Spinning particles in scalar-tensor gravity

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Abstract

We develop a new model of a spinning particle in Brans-Dicke space-time using a metric-compatible connection with torsion. The particle's spin vector is shown to be Fermi-parallel (by the Levi-Civita connection) along its worldline (an autoparallel of the metric-compatible connection) when neglecting spin-curvature coupling.

1 Introduction

Scalar fields are replete in string-inspired low-energy effective theories and occupy a prominent position in modern particle physics and cosmology. The most widely accepted implementations of mass generation and inflation employ scalar fields and it is not unreasonable to suggest that they should play a role on large scales. Indeed, a number of authors have suggested that Einsteinian gravity (General Relativity), a purely metric-based theory, is incomplete and should be augmented by a scalar component; one of the simplest examples of such a theory was proposed by Brans and Dicke [1]. Their theory was originally formulated as an action principle whose degrees of freedom are the spacetime metric and the Brans-Dicke scalar field φ. It was later shown [2] that Brans-Dicke theory could also be obtained from an action whose independent variables are a metric-compatible connection ∇, an orthonormal frame field and φ. Unlike the Levi-Civita connection $\bar{\nabla}$ of Einsteinian gravity, whose torsion vanishes identically, the field equations for ∇ yield a non-trivial torsion tensor in terms of φ. Although the action principle in [2] differs from that introduced by Brans and Dicke, the standard Brans-Dicke equations of motion are recovered when ∇ is expressed in terms of $\bar{\nabla}$ and φ.

In Einsteinian and Brans-Dicke gravity, electrically neutral spinless particles are *postulated* to follows autoparallels of $\bar{\nabla}$ (geodesics). However, the most natural connection in Brans-Dicke theory, from the perspective of [2], is ∇ and it is perfectly reasonable to suggest [3] that electrically neutral spinless massive particles should follow the autoparallels of ∇ rather than $\bar{\nabla}$. This hypothesis

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was examined in [3] where the autoparallels of ∇ and $\bar{\nabla}$ were used to compare the perihelion shifts of Mercury predicted by the two connections. The difference between the perihelion shifts predicted by the new theory and the original Brans-Dicke theory are sufficiently different to warrant further attention.

Analysis of inspiralling compact binaries using existing and future gravitational wave detectors employs matched filtering techniques. Signal extraction relies on accurate templates of the expected gravitational wave emission and relativistic spin-orbit and spin-spin coupling play a critical role (see [4] for a recent discussion). Existing templates rely on post-Newtonian treatments of Einsteinian gravity and may need modification if the effects of torsion and spin are significant. Furthermore, it is reasonable to suggest that the worldlines of spinning particles could be governed by ∇ rather than $\bar{\nabla}$; the purpose of this Letter is to investigate this possibility.

Spinning particles in Einsteinian gravity have long been a source of interest and consideration was recently given to spinning particles in theories of gravity with torsion [5]. The general approach followed in [5] has two main ingredients : (1) a set of relativistic balance laws and (2) a set of constitutive relations (akin to equations of state in gas dynamics). The balance laws are Noether identities arising from the diffeomorphism and gauge invariances of an action; they only involve *sources* (stress-energy-momentum tensor, spin tensor and other currents). The constitutive relations are model-dependent equations for the sources in terms of the true dynamical degrees of freedom and are needed to reduce the balance laws to a closed system of field equations.

Many workers (see [5] for a review) have invested effort in deriving the equations of motion for a spinning particle in General Relativity using the simplest source models. In this Letter, motivated by [3], we develop equations of motion for a spinning particle, based on a simple source model, that lead to the autoparallel equation of ∇ and the parallel transport law with respect to ∇. The equations for a spinning particle presented here are a natural alternative to the hypothesis that the Papapetrou-Dixon equations in Einsteinian gravity are also valid in the context of Brans-Dicke gravity.

2 Noether identities

Balance laws are generated from diffeomorphism and gauge invariances of an action. The approach adopted here is a generalisation of [5] and [6] to accommodate the Brans-Dicke scalar field φ. For simplicity, we will only consider electrically neutral matter and represent it using a single[1] p-form Φ.

Consider an effective action S

$$S[\Phi] = \int_{\mathcal{M}} \Lambda \tag{1}$$

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[1]It is straightforward to include more fields and charged matter but this would draw attention away from the essential features of our argument.

for uncharged matter Φ in a *background* spacetime $\mathcal{M}$ with metric g, metric-compatible connection ∇ and a *background* Brans-Dicke scalar field φ. The 4-form Λ is constructed tensorially from g, ∇, φ, Φ and, regardless of the detailed structure of Λ, it follows

$$\mathcal{L}_X \Lambda \simeq \tau_a \wedge \mathcal{L}_X e^a + S_a{}^b \wedge \mathcal{L}_X \omega^a{}_b + \rho \wedge \mathcal{L}_X \varphi + \mathcal{E} \wedge \mathcal{L}_X \Phi \tag{2}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to any vector field X on $\mathcal{M}$, $\simeq$ indicates equality up to an exact 4-form, $\{e^a\}$ is a g-orthonormal basis for 1-forms and $\{\omega^a{}_b\}$ are the connection 1-forms of ∇ associated with $\{e^a\}$ $(a, b, c = 0, 1, 2, 3)$. The precise details of the sources (the stress 3-forms τ_a, spin 3-forms $S_a{}^b$, 0-form ρ and $(4 - p)$-form $\mathcal{E}$) depend on the details of Λ; this will not concern us.

For a vector field X with compact support it follows

$$\int_{\mathcal{M}} \mathcal{L}_X \Lambda = \int_{\mathcal{M}} \left(\tau_a \wedge \mathcal{L}_X e^a + S_a{}^b \wedge \mathcal{L}_X \omega^a{}_b + \rho \wedge \mathcal{L}_X \varphi + \mathcal{E} \wedge \mathcal{L}_X \Phi \right). \tag{3}$$

Varying S with respect to Φ yields

$$\delta S = \int_{\mathcal{M}} \mathcal{E} \wedge \delta \Phi \tag{4}$$

where $\delta \Phi$ is an arbitrary variation of Φ with compact support. Thus, imposing the equations of motion $\mathcal{E} = 0$ for Φ leads to

$$\int_{\mathcal{M}} \mathcal{L}_X \Lambda = \int_{\mathcal{M}} \left(\tau_a \wedge \mathcal{L}_X e^a + S_a{}^b \wedge \mathcal{L}_X \omega^a{}_b + \rho \wedge \mathcal{L}_X \varphi \right). \tag{5}$$

Cartan's identity $\mathcal{L}_X \Lambda = \iota_X d\Lambda + d\iota_X \Lambda$, where d is the exterior derivative and ι_X is the interior product on forms with respect to X, yields

$$\int_{\mathcal{M}} \mathcal{L}_X \Lambda = 0 \tag{6}$$

because $d\Lambda = 0$ and X has compact support and using (5) it follows

$$\int_{\mathcal{M}} \left(\tau_a \wedge \mathcal{L}_X e^a + S_a{}^b \wedge \mathcal{L}_X \omega^a{}_b + \rho \wedge \mathcal{L}_X \varphi \right) = 0. \tag{7}$$

It can be shown [6]

$$\mathcal{L}_X e^a = D(\iota_X e^a) + \iota_X T^a - \delta_{SO(1,3)} e^a, \tag{8}$$
$$\mathcal{L}_X \omega^a{}_b = \iota_X R^a{}_b - \delta_{SO(1,3)} \omega^a{}_b \tag{9}$$

where $\delta_{SO(1,3)}$ indicates an infinitesimal $SO(1,3)$ frame transformation, D is the exterior covariant derivative on the orthonormal frame bundle, T^a the torsion 2-forms and $R^a{}_b$ the curvature 2-forms of ∇ associated with $\{e^a\}$. The action is invariant under $SO(1,3)$ frame transformations,

$$0 = \int_{\mathcal{M}} \delta_{SO(1,3)} \Lambda = \int_{\mathcal{M}} \left(\tau_a \wedge \delta_{SO(1,3)} e^a + S_a{}^b \wedge \delta_{SO(1,3)} \omega^a{}_b \right), \tag{10}$$

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and it follows

$$\int_{\mathcal{M}} \left(D\tau_c + \tau_a \wedge \iota_{X_c} T^a + S_a{}^b \wedge \iota_{X_c} R^a{}_b + \rho \, \iota_{X_c} d\varphi \right) \mathcal{W}^c = 0 \tag{11}$$

using (7-10) and $X = \mathcal{W}^a X_a$ with $\{X_a\}$ dual to $\{e^a\}$,

$$e^a(X_b) = \delta_b^a \tag{12}$$

where δ_b^a is the Kronecker delta. The action of $\delta_{SO(1,3)}$ on e^a and $\omega^a{}_b$ is

$$\delta_{SO(1,3)} e^a = -\mathcal{W}^a{}_b e^b, \tag{13}$$
$$\delta_{SO(1,3)} \omega^a{}_b = D\mathcal{W}^a{}_b \tag{14}$$

where $\mathcal{W}^a{}_b$ is an element of the Lie algebra $so(1,3)$ and has compact support on $\mathcal{M}$. Using (10), (13) and (14) it follows

$$\int_{\mathcal{M}} \left[DS_a{}^b - \frac{1}{2}(\tau_a \wedge e^b - \tau^b \wedge e_a) \right] \mathcal{W}^a{}_b = 0 \tag{15}$$

and since (11), (15) hold for all $\mathcal{W}^a$ and $\mathcal{W}^a{}_b$ with compact support we obtain the Noether identities

$$D\tau_c + \tau_a \wedge \iota_{X_c} T^a + S_a{}^b \wedge \iota_{X_c} R^a{}_b + \rho \, \iota_{X_c} d\varphi = 0, \tag{16}$$
$$DS_a{}^b - \frac{1}{2}(\tau_a \wedge e^b - \tau^b \wedge e_a) = 0 \tag{17}$$

relating the sources $\tau_a, S_a{}^b, \rho$.

3 Equations of motion for a spinning particle

Brans-Dicke theory can be obtained from a variational principle whose independent variables are $\{e^a, \omega^a{}_b, \varphi\}$ [2] and leads to the torsion 2-forms

$$T^a = \frac{1}{2} T^a{}_{bc} \, e^b \wedge e^c = e^a \wedge \frac{d\varphi}{\varphi} \tag{18}$$

for $\varphi \neq 0$. Equations (16-18) must be supplemented by further information in order to obtain a closed system. The following simple constitutive relations

$$\tau^a = \varphi \, P^a \star e^0, \tag{19}$$
$$S_a{}^b = \Sigma_a{}^b \star e^0, \tag{20}$$
$$\rho = -P^0 \star 1 \tag{21}$$

reduce to the model in [5] when φ is constant, where $\star$ is the Hodge map associated with g and $\star 1$ is the spacetime volume 4-form.

Equations describing a neutral spinning particle follow by taking moments of (16) and (17) in Fermi-normal coordinates on an open set $\mathcal{U} \subset \mathcal{M}$ containing the

image of the particle's worldline σ with proper time t (the unit tangent to σ is denoted $\dot{\sigma}$). The orthonormal co-frame $\{e^a\}$ is defined on σ such that[2] $e^0 = -\tilde{\dot{\sigma}}$ and $\{e^1, e^2, e^3\}$ are Fermi-parallel (with respect to ∇) along σ. Furthermore, $\{e^a\}$ is induced away from σ by parallel transport along radial spacelike autoparallels of ∇ whose tangents are orthogonal to $\dot{\sigma}$ (see [5], [7] for details). Moments of (16) and (17) are computed in Fermi-normal coordinates $\{x^1, x^2, x^3\}$ on $\mathcal{U}$ and to leading order

$$\frac{d\hat{P}^0}{dt} = -\boldsymbol{A} \cdot \boldsymbol{P}, \tag{22}$$

$$\frac{d\boldsymbol{P}_\mu}{dt} = -\frac{1}{\hat{\varphi}}\widehat{\frac{\partial \varphi}{\partial t}}\boldsymbol{P}_\mu - \boldsymbol{A}_\mu\,\hat{P}^0 - \hat{T}^a{}_{0\mu}\hat{P}_a - \hat{P}^0\frac{1}{\hat{\varphi}}\widehat{\frac{\partial \varphi}{\partial x^\mu}} + \frac{1}{\hat{\varphi}}\hat{R}_{ab\mu 0}\hat{\Sigma}^{ab}, \tag{23}$$

$$\frac{d\boldsymbol{h}}{dt} = \boldsymbol{A} \times \boldsymbol{s} + \frac{1}{2}\hat{\varphi}\boldsymbol{P}, \tag{24}$$

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{h} \times \boldsymbol{A} \tag{25}$$

where

$$R^a{}_b = \frac{1}{2}R^a{}_{bcd}e^c \wedge e^d, \tag{26}$$

$$\boldsymbol{A}_\mu = \hat{e}_\mu(\nabla_{\dot{\sigma}}\dot{\sigma}), \tag{27}$$

and hats indicate evaluation over the image of σ, i.e. at $\{x^\mu = 0\}$, and

$$\boldsymbol{P}_\mu = \hat{P}_\mu, \tag{28}$$

$$\boldsymbol{h}_\mu = \hat{\Sigma}^0{}_\mu, \tag{29}$$

$$\boldsymbol{s}_\mu = \frac{1}{2}\epsilon_{\mu\nu\omega}\hat{\Sigma}^{\nu\omega} \tag{30}$$

where $\epsilon_{\mu\nu\omega}$ is the Levi-Civita alternating symbol with $\mu, \nu, \omega = 1, 2, 3$. For given external fields φ and $R^a{}_{bcd}$, equations (22-25) and (18) are not sufficient to determine the worldline and spin of the particle. Thus, the above system is supplemented by the Tulczyjew-Dixon (subsidiary) conditions

$$\hat{P}_a\hat{\Sigma}^{ab} = 0 \tag{31}$$

i.e.

$$\boldsymbol{h}\hat{P}^0 = \boldsymbol{s} \times \boldsymbol{P}. \tag{32}$$

Furthermore, using (18) equation (23) can be simplified to

$$\frac{d\boldsymbol{P}_\mu}{dt} = -\boldsymbol{A}_\mu\,\hat{P}^0 + \frac{1}{\hat{\varphi}}\hat{R}_{ab\mu 0}\hat{\Sigma}^{ab}. \tag{33}$$

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[2] $\tilde{\dot{\sigma}}(Y) = g(\dot{\sigma}, Y)$ for all vectors Y.

Equations (22), (33), (24), (25) and (32) are a differential-algebraic system for the worldline and spin of a particle in a Brans-Dicke background spacetime.

Consider the weak-field limit of the above theory where the spin-curvature coupling in (33) is neglected,

$$\frac{d\boldsymbol{P}_\mu}{dt} = -\boldsymbol{A}_\mu \hat{P}^0. \tag{34}$$

Equations (22), (34), (24), (25) and (32) have the particular solutions

$$\boldsymbol{P} = 0, \tag{35}$$
$$\boldsymbol{h} = 0, \tag{36}$$
$$\boldsymbol{A} = 0, \tag{37}$$
$$\boldsymbol{s}_\mu = \text{const.}, \tag{38}$$
$$\hat{P}^0 = \text{const.} \tag{39}$$

and it immediately follows from (35-39)

$$\nabla_{\dot\sigma}\dot\sigma = 0, \tag{40}$$
$$\nabla_{\dot\sigma} S = 0, \tag{41}$$
$$g(S, \dot\sigma) = 0 \tag{42}$$

where $S = \boldsymbol{s}^\mu \partial/\partial x^\mu$ is the particle's spin vector. Clearly, for such solutions σ is an autoparallel of ∇ and S is parallel-transported with respect to ∇ along σ.

The metric-compatible connection ∇ and the Levi-Civita connection $\bar\nabla$ are related as

$$\widetilde{Z}(\nabla_X Y) = \widetilde{Z}(\bar\nabla_X Y) + \frac{1}{2}\big[\widetilde{X}(T(Z,Y)) + \widetilde{Y}(T(Z,X)) + \widetilde{Z}(T(X,Y))\big] \tag{43}$$

where the $(2,1)$ torsion tensor T and the torsion 2-forms T^a, equation (18), satisfy

$$e^a(T(X_b, X_c)) = \iota_{X_c}\iota_{X_b} T^a = \delta^a_b \frac{X_c\varphi}{\varphi} - \delta^a_c \frac{X_b\varphi}{\varphi}. \tag{44}$$

It follows (40-42) can be written[3]

$$\bar\nabla_{\dot\sigma}\dot\sigma = -\frac{\widetilde{d\varphi}}{\varphi} - \frac{d\varphi(\dot\sigma)}{\varphi}\dot\sigma, \tag{45}$$
$$\bar\nabla^F_{\dot\sigma} S = 0, \tag{46}$$
$$g(S, \dot\sigma) = 0 \tag{47}$$

where $\bar\nabla^F_{\dot\sigma} S$ is the Fermi-Walker derivative of the $\dot\sigma$-orthogonal spin vector S along σ,

$$\bar\nabla^F_{\dot\sigma} S = \bar\nabla_{\dot\sigma} S - \widetilde{S}(\bar\nabla_{\dot\sigma}\dot\sigma)\dot\sigma. \tag{48}$$

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[3] $g(\widetilde{d\varphi}, Y) = d\varphi(Y)$ for all vectors Y.

Equations (45-47) can also be written

$$A^a = -\left(g^{ab} + \frac{dx^a}{dt}\frac{dx^b}{dt}\right)\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^b}, \tag{49}$$

$$\frac{dS^a}{dt} + \Gamma^a_{bc}S^b\frac{dx^c}{dt} = g_{ab}A^aS^b, \tag{50}$$

$$g_{ab}S^a\frac{dx^b}{dt} = 0 \tag{51}$$

for the particle's world-line $x^a(t)$ and spin S^a where

$$A^a = \frac{d^2x^a}{dt^2} + \Gamma^a_{bc}\frac{dx^b}{dt}\frac{dx^c}{dt} \tag{52}$$

and Γ^a_{bc} are the Christoffel symbols of the Levi-Civita connection induced by the metric g_{ab}.

Equations (49-52) are a natural generalisation to Brans-Dicke theory of the conventional model of a freely-falling ideal gyroscope in Einsteinian gravity,

$$\bar{\nabla}_{\dot{\sigma}}\dot{\sigma} = 0, \tag{53}$$
$$\bar{\nabla}_{\dot{\sigma}}S = 0, \tag{54}$$
$$g(S,\dot{\sigma}) = 0 \tag{55}$$

i.e.

$$A^a = 0, \tag{56}$$
$$\frac{dS^a}{dt} + \Gamma^a_{bc}S^b\frac{dx^c}{dt} = 0, \tag{57}$$
$$g_{ab}S^a\frac{dx^b}{dt} = 0. \tag{58}$$

4 Conclusion

We have developed a simple model of a spinning particle on a Brans-Dicke background. Brans-Dicke theory is naturally formulated in terms of a metric-compatible connection ∇ with torsion and the present model, in the weak-field regime, exhibits solutions where the particle's worldline σ is an autoparallel of ∇ and its spin vector S is parallel (with respect to ∇) along σ. In terms of the Levi-Civita connection $\bar{\nabla}$, the worldline σ has non-zero acceleration and S is Fermi-parallel along σ.

Detailed knowledge of the collapse of compact spinning binaries is important for attempts to detect gravitational waves. Furthermore, it is invisaged that gravitational radiation will be used as a tool to study astrophysical objects and, given the plethora of scalar fields in low-energy string-inspired field theories, an effective Brans-Dicke scalar may play a significant role. Any predictive theory, in the present context, of the behaviour of a spinning particle on a Brans-Dicke

background relies on a source model that leads to a sensible Newtonian limit for the theory. The above (with φ constant) has the same Newtonian limit as the Papapetrou-Dixon equations in Einsteinian gravity.

The approach discussed here is a simple generalisation of the source model discussed in [5] and further work is necessary to elucidate the differences between such models when spin-curvature coupling is significant.

In the weak field limit, the particle behaves like an accelerating ideal gyroscope in Einsteinian gravity. It may be shown [7] that the precession rates (geodetic and Lense-Thirring) in Kerr spacetime given by (56-58) and the corresponding rates in Kerr-Brans-Dicke spacetime given by (49-52) are indistinguishable to leading order. This has implications for any attempt to use Gravity Probe B [8] to test the novel theory.

5   Acknowledgements

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