left\{(\gamma_\lambda R)_{\beta\gamma} A_\lambda + (\sigma_{\lambda\kappa} R)_{\beta\gamma} F_{\lambda\kappa}\right\} + \\ + &\left[m_1 B_1 + m_2 B_2 \gamma_5\right]_{\alpha\beta} \left\{R_{\beta\gamma}\varphi + (\gamma_5 R)_{\beta\gamma}\tilde{\phi} + (\gamma_5\gamma_\lambda R)_{\beta\gamma}\tilde{A}_\lambda\right\} = 0\,, \end{aligned} \tag{16}$$

$$\begin{aligned} &\left[i\gamma_\mu \partial_\mu + m_1 A_1 + m_2 A_2 \gamma_5\right]_{\gamma\beta} \left\{(\gamma_\lambda R)_{\alpha\beta} A_\lambda + (\sigma_{\lambda\kappa} R)_{\alpha\beta} F_{\lambda\kappa}\right\} - \\ - &\left[m_1 B_1 + m_2 B_2 \gamma_5\right]_{\alpha\beta} \left\{R_{\alpha\beta}\varphi + (\gamma_5 R)_{\alpha\beta}\tilde{\phi} + (\gamma_5\gamma_\lambda R)_{\alpha\beta}\tilde{A}_\lambda\right\} = 0\,, \end{aligned} \tag{17}$$

where $A_1 = \frac{\epsilon_1+\epsilon_3}{2}$, $A_2 = \frac{\epsilon_2+\epsilon_4}{2}$, $B_1 = \frac{\epsilon_1-\epsilon_3}{2}$, and $B_2 = \frac{\epsilon_2-\epsilon_4}{2}$. Thus, for spin 1 we have

$$\partial_\mu A_\lambda - \partial_\lambda A_\mu + 2m_1 A_1 F_{\mu\lambda} + i m_2 A_2 \epsilon_{\alpha\beta\mu\lambda} F_{\alpha\beta} = 0\,, \tag{18}$$

$$\partial_\lambda F_{\kappa\lambda} - \frac{m_1}{2} A_1 A_\kappa - \frac{m_2}{2} B_2 \tilde{A}_\kappa = 0\,, \tag{19}$$

with constraints

$$-i\partial_\mu A_\mu + 2m_1 B_1 \phi + 2m_2 B_2 \tilde{\phi} = 0\,, \tag{20}$$

$$i\epsilon_{\mu\nu\kappa\lambda}\partial_\mu F_{\nu\kappa} - m_2 A_2 A_\lambda - m_1 B_1 \tilde{A}_\lambda = 0\,, \tag{21}$$

$$m_1 B_1 \tilde{\phi} + m_2 B_2 \phi = 0\,. \tag{22}$$

If we remove A_λ and $\tilde{A}_\lambda$ from this set, we come to the final results for the AST field. Actually, we have twelve equations, see [16]. One can go even further. One can use the Barut equations for the BW input. So, we can get 16×16 combinations (depending on the eigenvalues of the corresponding sign operators), and we have different eigenvalues of masses due to $\partial_\mu^2 = \kappa m^2$.

Why do I think that the shown arbitrarieness of equations for the AST fields is related to 1) spin basis rotations; 2) the choice of normalization? (see ref. [17]) In the common-used basis three 4-potentials have parity eigenvalues -1 and one time-like (or spin-0 state), $+1$; the fields $\mathbf{E}$ and $\mathbf{B}$ have also definite parity properties in this basis. If we transfer to other basis, e.g., to the helicity basis [18] we can see that the 4-vector potentials and the corresponding fields are superpositions of a vector and an axial-vector [19]. Of course, they can be expanded in the fields in the "old" basis.

The detailed discussion of the generalized spin-1 case (as well as the problems related to normalization, indefinite metric and 4-vector fields) can be found in refs. [16, 17, 23].

3 Generalized Spin-2 Case

The spin-2 case can also be of some interest because it is generally believed that the essential features of the gravitational field are obtained from transverse components of the $(2,0)\oplus(0,2)$ representation of the Lorentz group. Nevertheless, questions of the redandant components of the higher-spin relativistic equations are not yet understood in detail [20].

We begin with the commonly-accepted procedure for the derivation of higher-spin equations below. We begin with the equations for the 4-rank symmetric spinor:

$$[i\gamma^\mu\partial_\mu - m]_{\alpha\alpha'}\,\Psi_{\alpha'\beta\gamma\delta} = 0\,, [i\gamma^\mu\partial_\mu - m]_{\beta\beta'}\,\Psi_{\alpha\beta'\gamma\delta} = 0\,, \tag{23}$$

$$[i\gamma^\mu\partial_\mu - m]_{\gamma\gamma'}\,\Psi_{\alpha\beta\gamma'\delta} = 0\,, [i\gamma^\mu\partial_\mu - m]_{\delta\delta'}\,\Psi_{\alpha\beta\gamma\delta'} = 0\,. \tag{24}$$

The massless limit (if one needs) should be taken in the end of all calculations.

We proceed expanding the field function in the complete set of symmetric matrices (as in the spin-1 case). In the beginning let us use the first two indices:

$$\Psi_{\{\alpha\beta\}\gamma\delta} = (\gamma_\mu R)_{\alpha\beta}\Psi^\mu_{\gamma\delta} + (\sigma_{\mu\nu}R)_{\alpha\beta}\Psi^{\mu\nu}_{\gamma\delta}\,. \tag{25}$$

We would like to write the corresponding equations for functions $\Psi^\mu_{\gamma\delta}$ and $\Psi^{\mu\nu}_{\gamma\delta}$ in the form:

$$\frac{2}{m}\partial_\mu\Psi^{\mu\nu}_{\gamma\delta} = -\Psi^\nu_{\gamma\delta}\,, \Psi^{\mu\nu}_{\gamma\delta} = \frac{1}{2m}\left[\partial^\mu\Psi^\nu_{\gamma\delta} - \partial^\nu\Psi^\mu_{\gamma\delta}\right]\,. \tag{26}$$

The constraints $(1/m)\partial_\mu \Psi^\mu_{\gamma\delta} = 0$ and $(1/m)\epsilon^{\mu\nu}{}_{\alpha\beta}\,\partial_\mu \Psi^{\alpha\beta}_{\gamma\delta} = 0$ can be regarded as the consequence of Eqs. (26). Next, we present the vector-spinor and tensor-spinor functions as

$$\Psi^\mu_{\{\gamma\delta\}} = (\gamma^\kappa R)_{\gamma\delta} G_\kappa{}^{\;\mu} + (\sigma^{\kappa\tau} R)_{\gamma\delta} F_{\kappa\tau}{}^{\mu}\,, \tag{27}$$

$$\Psi^{\mu\nu}_{\{\gamma\delta\}} = (\gamma^\kappa R)_{\gamma\delta} T_\kappa{}^{\;\mu\nu} + (\sigma^{\kappa\tau} R)_{\gamma\delta} R_{\kappa\tau}{}^{\mu\nu}\,, \tag{28}$$

i. e., using the symmetric matrix coefficients in indices γ and δ. Hence, the total function is

$$\begin{aligned}\Psi_{\{\alpha\beta\}\{\gamma\delta\}} &= (\gamma_\mu R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta} G_\kappa{}^{\;\mu} + (\gamma_\mu R)_{\alpha\beta}(\sigma^{\kappa\tau} R)_{\gamma\delta} F_{\kappa\tau}{}^{\mu} + \\ &+ \quad (\sigma_{\mu\nu} R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta} T_\kappa{}^{\;\mu\nu} + (\sigma_{\mu\nu} R)_{\alpha\beta}(\sigma^{\kappa\tau} R)_{\gamma\delta} R_{\kappa\tau}{}^{\mu\nu}\,;\end{aligned} \tag{29}$$

and the resulting tensor equations are:

$$\frac{2}{m}\partial_\mu T_\kappa{}^{\;\mu\nu} = -G_\kappa{}^{\;\nu}\,, \frac{2}{m}\partial_\mu R_{\kappa\tau}{}^{\mu\nu} = -F_{\kappa\tau}{}^{\nu}\,, \tag{30}$$

$$T_\kappa{}^{\;\mu\nu} = \frac{1}{2m}\left[\partial^\mu G_\kappa{}^{\;\nu} - \partial^\nu G_\kappa{}^{\;\mu}\right]\,, \tag{31}$$

$$R_{\kappa\tau}{}^{\mu\nu} = \frac{1}{2m}\left[\partial^\mu F_{\kappa\tau}{}^{\nu} - \partial^\nu F_{\kappa\tau}{}^{\mu}\right]\,. \tag{32}$$

The constraints are re-written to

$$\frac{1}{m}\partial_\mu G_\kappa{}^{\;\mu} = 0\,, \quad \frac{1}{m}\partial_\mu F_{\kappa\tau}{}^{\mu} = 0\,, \tag{33}$$

$$\frac{1}{m}\epsilon_{\alpha\beta\nu\mu}\partial^\alpha T_\kappa{}^{\;\beta\nu} = 0\,, \quad \frac{1}{m}\epsilon_{\alpha\beta\nu\mu}\partial^\alpha R_{\kappa\tau}{}^{\beta\nu} = 0\,. \tag{34}$$

However, we need to make symmetrization over these two sets of indices $\{\alpha\beta\}$ and $\{\gamma\delta\}$. The total symmetry can be ensured if one contracts the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ with *antisymmetric* matrices $R^{-1}_{\beta\gamma}$, $(R^{-1}\gamma^5)_{\beta\gamma}$ and $(R^{-1}\gamma^5\gamma^\lambda)_{\beta\gamma}$ and equate all these contractions to zero (similar to the $j = 3/2$ case considered in ref. [2, p. 44]. We obtain additional constraints on the tensor field functions:

$$G_\mu{}^{\;\mu} = 0\,, \quad G_{[\kappa\,\mu]} = 0\,, \quad G^{\kappa\mu} = \frac{1}{2} g^{\kappa\mu} G_\nu{}^{\;\nu}\,, \tag{35}$$

$$F_{\kappa\mu}{}^{\mu} = F_{\mu\kappa}{}^{\mu} = 0\,, \quad \epsilon^{\kappa\tau\mu\nu} F_{\kappa\tau,\mu} = 0\,, \tag{36}$$

$$T^\mu{}_{\mu\kappa} = T^\mu{}_{\kappa\mu} = 0\,, \quad \epsilon^{\kappa\tau\mu\nu} T_{\kappa,\tau\mu} = 0\,, \tag{37}$$

$$F^{\kappa\tau,\mu} = T^{\mu,\kappa\tau}\,, \quad \epsilon^{\kappa\tau\mu\lambda}(F_{\kappa\tau,\mu} + T_{\kappa,\tau\mu}) = 0\,, \tag{38}$$

$$R_{\kappa\nu}{}^{\mu\nu} = R_{\nu\kappa}{}^{\mu\nu} = R_{\kappa\nu}{}^{\nu\mu} = R_{\nu\kappa}{}^{\nu\mu} = R_{\mu\nu}{}^{\mu\nu} = 0\,, \tag{39}$$

$$\epsilon^{\mu\nu\alpha\beta}(g_{\beta\kappa} R_{\mu\tau,\nu\alpha} - g_{\beta\tau} R_{\nu\alpha,\mu\kappa}) = 0 \quad \epsilon^{\kappa\tau\mu\nu} R_{\kappa\tau,\mu\nu} = 0\,. \tag{40}$$

Thus, we encountered with the known difficulty of the theory for spin-2 particles in the Minkowski space. We explicitly showed that all field functions become to be equal to zero. Such a situation cannot be considered as a satisfactory one (because it does not give us any physical information) and can be corrected in several ways.

We shall modify the formalism [17]. The field function is now presented as

$$\Psi_{\{\alpha\beta\}\gamma\delta} = \alpha_1(\gamma_\mu R)_{\alpha\beta}\Psi^\mu_{\gamma\delta} + \alpha_2(\sigma_{\mu\nu}R)_{\alpha\beta}\Psi^{\mu\nu}_{\gamma\delta} + \alpha_3(\gamma^5\sigma_{\mu\nu}R)_{\alpha\beta}\widetilde{\Psi}^{\mu\nu}_{\gamma\delta}\,, \quad (41)$$

with

$$\Psi^\mu_{\{\gamma\delta\}} = \beta_1(\gamma^\kappa R)_{\gamma\delta}G_\kappa^{\ \mu} + \beta_2(\sigma^{\kappa\tau}R)_{\gamma\delta}F_{\kappa\tau}^{\ \ \mu} + \beta_3(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{F}_{\kappa\tau}^{\ \ \mu}, \quad (42)$$
$$\Psi^{\mu\nu}_{\{\gamma\delta\}} = \beta_4(\gamma^\kappa R)_{\gamma\delta}T_\kappa^{\ \mu\nu} + \beta_5(\sigma^{\kappa\tau}R)_{\gamma\delta}R_{\kappa\tau}^{\ \ \mu\nu} + \beta_6(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{R}_{\kappa\tau}^{\ \ \mu\nu}, \quad (43)$$
$$\widetilde{\Psi}^{\mu\nu}_{\{\gamma\delta\}} = \beta_7(\gamma^\kappa R)_{\gamma\delta}\widetilde{T}_\kappa^{\ \mu\nu} + \beta_8(\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{D}_{\kappa\tau}^{\ \ \mu\nu} + \beta_9(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}D_{\kappa\tau}^{\ \ \mu\nu}. \quad (44)$$

Hence, the function $\Psi_{\{\alpha\beta\}\{\gamma\delta\}}$ can be expressed as a sum of nine terms:

$$\begin{aligned}
&\Psi_{\{\alpha\beta\}\{\gamma\delta\}} = \alpha_1\beta_1(\gamma_\mu R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta}G_\kappa^{\ \ \mu} + \alpha_1\beta_2(\gamma_\mu R)_{\alpha\beta}(\sigma^{\kappa\tau}R)_{\gamma\delta}F_{\kappa\tau}^{\ \ \mu} +\\
+ \quad &\alpha_1\beta_3(\gamma_\mu R)_{\alpha\beta}(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{F}_{\kappa\tau}^{\ \ \mu} + +\alpha_2\beta_4(\sigma_{\mu\nu}R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta}T_\kappa^{\ \ \mu\nu} +\\
+ \quad &\alpha_2\beta_5(\sigma_{\mu\nu}R)_{\alpha\beta}(\sigma^{\kappa\tau}R)_{\gamma\delta}R_{\kappa\tau}^{\ \ \mu\nu} + \alpha_2\beta_6(\sigma_{\mu\nu}R)_{\alpha\beta}(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{R}_{\kappa\tau}^{\ \ \mu\nu} +\\
+ \quad &\alpha_3\beta_7(\gamma^5\sigma_{\mu\nu}R)_{\alpha\beta}(\gamma^\kappa R)_{\gamma\delta}\widetilde{T}_\kappa^{\ \ \mu\nu} + \alpha_3\beta_8(\gamma^5\sigma_{\mu\nu}R)_{\alpha\beta}(\sigma^{\kappa\tau}R)_{\gamma\delta}\widetilde{D}_{\kappa\tau}^{\ \ \mu\nu} +\\
+ \quad &\alpha_3\beta_9(\gamma^5\sigma_{\mu\nu}R)_{\alpha\beta}(\gamma^5\sigma^{\kappa\tau}R)_{\gamma\delta}D_{\kappa\tau}^{\ \ \mu\nu}\,. \qquad (45)
\end{aligned}$$

The corresponding dynamical equations are given by the set of equations

$$\frac{2\alpha_2\beta_4}{m}\partial_\nu T_\kappa^{\ \ \mu\nu} + \frac{i\alpha_3\beta_7}{m}\epsilon^{\mu\nu\alpha\beta}\partial_\nu\widetilde{T}_{\kappa,\alpha\beta} = \alpha_1\beta_1 G_\kappa^{\ \ \mu}\,; \quad (46)$$

$$\begin{aligned}
&\frac{2\alpha_2\beta_5}{m}\partial_\nu R_{\kappa\tau}^{\ \ \mu\nu} + \frac{i\alpha_2\beta_6}{m}\epsilon_{\alpha\beta\kappa\tau}\partial_\nu\widetilde{R}^{\alpha\beta,\mu\nu} + \frac{i\alpha_3\beta_8}{m}\epsilon^{\mu\nu\alpha\beta}\partial_\nu\widetilde{D}_{\kappa\tau,\alpha\beta} -\\
- \quad &\frac{\alpha_3\beta_9}{2}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\lambda\delta\kappa\tau}D^{\lambda\delta}_{\ \ \ \alpha\beta} = \alpha_1\beta_2 F_{\kappa\tau}^{\ \ \mu} + \frac{i\alpha_1\beta_3}{2}\epsilon_{\alpha\beta\kappa\tau}\widetilde{F}^{\alpha\beta,\mu}\,; \qquad (47)
\end{aligned}$$

$$2\alpha_2\beta_4 T_\kappa^{\ \ \mu\nu} + i\alpha_3\beta_7\epsilon^{\alpha\beta\mu\nu}\widetilde{T}_{\kappa,\alpha\beta} = \frac{\alpha_1\beta_1}{m}(\partial^\mu G_\kappa^{\ \ \nu} - \partial^\nu G_\kappa^{\ \ \mu})\,; \quad (48)$$

$$\begin{aligned}
&2\alpha_2\beta_5 R_{\kappa\tau}^{\ \ \mu\nu} + i\alpha_3\beta_8\epsilon^{\alpha\beta\mu\nu}\widetilde{D}_{\kappa\tau,\alpha\beta} + i\alpha_2\beta_6\epsilon_{\alpha\beta\kappa\tau}\widetilde{R}^{\alpha\beta,\mu\nu} -\\
- \quad &\frac{\alpha_3\beta_9}{2}\epsilon^{\alpha\beta\mu\nu}\epsilon_{\lambda\delta\kappa\tau}D^{\lambda\delta}_{\ \ \ \alpha\beta} = \frac{\alpha_1\beta_2}{m}(\partial^\mu F_{\kappa\tau}^{\ \ \nu} - \partial^\nu F_{\kappa\tau}^{\ \ \mu}) +\\
+ \quad &\frac{i\alpha_1\beta_3}{2m}\epsilon_{\alpha\beta\kappa\tau}(\partial^\mu\widetilde{F}^{\alpha\beta,\nu} - \partial^\nu\widetilde{F}^{\alpha\beta,\mu})\,. \qquad (49)
\end{aligned}$$

The essential constraints are:

$$\alpha_1\beta_1 G^\mu{}_\mu = 0\,, \quad \alpha_1\beta_1 G_{[\kappa\mu]} = 0; 2i\alpha_1\beta_2 F_{\alpha\mu}{}^\mu + \alpha_1\beta_3\epsilon^{\kappa\tau\mu}{}_\alpha \widetilde{F}_{\kappa\tau,\mu} = 0; \quad (50)$$

$$2i\alpha_1\beta_3\widetilde{F}_{\alpha\mu}{}^\mu + \alpha_1\beta_2\epsilon^{\kappa\tau\mu}{}_\alpha F_{\kappa\tau,\mu} = 0\,; 2i\alpha_2\beta_4 T^\mu{}_{\mu\alpha} - \alpha_3\beta_7\epsilon^{\kappa\tau\mu}{}_\alpha \widetilde{T}_{\kappa,\tau\mu} = 0; \quad (51)$$

$$2i\alpha_3\beta_7\widetilde{T}^\mu{}_{\mu\alpha} - \alpha_2\beta_4\epsilon^{\kappa\tau\mu}{}_\alpha T_{\kappa,\tau\mu} = 0; \quad (52)$$

$$i\epsilon^{\mu\nu\kappa\tau}\left[\alpha_2\beta_6\widetilde{R}_{\kappa\tau,\mu\nu} + \alpha_3\beta_8\widetilde{D}_{\kappa\tau,\mu\nu}\right] + 2\alpha_2\beta_5 R^{\mu\nu}{}_{\mu\nu} + 2\alpha_3\beta_9 D^{\mu\nu}{}_{\mu\nu} = 0; \quad (53)$$

$$i\epsilon^{\mu\nu\kappa\tau}\left[\alpha_2\beta_5 R_{\kappa\tau,\mu\nu} + \alpha_3\beta_9 D_{\kappa\tau,\mu\nu}\right] + 2\alpha_2\beta_6\widetilde{R}^{\mu\nu}{}_{\mu\nu} + 2\alpha_3\beta_8\widetilde{D}^{\mu\nu}{}_{\mu\nu} = 0; \quad (54)$$

$$2i\alpha_2\beta_5 R^{\mu\alpha}_{\beta\mu} + 2i\alpha_3\beta_9 D^{\mu\alpha}_{\beta\mu} + \alpha_2\beta_6\epsilon^{\nu\alpha}_{\lambda\beta}\widetilde{R}^{\lambda\mu}_{\mu\nu} + \alpha_3\beta_8\epsilon^{\nu\alpha}_{\lambda\beta}\widetilde{D}^{\lambda\mu}_{\mu\nu} = 0; \quad (55)$$

$$2i\alpha_1\beta_2 F^{\lambda\mu}{}_\mu - 2i\alpha_2\beta_4 T_\mu{}^{\mu\lambda} + \alpha_1\beta_3\epsilon^{\kappa\tau\mu\lambda}\widetilde{F}_{\kappa\tau,\mu} + \alpha_3\beta_7\epsilon^{\kappa\tau\mu\lambda}\widetilde{T}_{\kappa,\tau\mu} = 0\,; \quad (56)$$

$$2i\alpha_1\beta_3\widetilde{F}^{\lambda\mu}{}_\mu - 2i\alpha_3\beta_7\widetilde{T}_\mu{}^{\mu\lambda} + \alpha_1\beta_2\epsilon^{\kappa\tau\mu\lambda} F_{\kappa\tau,\mu} + \alpha_2\beta_4\epsilon^{\kappa\tau\mu\lambda} T_{\kappa,\tau\mu} = 0; \quad (57)$$

$$\begin{aligned}
& \alpha_1\beta_1(2G^\lambda{}_\alpha - g^\lambda{}_\alpha G^\mu{}_\mu) - 2\alpha_2\beta_5(2R^{\lambda\mu}{}_{\mu\alpha} + 2R_{\alpha\mu}{}^{\mu\lambda} + g^\lambda{}_\alpha R^{\mu\nu}{}_{\mu\nu}) + \\
+ \; & 2\alpha_3\beta_9(2D^{\lambda\mu}{}_{\mu\alpha} + 2D_{\alpha\mu}{}^{\mu\lambda} + g^\lambda{}_\alpha D^{\mu\nu}{}_{\mu\nu}) + 2i\alpha_3\beta_8(\epsilon_{\kappa\alpha}{}^{\mu\nu}\widetilde{D}^{\kappa\lambda}{}_{\mu\nu} - \\
- \; & \epsilon^{\kappa\tau\mu\lambda}\widetilde{D}_{\kappa\tau,\mu\alpha}) - 2i\alpha_2\beta_6(\epsilon_{\kappa\alpha}{}^{\mu\nu}\widetilde{R}^{\kappa\lambda}{}_{\mu\nu} - \epsilon^{\kappa\tau\mu\lambda}\widetilde{R}_{\kappa\tau,\mu\alpha}) = 0; \quad (58)
\end{aligned}$$

$$\begin{aligned}
& 2\alpha_3\beta_8(2\widetilde{D}^{\lambda\mu}{}_{\mu\alpha} + 2\widetilde{D}_{\alpha\mu}{}^{\mu\lambda} + g^\lambda{}_\alpha\widetilde{D}^{\mu\nu}{}_{\mu\nu}) - 2\alpha_2\beta_6(2\widetilde{R}^{\lambda\mu}{}_{\mu\alpha} + 2\widetilde{R}_{\alpha\mu}{}^{\mu\lambda} \\
+ \; & g^\lambda{}_\alpha\widetilde{R}^{\mu\nu}{}_{\mu\nu}) + +2i\alpha_3\beta_9(\epsilon_{\kappa\alpha}{}^{\mu\nu} D^{\kappa\lambda}{}_{\mu\nu} - \epsilon^{\kappa\tau\mu\lambda} D_{\kappa\tau,\mu\alpha}) - \\
- \; & 2i\alpha_2\beta_5(\epsilon_{\kappa\alpha}{}^{\mu\nu} R^{\kappa\lambda}{}_{\mu\nu} - \epsilon^{\kappa\tau\mu\lambda} R_{\kappa\tau,\mu\alpha}) = 0; \quad (59)
\end{aligned}$$

$$\begin{aligned}
& \alpha_1\beta_2(F^{\alpha\beta,\lambda} - 2F^{\beta\lambda,\alpha} + F^{\beta\mu}{}_\mu g^{\lambda\alpha} - F^{\alpha\mu}{}_\mu g^{\lambda\beta}) - \\
- \; & \alpha_2\beta_4(T^{\lambda,\alpha\beta} - 2T^{\beta,\lambda\alpha} + T_\mu{}^{\mu\alpha} g^{\lambda\beta} - T_\mu{}^{\mu\beta} g^{\lambda\alpha}) + \\
+ \; & \frac{i}{2}\alpha_1\beta_3(\epsilon^{\kappa\tau\alpha\beta}\widetilde{F}_{\kappa\tau}{}^\lambda + 2\epsilon^{\lambda\kappa\alpha\beta}\widetilde{F}_{\kappa\mu}{}^\mu + 2\epsilon^{\mu\kappa\alpha\beta}\widetilde{F}^\lambda{}_{\kappa,\mu}) - \\
- \; & \frac{i}{2}\alpha_3\beta_7(\epsilon^{\mu\nu\alpha\beta}\widetilde{T}^\lambda{}_{\mu\nu} + 2\epsilon^{\nu\lambda\alpha\beta}\widetilde{T}^\mu{}_{\mu\nu} + 2\epsilon^{\mu\kappa\alpha\beta}\widetilde{T}_{\kappa,\mu}{}^\lambda) = 0. \quad (60)
\end{aligned}$$

They are the results of contractions of the field function (45) with three antisymmetric matrices, as above. Furthermore, one should recover the relations (35-40) in the particular case when $\alpha_3 = \beta_3 = \beta_6 = \beta_9 = 0$ and $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \beta_4 = \beta_5 = \beta_7 = \beta_8 = 1$.

As a discussion we note that in such a framework we have physical content because only certain combinations of field functions would be equal to zero. In general, the fields $F_{\kappa\tau}{}^\mu$, $\widetilde{F}_{\kappa\tau}{}^\mu$, $T_\kappa{}^{\mu\nu}$, $\widetilde{T}_\kappa{}^{\mu\nu}$, and $R_{\kappa\tau}{}^{\mu\nu}$, $\widetilde{R}_{\kappa\tau}{}^{\mu\nu}$, $D_{\kappa\tau}{}^{\mu\nu}$, $\widetilde{D}_{\kappa\tau}{}^{\mu\nu}$ can correspond to different physical states and the equations above describe some kind of "oscillations" of one state to another. Furthermore, from the set of equations (46-49) one obtains the *second*-order equation for symmetric traceless tensor of the second rank ($\alpha_1 \neq 0$, $\beta_1 \neq 0$):

$$\frac{1}{m^2}\left[\partial_\nu \partial^\mu G_\kappa{}^\nu - \partial_\nu \partial^\nu G_\kappa{}^\mu\right] = G_\kappa{}^\mu \,. \tag{61}$$

After the contraction in indices κ and μ this equation is reduced to the set

$$\partial_\mu G^\mu{}_\nu = F_\nu \tag{62}$$

$$\frac{1}{m^2}\partial_\nu F^\nu = 0 \,, \tag{63}$$

i. e., to the equations connecting the analogue of the energy-momentum tensor and the analogue of the 4-vector potential. Further investigations may provide additional foundations to “surprising” similarities of gravitational and electromagnetic equations in the low-velocity limit, refs. [21, 22].

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