Optics of spin-1 particles from gravity-induced phases

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Abstract. The Maxwell and Maxwell-de Rham equations can be solved exactly to first order in an external gravitational field. The gravitational background induces phases in the wave functions of spin-1 particles. These phases yield the optics of the particles without requiring any thin lens approximation.

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1. Introduction

The deflection of photons from distant astronomical sources by intervening massive objects is rapidly becoming an important tool for experimental astrophysics and cosmology. At the same time a variety of optical, atomic and molecular interferometers in use or under development is approaching the threshold of measurability for inertial and gravitational effects.

Photons in gravitational fields are described by the generalized Maxwell equations

$$\nabla_\alpha \nabla^\alpha A_\mu = 0\,, \tag{1}$$

or, more generally, by the Maxwell-de Rahm equations

$$\nabla_\alpha \nabla^\alpha A_\mu - R_{\mu\alpha} A^\alpha = 0\,, \tag{2}$$

where ∇_α indicates covariant differentiation. We use units $\hbar = c = 1$.

In interferometry, where the phase shifts are proportional to the mass of the particle employed, one may wish to consider, beside photons, also massive, charge-less, spin-1 particles that satisfy the generalized Proca equation

$$\nabla_\alpha \nabla^\alpha A_\mu + m^2 A_\mu = 0\,, \tag{3}$$

in the hope that among the atoms and molecules to be used some indeed obey (3).

We have shown in previous work [1]-[6], that equations (1)-(3) can be solved exactly to first order in the metric deviation $\gamma_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, where $|\gamma_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu}$ is the Minkowski metric of signature -2. In all these solutions, the background gravitational field appears in a phase operator that acts on the wave function of the relative field-free equation. It is precisely from the corresponding gravity-induced phases that the optics of the particles considered follows.

The purpose of this paper is to show that the use of the solutions found presents advantages, in principle, over other current approximations. This is particularly true of gravitational lensing where large use is made of the thin lens approximation [7],[8] in which the lens is characterized by a surface mass density and the waves are scattered only at the thin lens plane. The phases can in fact be calculated by means of path integrals that can be integrated easily and exactly for most known metrics. Gravity acts along the complete length of the space-time paths and not only in a narrow region at the lens plane [9].

The plan of the paper is as follows. In Section II we give, for completeness, the solutions of (1)-(3) and show, in Section III, that they are gauge invariant. Geometrical and wave optics are discussed in Section IV. We summarize the results in Section V.

2. Solution of the spin-1 wave equation

To first order in $\gamma_{\mu\nu}$, (1)-(3) become

$$\nabla_\nu \nabla^\nu A_\mu \simeq (\eta^{\sigma\alpha} - \gamma^{\sigma\alpha}) A_{\mu,\alpha\sigma} - 2\Gamma_{\sigma,\mu\nu} A^{\sigma,\nu} - \frac{1}{2}\gamma_{\sigma\mu,\nu}{}^{\nu} A^\sigma = 0\,, \tag{4}$$

$$\nabla_\nu \nabla^\nu A_\mu - R_{\mu\alpha}A^\alpha \simeq (\eta^{\sigma\alpha} - \gamma^{\sigma\alpha})A_{\mu,\alpha\sigma} + 2\Gamma_{\sigma,\,\mu\nu}A^{\sigma,\nu} = 0\,, \tag{5}$$

$$(\nabla_\nu \nabla^\nu + m^2)A_\mu \simeq (\eta^{\sigma\alpha} - \gamma^{\sigma\alpha})A_{\mu,\alpha\sigma} + 2\Gamma_{\sigma,\,\mu\nu}A^{\sigma,\nu} - \frac{1}{2}\gamma_{\sigma\mu,\nu}{}^{\nu}A^\sigma + m^2 A_\mu = 0\,, \tag{6}$$

where ordinary differentiation of a quantity Φ is equivalently indicated by $\Phi_{,\alpha}$ or $\partial_\alpha\Phi$ and $\Gamma_{\sigma,\,\mu\nu} = 1/2(\gamma_{\sigma\mu,\nu} + \gamma_{\sigma\nu,\mu} - \gamma_{\mu\nu,\sigma})$ are the Christoffel symbols. In what follows we also use the notations $K^\alpha \equiv x^\alpha - z^\alpha$ and $B_{[\alpha\lambda\ldots\beta]} \equiv B_{\alpha\lambda\ldots\beta} - B_{\beta\lambda\ldots\alpha}$. In deriving (4)-(6), we have used the Lanczos-De Donder gauge condition

$$\gamma_{\alpha\nu,}{}^{\nu} - \frac{1}{2}\gamma^\sigma_{\sigma,\alpha} = 0\,. \tag{7}$$

The field $A_\mu(x)$ also satisfies the condition

$$\nabla_\mu A^\mu = 0\,. \tag{8}$$

Though (8) is strictly required only when $m = 0$, it is convenient to impose it also in the case of a massive spin-1 particle. Equations (4) and (6) can be handled simultaneously. As shown in Appendix A, their solution is

$$A_\mu(x) \simeq e^{-i\xi}a_\mu(x) \approx (1 - i\xi)a_\mu(x) = a_\mu(x) + \frac{1}{2}\int_P^x dz^\lambda \gamma_{\alpha\lambda}(z)\partial^\alpha a_\mu(x) \tag{9}$$

$$-\frac{1}{4}\int_P^x dz^\lambda \gamma_{[\alpha\lambda,\beta]}(z)K^{[\alpha}\partial^{\beta]}a_\mu(x) + \int_P^x dz^\lambda \Gamma_{\sigma,\,\mu\lambda}(z)a^\sigma(x)\,,$$

where a_μ satisfies the equation $\partial_\nu\partial^\nu a_\mu = 0$ in the case of (4), and $(\partial_\nu\partial^\nu + m^2)a_\mu = 0$ when (9) is a solution of (6). The solutions $a_\mu(x)$ used in this paper (Section IV) are either plane waves or spherical waves multiplied by $e^{-ik_0x^0}$. For this reason, it is convenient to group i with ξ.

Similarly, it can be verified by substitution that the Maxwell-de Rahm equation (5) has the solution [4]

$$A_\mu(x) \simeq e^{-i\eta}a_\mu(x) \approx (1 - i\eta)a_\mu(x) = a_\mu + \int_P^x dz^\lambda \gamma_{\alpha\mu}(z)\partial_\lambda a^\alpha(x) - \tag{10}$$

$$-\frac{1}{4}\int_P^x dz^\lambda \left[\gamma_{[\alpha\lambda,\beta]}(z)K^{[\alpha}\partial^{\beta]}\eta_{\beta\mu} - 2\gamma_{\alpha\lambda}\partial^\alpha\eta_{\beta\mu} + 2\gamma_{[\mu\lambda,\beta]}(z)\right]a^\beta(x)\,,$$

where again $\partial_\alpha\partial^\alpha a_\mu = 0$ and

$$-i\eta a_\mu(x) = -i\xi a_\mu(x) - \frac{\epsilon}{2}\int_P^x dz^\lambda[\gamma_{\alpha\mu,\lambda}(z)a^\alpha(x) - 2\gamma_{\alpha\mu}(z)\partial_\lambda]a^\alpha(x)\,. \tag{11}$$

The last two terms are typical of the Maxwell-de Rahm solution and have been tagged with the parameter ϵ ($\epsilon = 1$) for bookkeeping purposes. Solutions (9) and (10) must also satisfy (8). On using $\nabla^\alpha A_\alpha \simeq \eta^{\alpha\beta}A_{\alpha,\beta} - \gamma^{\alpha\beta}a_{\alpha,\beta}$ and (A.3), we obtain

$$\nabla^\alpha A_\alpha = f(x) \tag{12}$$

$$-f(x) \equiv \frac{1}{2}\gamma_{\sigma\mu}(a^{\sigma,\mu} + a^{\mu,\sigma}) - \frac{1}{2}\partial^\mu(\gamma_{\sigma\mu}a^\sigma) + \epsilon\left[\frac{dx^\lambda}{ds}\gamma_{\alpha\mu}(x)\partial^\mu\partial_\lambda a^\lambda(x)\right]_{s_1}^{s_2}.$$

By performing the gauge transformation $A_\sigma \to A_\sigma + \Lambda_{,\sigma}$, we find $\nabla^\sigma A_\sigma \to \nabla^\sigma A_\sigma + \nabla^\sigma \nabla_\sigma \Lambda + f = 0$, from which we obtain $\nabla^\sigma A_\sigma = 0$ provided

$$\nabla^\sigma \nabla_\sigma \Lambda + f = 0\,. \tag{13}$$

Finally, the addition of a term $m^2 A_\mu$ to the l.h.s. of (2) invalidates the solution (10) that can not, therefore, be extended to include de Rahm's term, except when $2\epsilon m^2 \gamma_{\mu\nu} a^\nu$ can be considered a second order quantity. It is known, on the other hand, that the term $R_{\mu\alpha} A^\alpha$ can be dropped in lensing when the wavelength λ of A^α is much smaller than the typical radius of curvature of the gravitational background, or when, in the weak field approximation, $R_{\mu\alpha} = -1/2 \gamma_{\mu\alpha,\nu}{}^{\nu} = 0$.

3. Gauge invariance

The first two integrals in (9) and (10) represent by themselves a solution of the Klein-Gordon equation $(\nabla_\mu \nabla^\nu + m^2)\phi = 0$. The additional terms must therefore be related to spin. In fact (9) can be written as

$$A_\mu = a_\mu - \frac{1}{2}\int_P^x dz^\lambda \gamma_{[\alpha\lambda,\beta]} K^\alpha \partial^\beta a_\mu + \frac{1}{2}\int_P^x dz^\lambda \gamma_{\alpha\lambda} \partial^\alpha a_\mu + \tag{14}$$

$$+\frac{i}{2}\int_P^x dz^\lambda \gamma_{[\alpha\lambda,\beta]} S^{\alpha\beta} a_\mu + \frac{i}{2}\int_P^x dz^\lambda \gamma_{\alpha\beta,\lambda} T^{\alpha\beta} a_\mu(x)\,,$$

where the tensors

$$(S^{\alpha\beta})_{\mu\nu} = -\frac{i}{2}(\delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu)\,, \quad (T^{\alpha\beta})_{\mu\nu} = -\frac{i}{2}(\delta^\alpha_\mu \delta^\beta_\nu + \delta^\beta_\mu \delta^\alpha_\nu) \tag{15}$$

have been written in matrix form as $S^{\alpha\beta}$ and $T^{\alpha\beta}$ and $S^{\alpha\beta} a_\mu = (S^{\alpha\beta})_{\mu\nu} a^\nu$, where a^ν is a column matrix. The matrices $S_i = 2\epsilon_{ijk} S^{jk}$ satisfying the commutation relation $[S_i, S_j] = i\epsilon_{ijk} S_k$, can be recognized as rotation matrices. These, and the matrix a^ν, are represented by

$$S_1 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, S_2 = i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$S_3 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, a^\nu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}.$$

We give the matrices $S^{\alpha\beta}$ in Appendix B and show the equivalence of matrix and tensor expressions. In the same Appendix we also show that by applying Stokes theorem to the line integrals on the r.h.s. of (14) extended to a closed space-time path Γ, we obtain, to first order [10],

$$A_\mu = \left(1 - \frac{i}{4}\int_\Sigma d\tau^{\sigma\delta} R_{\sigma\delta\alpha\beta} J^{\alpha\beta}\right) a_\mu \,, \qquad (16)$$

where $J^{\alpha\beta} = L^{\alpha\beta} + S^{\alpha\beta}$ is the total angular momentum of the spin-1 particle, $R_{\mu\nu\alpha\beta} = 1/2(\gamma_{\mu\beta,\nu\alpha} + \gamma_{\nu\alpha,\mu\beta} - \gamma_{\mu\alpha,\nu\beta} - \gamma_{\nu\beta,\mu\alpha})$ is the linearized Riemann tensor, and Σ is the surface bound by Γ. Equation (16) is unambiguous because, to first order, the values of $\gamma_{\alpha\beta}$ on Σ can be obtained from those on Γ independently of the path followed [10]. The term of (14) that contains $S_{\alpha\beta}$ gives rise to the Skrotskii effect [11],[12]. It follows from (16) that (9) is covariant and also invariant under the gauge transformations $\gamma_{\mu\nu} \to \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$ induced by the coordinate transformations $x_\mu \to x_\mu + \xi_\mu$ which are still allowed in the weak field approximation. The quantities ξ_μ, also small of first order, must satisfy the equation $\xi_{\mu,\sigma}{}^{\sigma} = 0$ because of the Lanczos-De Donder condition. It also follows from (16) that the term containing $T^{\alpha\beta}$ in (14) does not contribute to integrations over closed paths, behaves as a gauge term and may therefore be dropped.

The difference between (10) and (9) is represented by the two terms labelled by ϵ in (11). Of these, the first one vanishes when integrated over a closed path. To evaluate the second term we apply the transformations $\gamma_{\mu\nu} \to \gamma_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$, integrate over a closed path and choose, for simplicity, $\xi_\alpha = \xi_{\alpha 0} e^{-ilx}$ and $a_\alpha = a_{\alpha 0} e^{-ikx}$, where $l_\alpha l^\alpha = 0$, $k_\alpha k^\alpha = 0$. We obtain

$$-i\epsilon \oint dz^\lambda k_\lambda (\xi_{\alpha,\mu}(z) + \xi_{\mu,\alpha}(z)) a^\alpha(x) = -\frac{\epsilon}{2} l_{[\mu}\xi_{\alpha]\,0} a^\alpha(x) \oint dz^\lambda l_\lambda +$$

$$+ i\epsilon \left[l_\mu \oint dz^\lambda (\partial_\lambda a^\alpha(x) \xi_{\alpha 0}) + \xi_{\mu 0} \oint dz^\lambda (\partial_\lambda a^\alpha(x) l_\alpha) \right] = 0 \,,$$

because both integrands are of the type du and return to the initial value at the final point. This shows that the addition of the de Rahm term to the solution does not affect its invariance under the gauge transformations of $\gamma_{\mu\nu}$.

We must now consider the effect of the electromagnetic gauge transformations $a_\mu \to a_\mu + \Lambda_{,\mu}$ which have already been used to obtain (13). Equation (9) can be explicitly re-written in term of the gauge invariant electromagnetic field tensors $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{[\nu,\mu]}$, and $f_{\mu\nu} = a_{[\nu,\mu]}$ as follows

$$F_{\mu\nu} = f_{\mu\nu} - \frac{1}{2}\int_P^x dz^\lambda \gamma_{[\alpha\lambda,\beta]}(z) k^\alpha \partial^\beta f_{\mu\nu}(x) + \int_P^x dz^\lambda \gamma_{[\mu\lambda,}{}^{\beta]}(z) f_{\nu\beta}(x)$$

$$+ \frac{1}{2}\int_P^x dz^\lambda \gamma_{\alpha\lambda}(z) \partial^\alpha f_{\mu\nu}(x) + \gamma_\nu^\beta(x) f_{\mu\beta}(x) \,.$$

The additional terms that the Maxwell-de Rham solution (10) has relative to (9) can be re-written in the form

$$-\frac{\epsilon}{2}\int_P^x dz^\lambda \left[\gamma_{\beta\mu,\lambda}(z) a^\beta(x) - 2\gamma_\mu^\alpha(z) f_{\lambda\alpha}(x) - 2\gamma_\mu^\alpha(z) a_{\lambda,\alpha}(x)\right] \,. \qquad (17)$$

The second term of (17) is already invariant because it contains $f_{\lambda\alpha}$. The first term vanishes when the integration is over closed space-time paths. The last term transforms according to

$$\epsilon\int_P^x dz^\lambda \gamma^\alpha_\mu(z)a_{\lambda,\alpha}(x)\rightarrow \epsilon\int_P^x dz^\lambda \gamma^\alpha_\mu(z)(a_{\lambda,\alpha}+\Lambda_{,\lambda\alpha})\,, \tag{18}$$

can be re-written in the form

$$\epsilon\int_P^x dz^\lambda \frac{\partial^2}{\partial x^\lambda \partial x^\alpha}(\gamma_{\alpha\mu}(z)\Lambda(x))\,,$$

and vanishes when the integration is over closed space-time paths. Equation (10) is therefore invariant under electromagnetic gauge transformations.

4. Optics

4.1. Lensing

The correct general relativistic deviation of light rays in the gravitational field of a star follows immediately from the first two integrals in (9) and (10). We choose, e.g., a gravitational background represented by the Lense-Thirring metric, $\gamma_{00}=2\phi$, $\gamma_{ij}=2\phi\delta_{ij}$, $\phi=-GM/r$, and $\gamma_{0i}\equiv h_i=2GJ_{ij}x^j/r^3$, where $x^j=(x,y,z)$, $r=\sqrt{x^2+y^2+z^2}$ and J_{ij} is related to the angular momentum of the gravitational source. In particular, if the source rotates with angular velocity $\vec{\omega}=(0,0,\omega)$, then $h_1=4GMR^2\omega y/5r^3$, $h_2=-4GMR^2\omega x/5r^3$, .

Without loss of generality, we assume that photons propagate along the z-direction, hence $k^\alpha\simeq(k,0,0,k)$. Moreover, photons propagate along null geodesics, from which it follows that $ds^2=0$, or $dt=dz$. Using plane waves for $a_\mu(x)=a^0_\mu exp(-ik_\alpha x^\alpha)$, the solution has the form $A_\mu=e^{-i\chi}a^0_\mu$, where

$$\chi=k_\alpha x^\alpha-\frac{1}{4}\int_P^x dz^\lambda\gamma_{[\alpha\lambda,\beta]}(z)K^{[\alpha}k^{\beta]}+\frac{1}{2}\int_P^x dz^\lambda\gamma_{\alpha\lambda}(z)k^\alpha\,. \tag{19}$$

We can define the photon momentum as

$$\tilde{k}_\alpha=\frac{\partial\chi}{\partial x^\alpha}\,. \tag{20}$$

It is easy to show that χ satisfies the eikonal equation $g^{\alpha\beta}\chi_{,\alpha}\chi_{,\beta}=0$. We can also prove, by direct substitution, that the inclusion of de Rahm's terms (see (11)) in (19) does not invalidate the eikonal equation. Geometrical optics is not therefore affected by the term $R_{\mu\alpha}A^\alpha$ in (2).

For the Lense-Thirring metric, χ is given by

$$\begin{aligned}\chi\simeq\; &-\frac{k}{2}\int_P^Q\{(x-x')\phi_{,z'}dx'+(y-y')\phi_{,z'}dy'-\\ &-2[(x-x')\phi_{,x'}+(y-y')\phi_{,y'}]dz'\}+k\int_P^Q dz'\phi-\\ &-\frac{k}{2}\int_P^Q\{[(x-x')(h_{1,z'}-2h_{3,x'})+(y-y')(h_{2,z'}-2h_{3,y'})]\,dz'-\end{aligned} \tag{21}$$

$$- \ [(x-x')h_{1,x'} + (y-y')h_{1,y'}]\,dx' - [(x-x')h_{2,x'} + (y-y')h_{2,y'}]\,dy'\}$$
$$+ \ \frac{k}{2}\int_P^Q [2h_3 dz' + h_1 dx' + h_2 dy']\ , \tag{22}$$

where P is the point at which the photons are generated, and Q is a generic point along their trajectory. The components of the photon momentum are therefore

$$\tilde{k}_1 = 2k\int_P^Q \left(-\frac{1}{2}\frac{\partial\phi}{\partial z}\,dx - \frac{1}{2}\frac{\partial h_2}{\partial x}\,dy + \frac{\partial(\phi+h_3)}{\partial x}dz\right) -$$
$$- \frac{k}{2}(h_1(Q) - h_1(P))\,, \tag{23}$$
$$\tilde{k}_2 = 2k\int_P^Q \left(-\frac{1}{2}\frac{\partial\phi}{\partial z}\,dy + \frac{1}{2}\frac{\partial h_1}{\partial y}\,dx + \frac{\partial(\phi+h_3)}{\partial y}dz\right) +$$
$$+ \frac{k}{2}(h_2(Q) - h_2(P))\,, \tag{24}$$
$$\tilde{k}_3 = k(1+\phi+h_3)\,. \tag{25}$$

We also have $\tilde{\mathbf{k}} = \tilde{\mathbf{k}}_\perp + \tilde{k}_3\,\mathbf{e}_3\,,\quad \tilde{\mathbf{k}}_\perp = \tilde{k}_1\,\mathbf{e}_1 + \tilde{k}_1\,\mathbf{e}_2\,,$ where $\tilde{\mathbf{k}}_\perp$ is the component of the momentum orthogonal to the direction of propagation of the photons.

Since only phase differences are physical, it is convenient to choose the space-time path by placing the photon source at distances that are very large relative to the dimensions of the lens, and the generic point is located along the z direction. We therefore replace Q with z, where $z \gg x, y$. Using the expression for $h_{1,2}$ we find that their contribution is negligible and (23)-(25) simplify to

$$\tilde{k}_1 = 2k\int_{-\infty}^{z} \frac{\partial(\phi+h_3)}{\partial x}dz\,, \tag{26}$$
$$\tilde{k}_2 = 2k\int_{-\infty}^{z} \frac{\partial(\phi+h_3)}{\partial y}dz\,, \tag{27}$$
$$\tilde{k}_3 = k(1+\phi+h_3)\,. \tag{28}$$

From (26)-(28) we can determine the deflection angle θ. Let us analyze first the case of non-rotating lenses, i.e. $h_3 = 0$. We get

$$\tilde{k}_1 \sim k\,\frac{2GM}{R^2}\,x\left(1+\frac{z}{r}\right)\ , \tag{29}$$
$$\tilde{k}_2 \sim k\,\frac{2GM}{R^2}\,y\left(1+\frac{z}{r}\right)\ , \tag{30}$$
$$\tilde{k}_3 = k(1+\phi+h_3)\,, \tag{31}$$

where $R = \sqrt{x^2+y^2}$. By defining the deflection angle as $\tan\theta = \tilde{k}_\perp/\tilde{k}_3$, it follows that

$$\tan\theta \sim \theta \sim \frac{2GM}{R}\left(1+\frac{z}{r}\right)\ . \tag{32}$$

In the limit $z \to \infty$ we obtain the usual Einstein result $\theta_M \sim 4GM/R$. The remaining term of (14) contains $S_{\alpha\beta}$ and therefore is a spin effect. It is usually referred to as the Skrotskii effect and represents a rotation of the plane of polarization of the incoming photon beam.

A general expression for the index of refraction n can also be derived from (19),(20) and $n = \tilde{k}/\tilde{k}_0 = \chi_{,3}/\chi_{,0}$.

4.2. Wave effects in gravitational lensing

We consider the propagation of light waves in a background metric represented by $\gamma_{00} = 2U(\rho)$, $\gamma_{ij} = 2U(\rho)\delta_{ij}$, where $U(\rho) = -GM/\rho$ is the gravitational potential of the lens. Wave optics effects can be calculated by using the type of double slit arrangement indicated in Fig.1. We neglect all spin effects. This limits the calculation of the phase difference to the first two terms in (9) and (10). We also assume for simplicity that $k^1 = 0$, so that propagation is entirely in the (x^2, x^3)-plane and the set-up is planar. We use a solution of the equation $\partial^\gamma\partial_\gamma a_\alpha = 0$ of the form $a_\mu = a^0_\mu e^{-ik_\sigma x^\sigma}/r = a^0_\mu e^{-ik_0 x^0} e^{-ik_i x^i}/r = a^0_\mu e^{-ik_0 x^0} e^{ikr}/r$, valid everywhere outside the light source ($r = 0$), which is the region of physical interest in lensing. We neglect the contribution from the de Rahm term because, for the metric used, $R_{\sigma\mu}A^\sigma = -1/2\gamma_{\sigma\mu,\nu}{}^{\nu}A^\sigma = -\nabla^2(GM/\rho)A_\mu$ which vanishes for $\rho \neq 0$.

The corresponding wave amplitude ϕ is therefore

$$\phi(x) = \frac{e^{-ik_\sigma x^\sigma}}{2r}\left\{2 + \int_S^x \left[dz^0\gamma_{00}\Pi^0 + dz^2\gamma_{22}\Pi^2 + dz^3\gamma_{33}\Pi^3\right] - \right. \tag{33}$$

$$\left.\int_S^x \left[dz^0\left(\gamma_{00,2}K^{[0}\Pi^{2]} + \gamma_{00,3}K^{[0}\Pi^{3]}\right) + dz^2\gamma_{22,3}K^{[2}\Pi^{3]} + dz^3\gamma_{33,2}K^{[3}\Pi^{2]}\right]\right\}$$

where $\Pi^0 = -ik$, $\Pi^i = -ik^i - x^i/r$, and we have taken into account the fact that γ_{11} plays no role in the planar arrangement chosen. The change in phase must now be calculated along the different paths SP+PO and SL+LO according to $\Delta\tilde{\phi} = (\phi_{SLO} - \phi_{SPO})/(e^{-ik_\sigma x^\sigma}/r)$ and taking into account the values of Π^i in the various intervals. The total change in phase is $\Delta\tilde{\phi} = \Delta\tilde{\phi}_{SL} + \Delta\tilde{\phi}_{LO} - \Delta\tilde{\phi}_{SP} - \Delta\tilde{\phi}_{PO}$. The details of the calculation of $\Delta\tilde{\phi}$ along the path segments are given in Appendix C. All integrations in (C.2), (C.4). (C.6) and (C.8) can be performed exactly. All results can be expressed in terms of physical variables r_s, r_0, b^+, b^-, and s and lensing variables D_s, D_{ds}, D_d, θ^+, θ^-, and β. We obtain

$$\begin{aligned}\Delta\tilde{\phi} = \tilde{y}\Big\{&\ln\left(-\sqrt{D^2_{dS} + (s+b^-)^2} + b^-\cos\gamma + r_S\right) - \ln\left(b^-\left(1+\cos\gamma\right)\right)\\ &+ \ln\left(b^+\left(1-\cos\varphi^+\right)\right) - \ln\left(r_S - r_L - b^+\cos\varphi^+\right)\\ &+ \ln\left(b^- + r_0\cos\theta^- - \sqrt{b^{-\,2} + r_0^2}\right) - \ln\left(r_0\left(1+\cos\theta^-\right)\right)\\ &+ \ln\left(r_0\left(1+\cos\theta^+\right)\right) - \ln\left(b^+ + r_0\cos\theta^+ - \sqrt{b^{+\,2}+r_0^2}\right)\Big\}\,,\end{aligned} \tag{34}$$

where $r_S^2 = b^{+\,2} + r_L^2 + 2b^+ r_L\cos\varphi^+$, $r_L^2 = D^2_{dS} + (s-b^+)^2$, $\varphi^+ + \alpha^+ + \alpha^- + \gamma - \theta^+ - \theta^- = \pi$ and $\tilde{y} = 2GMk$. Equation (34) agrees with that obtained in [9] for plane waves and can be applied to the optics of any particle when spin effects are neglected.

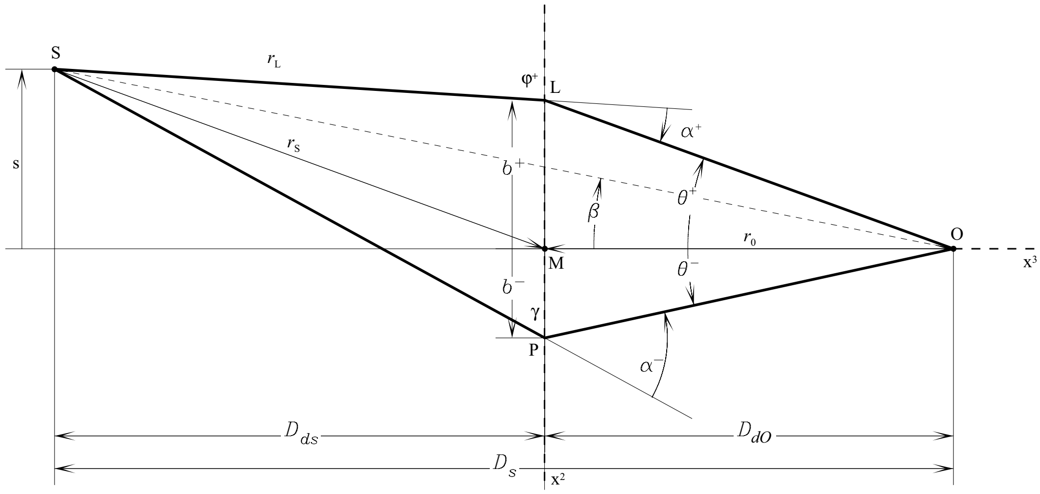

Figure 1. Geometry of a two-image gravitational lens or, equivalently, of a double slit interference experiment. The solid lines represent the particle paths between the particle source at S and the observer at O. M is the spherically symmetric gravitational lens. S, M, O and the particle paths lie in the same plane. The physical variables are $r_S, r_0, b^{\pm}, s$, while the lensing variables are indicated by $D_{dS}, D_{dO}, D_S, \theta^{\pm}, \beta$. The deflection angles are indicated by $\alpha^{\pm}$.

The contributions $\tilde{\chi}$ of the real parts of Π^2 and Π^3 to (C.2), (C.4), (C.6) and (C.8) are listed below by path segment

$$\tilde{\chi}_{SL} = \frac{GM}{r_L^2}\left\{ b^+ \frac{-2+3\cos^2\varphi^+}{\cos\varphi^+} \ln\frac{r_S - r_L - b^+\cos\varphi^+}{b^+\left(1-\cos\varphi^+\right)} - \right. \\ \left. - \frac{2r_L b^+ \sin^2\varphi^+}{r_S\cos\varphi^+}\right\}, \tag{35}$$

$$\tilde{\chi}_{LO} = \frac{GM}{R_1^2}\left\{ r_0 \frac{2-3\cos^2\theta^+}{\cos\theta^+} \ln\frac{b^+ - \sqrt{r_0^2 + b^{+\,2}} + b^+\cos\theta^+}{r_0\left(1+\cos\theta^+\right)} + \right. \\ \left. + \frac{2r_0\sqrt{r_0^2+b^{+\,2}}\sin^2\theta^+}{b^+\cos\theta^+}\right\}, \tag{36}$$

$$\tilde{\chi}_{SP} = \frac{GM}{R^2}\left\{ b^- \frac{2-3\cos^2\gamma}{\cos\gamma} \ln\frac{r_S - \sqrt{D_{dS}^2 + \left(s+b^-\right)^2} + b^-\cos\gamma}{b^-\left(1+\cos\gamma\right)} + \right. \\ \left. + \frac{2b^-\sqrt{D_{dS}^2 + \left(s+b^-\right)^2}\sin^2\gamma}{r_S\cos\gamma}\right\}, \tag{37}$$

$$\tilde{\chi}_{PO} = \frac{GM}{R_2^2}\left\{ r_0 \frac{2-3\cos^2\theta^-}{\cos\theta^-} \ln\frac{b^- - \sqrt{r_0^2 + b^{-\,2}} + r_0\cos\theta^-}{r_0\left(1+\cos\theta^-\right)} + \right. \\ \left. + \frac{2r_0\sqrt{r_0^2+b^{-\,2}}\sin^2\theta^-}{b^-\cos\theta^-}\right\}. \tag{38}$$

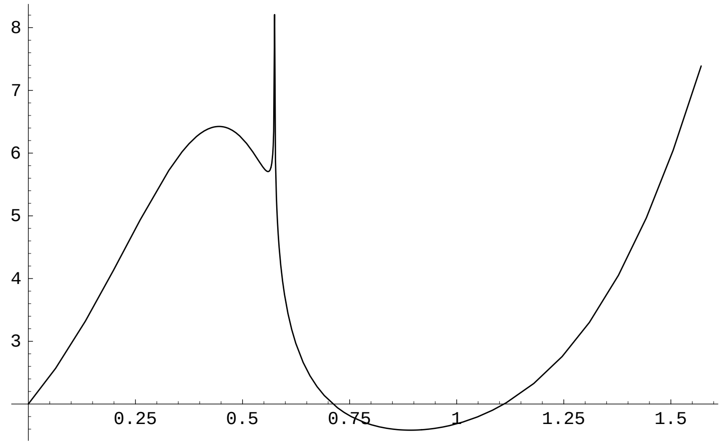

Figure 2. The plot of $F/(4GM/R_1)$ versus θ has a singularity at $\theta = 0.57483$ where the argument of the logarithm in (42) vanishes. For all other values of θ and for reasonable values of M and R_1, the amplification factor $F \approx 1$.

Contrary to $\Delta\tilde{\phi}$, $\tilde{\chi} = \tilde{\chi}_{SL} + \tilde{\chi}_{LO} + \tilde{\chi}_{SP} + \tilde{\chi}_{PO}$ is completely independent of $\tilde{y}$.

The meaning of (35)-(38) becomes evident by recalling that the total wave amplitude has the form

$$\phi(x) = \frac{e^{ikr}}{r} \exp\left[-ik_0 x^0 - i\xi\,(x) + \tilde{\chi}\,(x)\right] \tag{39}$$

and that the ratio

$$F \equiv \frac{\phi\phi^*}{\phi_0\phi_0^*} = e^{\tilde{\chi}+\tilde{\chi}^*} \tag{40}$$

is known as the amplification factor. As an example, let us consider the simpler case of particles coming from $x^3 = -\infty$ ($\varphi^+ = \gamma = \pi/2$) with $b^+ = b^- = s \equiv b\,, r_0 = D_{dS}$. Then Fig.1 gives $\theta^+ = \theta^- \equiv \theta$, and $R_1 = R_2$. In this case $\tilde{\chi}_{SL} = -\tilde{\chi}_{SP}\,, \tilde{\chi}_{LO} = \tilde{\chi}_{PO}$ and $\Delta\tilde{\phi} = 0$. A simple calculation shows that the probability density of finding a particle at O is

$$\phi\phi^* \propto \frac{4}{r^2} e^{\tilde{\chi}+\tilde{\chi}^*} \cos^2\frac{\Delta\tilde{\phi}}{2} = \frac{4}{r^2} e^{\tilde{\chi}+\tilde{\chi}^*}\,, \tag{41}$$

where

$$\tilde{\chi} = \frac{4GM}{R_1}\left\{\sin\theta + \frac{2-3\cos^2\theta}{2}\ln\frac{\sin\theta - 1 + \sin\theta\cos\theta}{\cos\theta(1+\cos\theta)}\right\}\,. \tag{42}$$

The quantity $F/(4GM/R_1)$ as a function of θ is represented in Fig.2. It has a singularity at the value $\tan\theta = 0.647799$ for which the argument of the logarithm in (42) vanishes. We find $F \approx 1$ for all other values of θ and for all reasonable values of the parameters involved.

5. Conclusions

Covariant wave equations for massless and massive spin-1 particles can be solved exactly to first order in $\gamma_{\mu\nu}$. The solutions are covariant and invariant with respect to the gauge transformations of a_μ and $\gamma_{\mu\nu}$ and are known when a solution of the free wave equation is known.

The external gravitational field only appears in the phase of the wave function. We have shown elsewhere [2],[9] that the spin-gravity coupling and Mashhoon's helicity-rotation interaction [13],[14] follow from $-i\xi a_\mu$ and $-i\eta a_\mu$. According to equations (10), (11) and (14), the spin term $S_{\alpha\beta}$ finds its origin in the anti-symmetric part of the space-time connection. In the case of fermions, $S_{\alpha\beta}$ is accounted for by the spinorial connection [6]. The term of (14) that contains $S_{\alpha\beta}$ gives rise to the Skrotskii effect.

If the solution of the free wave equation is a plane wave, then the change in phase is entirely given by $\Delta\tilde{\varphi}$. It is from the gravity-induced phase that we have derived the optics of the particles. The phase can be calculated by means of path integrations that can be frequently performed in an exact way, when the metric of the gravitational background is assigned.

From the phases we have derived the geometrical optics of the particles and verified that their deflection is that predicted by general relativity. In addition, the background gravitational field acts as a medium whose index of refraction can be calculated for any metric from (19), (20) and $n = \tilde{k}/\tilde{k}_0$.

Wave optics can also be extracted from the gravity-induced phases. In lensing this is normally accomplished by means of a thin lens model whereby the gravitational potential vanishes everywhere except on the lens plane. In the present approach the gravitational field acts all along the particle's trajectory from source to observer and no thin lens approximation is required.

The exact expression (34) for the phase change $\Delta\tilde{\phi}$, gives rise to interference and diffraction phenomena. In gravitational lensing, wave effects for a point source depend on the parameter $\tilde{y}$ and different values require, in general, different approximations to the solution of the wave equation. In particular, diffraction effects are expected to be considerable when $\tilde{y} \simeq 1$. In the present approach and in the configuration of Fig. 1, (34) applies regardless of the value of $\tilde{y}$ when $\Delta\tilde{\phi}$ is real.

The extension of our results to include spherical wave solutions of Helmholtz equation yields an amplification factor (40) whose value can be calculated exactly. We find that essentially $F \approx 1$ for all reasonable values of the parameters involved.

Because spin has been neglected in the example given, the same results can also be applied to the case of gravitational lensing of gravitational waves[15].

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Appendix A.

In order to prove that (9) is a solution of (6) and therefore also of (4) when $m = 0$, we make use of the formula

$$\partial_\tau \int_P^x dz^\lambda G_\lambda(z,x) = G_\tau(x,x) + \int_P^x dz^\lambda \partial_\tau G_\lambda(z,x)\,. \tag{A.1}$$

By writing (9) in the form

$$A_\mu(x) \simeq (1 - i\xi(x))a_\mu(x) \tag{A.2}$$

and differentiating $-i\xi a_\mu$ once according to (A.1), we obtain

$$\partial_\tau(-i\xi a_\mu) = -\frac{1}{4}\int_P^x dz^\lambda \gamma_{[\alpha\lambda,\beta]}(z)[\delta_\tau^{[\alpha}\partial^{\beta]}a_\mu(x) + K^{[\alpha}\partial_\tau\partial^{\beta]}a_\mu(x)] \tag{A.3}$$
$$+\frac{1}{2}\gamma_{\alpha\tau}(x)\partial^\alpha a_\mu(x) + \frac{1}{2}\int_P^x dz^\lambda \gamma_{\alpha\lambda}(z)\partial_\tau\partial^\alpha a_\mu(x)$$
$$+\Gamma_{\sigma,\,\mu\tau}(x)a^\sigma(x) + \int_P^x dz^\lambda \Gamma_{\sigma,\,\mu\lambda}(z)\partial_\tau a^\sigma(x)\,.$$

Differentiating (A.3) once more and contracting the indices of differentiation, we find

$$\partial^\tau\partial_\tau(-i\xi a_\mu) = -\frac{1}{2}(\gamma^\sigma_{[\sigma,\beta]}\partial^\beta a_\mu - \frac{1}{2}\int_P^x dz^\lambda \gamma_{[\tau\lambda,\beta]}\partial^\tau\partial^\beta a_\mu \tag{A.4}$$
$$-\frac{1}{2}\int_P^x dz^\lambda \gamma_{[\alpha\lambda,\beta]}K^\alpha\partial^\tau\partial_\tau\partial^\beta a_\mu + \frac{1}{2}\gamma^\tau_{\alpha,\tau}\partial^\alpha a_\mu + \gamma_{\alpha\tau}\partial^\tau\partial^\alpha a_\mu$$
$$+\frac{1}{2}\int_P^x dz^\lambda \gamma_{\alpha\lambda}\partial^\tau\partial_\tau\partial^\alpha a_\mu + \partial^\tau(\Gamma_{\sigma,\,\mu\tau})a^\sigma + 2\Gamma_{\sigma,\,\mu\tau}\partial^\tau a^\sigma + \int_P^x dz^\lambda \Gamma_{\sigma,\,\mu\lambda}\partial^\tau\partial_\tau a^\sigma$$

We assume that $a_\mu(x)$ is well behaved and that successive differentiations applied to it commute. Then the second term on the r.h.s. of (A.4) vanishes because two symmetric indices are contracted on two anti-symmetric indices. The third, sixth and last terms may be written as $-m^2(-i\xi)a_\mu$. We find

$$\partial^\tau\partial_\tau(-i\xi a_\mu) = -m^2(-i\xi)a_\mu - \frac{1}{2}(\gamma^\sigma_{\sigma,\beta} - 2\gamma^\tau_{\beta,\tau})\partial^\beta a_\mu + \gamma_{\alpha\tau}\partial^\tau\partial^\alpha a_\mu - \tag{A.5}$$
$$+\partial^\tau(\Gamma_{\sigma,\,\mu\tau})a^\sigma + 2\Gamma_{\sigma,\,\mu\tau}\partial^\tau a^\sigma\,.$$

The second term on the r.h.s. of (A.5) may be dropped on account of (7). We now substitute the solution (A.2) into the r.h.s. of (6) and keep only first order terms. We get

$$\nabla_\nu\nabla^\nu A_\mu + m^2 A_\mu \simeq (\eta^{\sigma\alpha} - \gamma^{\sigma\alpha})\partial_\sigma\partial_\alpha a_\mu + \partial^\sigma\partial_\sigma(-i\xi)a_\mu \tag{A.6}$$
$$+2\Gamma_{\sigma,\,\mu\nu}\partial^\nu a^\sigma - \frac{1}{2}\gamma_{\sigma\mu,\nu}{}^\nu a^\sigma + m^2(1 - i\xi)a_\mu\,,$$

and, on using (A.5),

$$\nabla_\nu\nabla^\nu A_\mu + m^2 A_\mu \simeq -\frac{1}{2}[(\gamma^\sigma_{\sigma,\beta} - 2\gamma^\tau_{\beta,\tau})\partial^\beta a_\mu + (\gamma_{\mu\tau,\sigma}{}^\tau - \gamma_{\sigma\tau,\mu}{}^\tau)a^\sigma] = \tag{A.7}$$
$$= -\frac{1}{2}[(\gamma^\sigma_{\sigma,\beta} - 2\gamma^\tau_{\beta,\tau})\partial^\beta a_\mu + (-\gamma^\tau_{\tau,\mu\sigma} + \gamma^\tau_{\tau,\sigma\mu})a^\sigma] = 0$$

on account of (7).

Appendix B.

The matrices $S^{\alpha\beta}$ can be easily obtained from their definition (15) where μ and ν label, respectively, rows and columns. We find, for instance,

$$S^{01} = -\frac{i}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, S^{02} = -\frac{i}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us define the quantity $A_{\alpha\beta} \equiv dz^{\lambda}\gamma_{[\alpha\lambda,\beta]}$. On using (15), we obtain

$$A_{\alpha\beta}(S^{\alpha\beta})^{\mu\nu}a_{\nu} = \tag{B.1}$$

$$-\frac{i}{2}A_{\alpha\beta}\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\beta\mu}\eta^{\alpha\nu}\right)a_{\nu} = -i\left(A^{\mu 0}a_0 + A^{\mu 1}a_1 + A^{\mu 2}a_2 + A^{\mu 3}a_3\right),$$

which is a four-vector and can also be understood as a single-column matrix whose rows are obtained by setting $\mu = 0, 1, 2, 3$. On the other hand, using the matrices $S^{\alpha\beta}$, we also find

$$(A_{\alpha\beta}S^{\alpha\beta})(a^{\mu}) = \tag{B.2}$$

$$= 2\left[A_{01}S^{01} + A_{02}S^{02} + A_{03}S^{03} + A_{12}S^{12} + A_{13}S^{13} + A_{23}S^{23}\right]a^{\mu} =$$

$$= -i\begin{pmatrix} A_{01}a^1 + A_{02}a^2 + A_{03}a^3 \\ -A_{01}a^0 + A_{12}a^2 + A_{13}a^3 \\ -A_{02}a^0 - A_{12}a^1 + A_{23}a^3 \\ -A_{03}a^0 - A_{13}a^1 - A_{23}a^2 \end{pmatrix},$$

which coincides with the final matrix in (B.1). Therefore a_{ν} is understood as a four-vector in (B.1), while in (B.2) (a^{μ}) is a single-column matrix. This proves the equivalence of the matrix and tensor expressions used in (14) and (15). We now apply Stokes theorem to

$$A_{\mu}(x) \simeq a_{\mu}(x) + \oint_{\Gamma} dz^{\lambda}G_{\lambda}(z,x)a_{\mu}(x)\,, \tag{B.3}$$

where Γ is a closed path in Minkowski space. Following textbook procedures [10], we obtain

$$A_{\mu} \simeq a_{\mu} + \frac{1}{2}\int_{\Sigma} d\tau^{\sigma\delta}\left(\frac{\partial}{\partial z^{\delta}}G_{\sigma} - \frac{\partial}{\partial z^{\sigma}}G_{\delta}\right)a_{\mu}\,, \tag{B.4}$$

where

$$G_{\sigma}a_{\mu} = -\frac{1}{2}\gamma_{[\alpha\sigma,\beta]}(z)K^{\alpha}\partial^{\beta}a_{\mu}(x) + \frac{1}{2}\gamma_{\alpha\sigma}(z)\partial^{\alpha}a_{\mu}(x) + \tag{B.5}$$

$$+\frac{i}{4}\gamma_{[\alpha\sigma,\beta]}(z)S^{\alpha\beta}a_{\mu}(x) + \frac{i}{4}\gamma_{\alpha\beta,\sigma}(z)T^{\alpha\beta}a_{\mu}(x)\,,$$

and Σ is a surface bound by Γ. We then find

$$G_{\sigma,\delta}a_\mu = [-\frac{1}{2}\gamma_{[\alpha\sigma,\beta]\delta}K^\alpha\partial^\beta + \frac{1}{2}\gamma_{[\delta\sigma,\beta]}\partial^\beta +$$

$$+\frac{1}{2}\gamma_{\alpha\sigma,\delta}\partial^\alpha + \frac{i}{4}\gamma_{[\alpha\sigma,\beta]\delta}S^{\alpha\beta} + \frac{i}{4}\gamma_{\alpha\beta,\sigma\delta}T^{\alpha\beta}]a_\mu\,,$$

and again

$$d\tau^{\sigma\delta}(G_{\sigma,\delta}-G_{\delta,\sigma})\frac{a_\mu}{2} = \frac{1}{4}[-(\gamma_{\alpha\sigma,\beta\delta}+\gamma_{\beta\delta,\alpha\sigma}-\gamma_{\alpha\delta,\beta\sigma}-\gamma_{\beta\sigma,\alpha\delta})(x^\alpha - z^\alpha)\partial^\beta \tag{B.6}$$

$$+\frac{i}{2}(\gamma_{\alpha\sigma,\beta\delta}+\gamma_{\beta\delta,\alpha\sigma}-\gamma_{\alpha\delta,\beta\sigma}-\gamma_{\beta\sigma,\alpha\delta})S^{\alpha\beta}]a_\mu d\tau^{\sigma\delta} =$$

$$= d\tau^{\sigma\delta}\frac{1}{2}[-R_{\alpha\beta\delta\sigma}(x^\alpha - z^\alpha)\partial^\beta + \frac{i}{2}R_{\alpha\beta\delta\sigma}S^{\alpha\beta}]a_\mu\,.$$

By introducing the angular momentum generators of the Lorentz group

$$L^{\alpha\beta} = (x^\alpha - z^\alpha)(i\partial^\beta) - (x^\beta - z^\beta)(i\partial^\alpha) \tag{B.7}$$

into (B.6) and using (B.4), we find

$$A_\mu \simeq a_\mu - \frac{i}{4}\int_\Sigma d\tau^{\sigma\delta}R_{\alpha\beta\sigma\delta}(L^{\alpha\beta}+S^{\alpha\beta})a_\mu\,, \tag{B.8}$$

which corresponds to (16).

Appendix C.

In order to calculate $\Delta\tilde{\phi}$, it is convenient to transform all space integrations into integrations over z^0. Along SL we have

$$U = \frac{-GM}{q_{SL}(z^0)^{1/2}}\,, \quad q_{SL}(z^0) \equiv (r_L - z^0)^2 + b^{+\,2} + 2(r_L - z^0)b^+\cos\varphi^+\,, \tag{C.1}$$

$$k^2 = k\cos\varphi^+, \quad k^3 = k\sin\varphi^+\,,$$

$$\Pi^2 = -ik\cos\varphi^+ + \frac{\cos\varphi^+}{r_L}\,, \quad \Pi^3 = -ik\sin\varphi^+ + \frac{\sin\varphi^+}{r_L}\,, \quad \text{at} \quad z^0 = r_L\,,$$

where $r_L\sin\varphi^+ = D_{dS}$. We find

$$-\frac{\Delta\tilde{\phi}_{SL}}{GM} = \int_0^{r_L} dz^0 q_{SL}(z^0)^{-1/2}[\Pi^0 + \cos\varphi^+\Pi^2 + \sin\varphi^+\Pi^3] + \tag{C.2}$$

$$+\,2\int_0^{r_L} dz^0 q_{SL}(z^0)^{-3/2}(-z^0 + r_L + b^+\cos\varphi^+)(r_L - z^0)\,\times$$

$$\times\,[-\Pi^2\frac{\sin^2\varphi^+}{\cos\varphi^+} - \Pi^3\frac{\cos^2\varphi^+}{\sin\varphi^+} + \Pi^0]\,,$$

where Π^2 and Π^3 must take the values specified in (C.1) at the upper integration limit. Analogously, for LO we get

$$U = -\frac{GM}{q_{LO}(z^0)^{1/2}}\,, k^2 = k\sin\theta^+\,, k^3 = k\cos\theta^+\,, R_1 = \sqrt{r_0^2 + b^{+2}}\,, \tag{C.3}$$

$$q_{LO}(z^0) \equiv (\sqrt{r_0^2 + b^{+\,2}} - z^0 + r_L)^2 + r_0^2 -$$

$$-2r_0(\sqrt{r_0^2 + b^{+\,2}} - z^0 + r_L)\cos\theta^+$$

$$\Pi^2 = -ik\sin\theta^+ + \frac{\sin\theta^+}{R_1}\,, \quad \Pi^3 = -ik\cos\theta^+ + \frac{\cos\theta^+}{R_1}\,, \text{at } z^0 = r_L + R_1$$

and the change in phase is

$$-\frac{\Delta\tilde{\phi}_{LO}}{GM} = \int_{r_L}^{r_L+R_1} dz^0 q_{LO}^{-1/2}[\Pi^0 + \sin\theta^+\Pi^2 + \cos\theta^+\Pi^3] + \qquad \text{(C.4)}$$

$$+ 2\int_{r_L}^{r_L+R_1} dz^0 q_{LO}(z^0)^{-3/2}(z^0 - R_1 + r_0\cos\theta^+) \times$$

$$\times (r_L + R_1 - z^0)\left[-\Pi^2\frac{\cos^2\theta^+}{\sin\theta^+} - \Pi^3\frac{\sin^2\theta^+}{\cos\theta^+} + \Pi^0\right].$$

Again, Π^2 and Π^3 take the values calculated in (C.3) at the upper integration limit.

For SP we find

$$U = -\frac{GM}{q_{SP}(z^0)^{1/2}}\,, \quad q_{SP}(z^0) \equiv b^{-\,2} + (R - z^0)^2 - 2(R - z^0)b^-\cos\gamma\,, \qquad \text{(C.5)}$$

$$k^2 = k\cos\gamma, \quad k^3 = k\sin\gamma\,,$$

$$\cos\gamma = \frac{R^2 + b^{-\,2} - r_s^2}{2b^-R}\,, R = \sqrt{D_{dS} + (s + b^-)^2}\,,$$

$$\Pi^2 = -ik\cos\gamma + \frac{\cos\gamma}{R}\,, \quad \Pi^3 = -ik\sin\gamma + \frac{\sin\gamma}{R}\,, \quad \text{at} \quad z^0 = R$$

and the corresponding change in phase is given by

$$-\frac{\Delta\tilde{\phi}_{SP}}{GM} = \int_0^R dz^0 q_{SP}(z^0)^{-1/2}[\Pi^0 + \cos\gamma\Pi^2 + \sin\gamma\Pi^3] \qquad \text{(C.6)}$$

$$+ 2\int_0^R dz^0 q_{SP}(z^0)^{-3/2}(-z^0 + R - b^-\cos\gamma)(R - z^0) \times$$

$$\times \left[-\Pi^2\frac{\sin^2\gamma}{\cos\gamma} - \Pi^3\frac{\cos^2\gamma}{\sin\gamma} + \Pi^0\right].$$

Finally, for PO we obtain

$$U = -\frac{GM}{q_{PO}(z^0)^{1/2}}\,, \quad k^2 = -k\sin\theta^-, \quad k^3 = k\cos\theta^-\,, \qquad \text{(C.7)}$$

$$q_{PO}(z^0) \equiv r_0^2 + (R_2 + R - z^0)^2 - 2r_0(R_2 + R - z^0)\cos\theta^-\,,$$

$$R_2 = \sqrt{r_0^2 + b^{-\,2}}\,,$$

$$\Pi^2 = ik\sin\theta^- - \frac{\sin\theta^-}{R_2}\,, \quad \Pi^3 = -ik\cos\theta^- + \frac{\cos\theta^-}{R_2}\,, \quad \text{at} \quad z^0 = R + R_2$$

and the corresponding change in phase is

$$
\begin{aligned}
-\frac{\Delta\tilde{\phi}_{PO}}{GM} = & \int_R^{R+R_2} dz^0 q_{PO}^{-1/2}[\Pi^0 - \sin\theta^-\Pi^2 + \cos\theta^-\Pi^3] \\
& + 2\int_R^{R+R_2} dz^0 q_{PO}(z^0)^{-3/2}(z^0 - R_2 - R + r_0\cos\theta^-) \times \\
& \times (R + R_2 - z^0)\left[\Pi^2\frac{\cos^2\theta^-}{\sin\theta^-} - \Pi^3\frac{\sin^2\theta^-}{\cos\theta^-} + \Pi^0\right] .
\end{aligned} \tag{C.8}
$$

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