Gravitational solitons and C^0 vacuum metrics in five-dimensional Lovelock gravity

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Abstract

Junction conditions for vacuum solutions in five-dimensional Einstein-Gauss-Bonnet gravity are studied. We focus on those cases where two spherically symmetric regions of space-time are joined in such a way that the induced stress tensor on the junction surface vanishes. So a spherical vacuum shell, containing no matter, arises as a boundary between two regions of the space-time. A general analysis is given of solutions that can be constructed by this method of geometric surgery. Such solutions are a generalized kind of spherically symmetric empty space solutions, described by metric functions of the class C^0. New global structures arise with surprising features. In particular, we show that vacuum spherically symmetric wormholes do exist in this theory. These can be regarded as gravitational solitons, which connect two asymptotically (Anti) de-Sitter spaces with different masses and/or different effective cosmological constants. We prove the existence of both static and dynamical solutions and discuss their (in)stability under perturbations that preserve the symmetry. This leads us to discuss a new type of instability that arises in five-dimensional Lovelock theory of gravity for certain values of the coupling of the Gauss-Bonnet term. The issues of existence and uniqueness of solutions and determinism in the dynamical evolution are also discussed.

Contents

1 Introduction

In this article we shall be concerned with the Einstein-Gauss-Bonnet theory of gravity, whose action is given by the Einstein-Hilbert term plus the Einstein cosmological constant term Λ, and both supplemented with the Gauss-Bonnet term, quadratic in the curvature. The action of the theory reads[1]

$$ S = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} \left(\mathcal{R} - 2\Lambda + \alpha \left(\mathcal{R}^2 - 4\mathcal{R}_{AB}\mathcal{R}^{AB} + \mathcal{R}_{ABCD}\mathcal{R}^{ABCD} \right) \right), \qquad (1) $$

where $\kappa^2 = 8\pi G$ and α represents the coupling constant of the quadratic term. The quadratic term is often called the Gauss-Bonnet term because it is the dimensional extension of the Gauss-Bonnet topological invariant in four dimensions.

In five dimensions, the action (1) contains all of the non-zero terms of the Lovelock series. It is thus the most general metric torsion-free theory of gravity which leads to conserved equations of motion which are second order in derivatives [1]. The perturbation theory about the maximally symmetric vacuum is free of ghosts [2, 3] which suggests that it could appear as a higher order correction to Einstein's theory in the effective action coming from some more fundamental quantum theory. In fact, the Gauss-Bonnet term naturally arises as a higher order correction to gravity within string theory. Although the fourth-order derivative corrections are known to appear as the next-to-leading-order correction in the Type II strings[6], the quadratic corrections are present in both the heterotic and bosonic string theory [2, 4, 5]. In those cases, the coupling of the Gauss-Bonnet term is given by α' multiplied by a function of the dilaton, and so corresponding to powers of the string coupling. The five-dimensional Gauss-Bonnet term also arises in the Calabi-Yau compactification of M-theory, where the coupling of the second-order corrections turns out to be given in terms of the Kähler moduli of the six-dimensional compact manifold [7].

The presence of the Gauss-Bonnet term introduces some exotic features not found in General Relativity. One such feature is related to the problem of causality; this was treated in Ref. [8] in the Hamiltonian formalism (see also Ref. [9] for an alternative treatment of the Cauchy problem). Because of the non-linearity of the theory, the canonical momenta are not linear in the extrinsic curvature; and there exist quite generically points in the phase space where the Hamiltonian turns out to be multiple-valued. In such a situation, there is a breakdown in the deterministic evolution of the metric from the initial data. This can also be seen explicitly using the junction conditions [10, 11]. In fact, it can be shown that there exist vacuum solutions where the extrinsic curvature can jump spontaneously at some spacelike hypersurface in a way that is not predicted by the initial data[2]. This breakdown in predictability is induced by the presence of terms in the junction conditions which, unlike the Israel conditions valid for Einstein's theory, contain non-linear contributions coming from the Gauss-Bonnet term.

On the other hand, the timelike version of such a jump in the extrinsic curvature is also of great interest. This is realized by the existence of a kind of gravitational solitons in the theory, which resemble a kink solution. These solitons correspond to spacetimes that contain timelike hypersurfaces where the metric is C^0 continuous but where the extrinsic curvature jumps. Although the Riemann curvature tensor contains delta-function singularities on the hypersurface, these spacetimes can still be vacuum solutions because of a nontrivial cancelation coming from additional terms in the junction conditions. Some explicit examples have appeared in the literature [12], and a spherically symmetric realization of such solutions were studied in detail in Ref. [13] for the case of pure Gauss-Bonnet gravitational theory. Here, the systematical analysis made in Ref. [13] will be extended to the more phenomenologically important case where Einstein-Hilbert term and cosmological constant are included in the gravitational action. We will show that vacuum shell solutions are indeed found in Einstein-Gauss-Bonnet theory described by the action (1).

So then we will consider the junction conditions for spherical thin shells in Einstein-Gauss-Bonnet theory in the case that the induced stress tensor on the shell vanishes. Then, we will show that geometries associated with two different spherically symmetric spaces can be joined without

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[1]Capital Roman letters A, B etc. have been employed for five-dimensional tensor indices. $\mathcal{R}^A_{\ BCD}$ is the five-dimensional Riemann tensor.

[2]The junction condition in vacuum gives precisely that the jump in the canonical momenta is zero. The existence of solutions with non-zero jump in the extrinsic curvature at a spacelike shell is therefore equivalent to the problem of a multiple-valued Hamiltonian.

resorting to the introduction of matter fields as a source. Depending on the orientation of the two spaces, different global structures may arise. For instance, for one choice of orientation we get vacuum wormholes in five-dimensions. These wormholes are gravitational solitons that connect two regions with different masses which can be asymptotically either flat, Anti de Sitter (AdS) or de Sitter (dS) depending on the sign of the effective cosmological constant in each region. Other choices of orientation are possible, such as spherical bubbles, inside of which the value of the effective cosmological drastically changes. All the cases we will study in detail are such that the singular hypersurfaces where the jump in extrinsic curvature is located correspond to a sphere. We will call them "vacuum shells".

This paper is organised as follows. We begin section 2 by presenting some preliminary material that will be used in the rest of the paper. First, we review basic aspects of the spherically symmetric solution of Einstein-Gauss-Bonnet theory: the well known Boulware-Deser metric [14, 22]. Secondly, we review the junction conditions for this theory. We discuss both the cases where the junction hypersurface is of timelike and spacelike signature, we describe the different orientations allowed. At the end of the second section we derive the equation that contains all the information about the junction of two spherically symmetric vacuum solutions in the five-dimensional Einstein-Gauss-Bonnet theory. In section 3 we focus our attention on the static case corresponding to the timelike time-independent junctions; also we study the instantaneous case corresponding to the spacelike analogue. We explore the space of parameters of the theory for which solutions describing wormhole-like and bubble-like geometries exist. We see that such vacuum shells can also contain interior regions where naked singularities arise. In section 4 we survey the catalogue of these curious geometries, and we discuss the qualitative aspects of static solutions, emphasizing the most relevant properties. In section 5 we analyze the dynamical solutions. This includes a discussion of the (in)stability of the static solutions under perturbations that preserve the symmetry. Also some general results about the behavior of time-dependent solutions are given in Propositions 11 and 13. In section 6 we give an exhaustive parametrization of the space of constant radius solutions. Section 7 is devoted to a discussion of the C^0 class metrics and the topology of the solutions. We also discuss there the uniqueness and staticity of the spherically symmetric solutions, concerning the global validity of the Birkhoff-type theorems in Lovelock gravity. Section 8 contains the conclusions.

With respect to the style of presentation, we have chosen to organize our results in a series of remarks, propositions and theorems in order to highlight key facts, but descriptions such as 'theorem' should not be taken in the most strict mathematical sense.

2 The setup

First, we will present some introductory material and notation and conventions. The spherically symmetric solutions of Einstein-Gauss-Bonnet gravitational theory will be reviewed. Then we will discuss the junction conditions in this theory.

Then we will show how these junction conditions permit to join two spherically symmetric spaces without resorting to the introduction of matter source.

2.1 The bulk metric

Let us consider the Einstein-Gauss-Bonnet theory. The field equations associated with the action (1) coupled to some matter action take the form

$$G_B^A + \Lambda \delta_B^A + \alpha H_B^A = \kappa^2 T_B^A \,, \tag{2}$$

where T_B^A is the stress tensor, $G_B^A \equiv -\frac{1}{4} \delta_{BEF}^{ACD} \mathcal{R}_{AB}^{EF} = \mathcal{R}_{\ B}^A - \frac{1}{2} \delta_B^A \mathcal{R}$ is the Einstein tensor and

$$H_B^A \equiv -\frac{1}{8} \delta_{BD_1...D_4}^{AC_1...C_4} \mathcal{R}^{D_1 D_2}_{\ \ C_1 C_2} \mathcal{R}^{D_3 D_4}_{\ \ C_3 C_4} \,,$$

and where the antisymmetrized Kronecker delta is defined as $\delta_{B_1...B_p}^{A_1...A_p} \equiv p! \delta_{[B_1}^{A_1} \cdots \delta_{B_p]}^{A_p}$.

We are mainly interested in the static spherically symmetric solution (without matter) to Einstein-Gauss-Bonnet theory in five dimensions. In this case, of space-times fibered over (constant radius)

3-spheres, the solutions correspond to the analogues of the Schwarzschild geometry, and its form was found by D. Boulware and S. Deser in Ref. [14]. More generally, the solutions that correspond to fiber bundles over 3-surfaces of constant negative (or vanishing) curvature were subsequently studied in Ref. [16] (and also Ref. [17] in a special class of Lovelock theories in arbitrary dimension [18]). Let us discuss these solutions here. First, let us write the ansatz for the metric as follows

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \, d\Omega_k^2 \,, \tag{3}$$

where $d\Omega_k^2$ is the metric of the constant curvature three-manifold (of normalized curvature $k = +1$, -1 or 0). From $T_0^0 = 0$ (the other field equations are equivalent to it) one obtains

$$f'\{r^2 + 4\alpha(k - f)\} = -2r^3\frac{\Lambda}{3} + 2r(k - f). \tag{4}$$

This is integrated for $(k - f)$ to give

$$f(r) = k + \frac{r^2}{4\alpha}\left(1 + \xi\sqrt{1 + \frac{4\Lambda\alpha}{3} + \frac{16M\alpha}{r^4}}\right) \tag{5}$$

where $\xi^2 = 1$. The case $\xi = +1$ corresponds to the "exotic branch" of the Boulware-Deser metrics which for $\Lambda = 0$ and $M = 0$ gives a "microscopic" anti-de Sitter or de Sitter metric, with $f(r) = 1 + r^2/2\alpha$. It is usually argued that this exotic branch turns out to be an unstable vacuum of the theory, containing ghost excitations [14, 2]. Unlike the case $\xi = -1$, this branch does not have a well defined $\alpha \to 0$ limit. As in the case of Schwarzschild solution, M here is a constant of integration, and is also associated with the mass of the solution. In fact, when there is an asymptotic region at the infinity of the coordinate r, i.e. $1 + \frac{4\alpha\Lambda}{3} \geq 0$, the total energy w.r.t. each constant curvature background is calculated to be

$$\text{mass} = M\,\frac{6\pi^2}{\kappa^2}\,, \tag{6}$$

so that, in general, we will call M the mass parameter or simply the mass of the metric[3].

The general features of the black holes (3)-(5), such as horizons structure, singularities, etc, were studied systematically in Ref. [19]; for further details see the Appendix. Unlike General Relativity, the Einstein-Gauss-Bonnet theory admits massive solutions with no horizon but with a naked singularity at the origin. From (5) we see that this always happens for the exotic branch $\xi = +1$, and might also happen for the branch $\xi = -1$, provided $M < \alpha$. A related feature occurs for electrically charged solutions [20, 21]. Among other interesting properties, it can be seen that charged black holes in Einstein-Gauss-Bonnet theory have a single horizon if the mass reaches a certain critical value. Another substantial difference between the Schwarzschild solution and the Boulware-Deser solution concerns thermodynamics. Unlike black holes in General Relativity, the Einstein-Gauss-Bonnet black holes turn out to be eternal. The thermal evaporation process leads to eternal remnants due to a change of the sign in the specific heat for sufficiently small black holes. This and the other unusual phenomena discussed above are ultimately due to the ultraviolet corrections introduced by the Gauss-Bonnet term.

The discussion about a spherically symmetric solution of a given theory of gravity immediately raises the obvious question about its uniqueness. Regarding this, there is a subtlety that deserves to be pointed out. The uniqueness of the Boulware-Deser solution, discussed previously in Refs. [22, 23] (see [24] for a uniqueness result in axi-dilaton gravity with Gauss-Bonnet term), is only valid under certain assumptions. This was formalized in a theorem proven by R. Zegers [25], and which also holds for generic Lovelock theory in any dimension. Let us state the result as applies for Einstein-Gauss-Bonnet theory in five dimensions:

Theorem 1 (Ref. [25]). *Any solution with spherical (or planar or hyperbolic) symmetry in the second-order Einstein-Gauss-Bonnet theory of gravity has to be locally static and given by the Boulware-Deser solution provided two key conditions are satisfied: i) The coefficients of the Lovelock expansion are generic enough, which means that the exceptional combination $\alpha\Lambda = -3/4$ is excluded; ii) the solution is C^2 smooth.*

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[3]It should be kept in mind that the masses M in each branch ξ, by being the total energy w.r.t. the $M = 0$ spacetime in *that branch*, can not be directly compared.

Condition *i)* is certainly a necessary assumption. Indeed, the non-uniqueness in the case of $\alpha\Lambda = -3/4$, corresponding to the (A)dS-invariant Chern-Simons theory, is a well-known result and was explicitly shown in Refs. [23, 26]. In this paper, we will see that condition *ii)* is also necessary. In fact, the vacuum shell solutions we will present are C^0 spacetimes which are only piecewise of the Boulware-Deser form.

In order to analyze C^0 solutions, we will need to use the junction conditions in the theory, which will now be discussed.

2.2 Junction Conditions

The next ingredient in our discussion is the junction conditions in Einstein-Gauss-Bonnet theory. These are the analogues of the Israel conditions [28] in General Relativity, and were worked out in Refs. [10, 11]. In particular, the junction conditions will be employed to join two different spherically symmetric spaces.

We will organize the discussion as follows: First, we will discuss the timelike junction condition; namely, the case where the surgery is performed on a timelike hypersurface, which we shall call a timelike shell. After studying this we will briefly discuss its spacelike analogue.

2.2.1 Timelike shell

Let Σ be a timelike hypersurface separating two bulk regions of spacetime, region $\mathcal{V}_L$ and region $\mathcal{V}_R$ ("left" and "right"). Conveniently, we introduce the coordinates (t_L, r_L) and (t_R, r_R) and the metrics

$$ds_L^2 = -f_L\, dt_L^2 + \frac{dr_L^2}{f_L} + r_L^2 d\Omega^2\,, \tag{7}$$

$$ds_R^2 = -f_R\, dt_R^2 + \frac{dr_R^2}{f_R} + r_R^2 d\Omega^2\,, \tag{8}$$

in the respective regions. We shall be interested in the case where the bulk regions are empty of matter so $f_L(r_L)$ and $f_R(r_R)$ are the Boulware-Deser metric functions given by equation (5). In general, the mass parameter M_R will be different from M_L. Moreover, we will also consider the possibility of having ξ_R different from ξ_L, so that the two different branches of the Boulware-Deser solution can be considered to the two spaces to be joined.

It is convenient to parameterize the shell's motion in the $r-t$ plane using the proper time τ on Σ. In region $\mathcal{V}_L$ we have $r_L = a(\tau)$, $t_L = T_L(\tau)$ and in region $\mathcal{V}_R$ we have $r_R = a(\tau)$, $t_R = T_R(\tau)$. The induced metric on Σ induced from region $\mathcal{V}_L$ is the same as that induced from region $\mathcal{V}_R$, and is given by

$$d\hat{s}^2 = -d\tau^2 + a(\tau)^2 d\Omega^2\,. \tag{9}$$

This guarantees the existence of a coordinate system where the metric is continuous (C^0).

Here, $d\Omega^2$ will be chosen to be the line element of a 3-manifold with (intrinsic) curvature $k = \pm 1, 0$ i.e. it is a unit sphere, a hyperboloid or flat space respectively. Although our interest will be mainly focused on the spherical shell similar features to those we will discuss hold also for the cases $k = 0$ and $k = -1$. The hypersurface Σ is the shell's world-volume, i.e. the four-dimensional history of the shell in spacetime. The intrinsic geometry is well defined on Σ and given by (9). However, since the metric is only C^0 and not necessarily differentiable, the geometry of the embedding of Σ into $\mathcal{V}_L$ is independent of the embedding of Σ into $\mathcal{V}_R$. The geometric information about the embedding is quantified by the extrinsic curvature as well as the orientation of Σ with respect to each bulk region.

To be precise, let us consider the following conventions for a timelike shell outside of any event horizon:

- The hypersurface Σ has a single unit normal vector $\boldsymbol{n}$ which points from left to right.

- The orientation factor η of each bulk region is defined as follows: $\eta = +1$ if the radial coordinate r points from left to right, while $\eta = -1$ if the radial coordinate r points from right to left.

This is depicted in Fig. 1. Notice that the wormhole depicted on the left of that figure is not the only possibility for $\eta_L \eta_R < 0$. While this geometry roughly speaking corresponds to joining
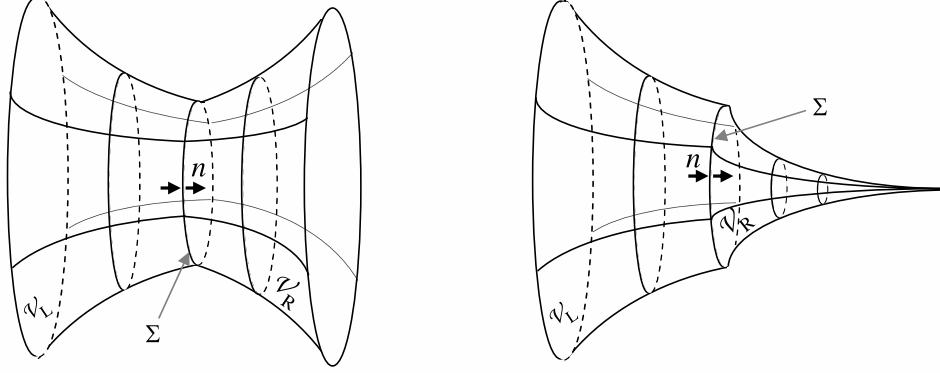
Vacuum wormhole solution, corresponding to a juncture with $\eta_L\eta_R<0$. Two disconnected asymptotically (A)dS regions.

False vacuum bubble solution, corresponding to a juncture with orientation $\eta_L\eta_R>0$. A naked singularity arises at the origin.

Figure 1: The figure on the left corresponds to a wormhole-like solution, defining the orientation $\eta_L\eta_R < 0$. The throat connects two different asymptotically (Anti) de-Sitter spaces. The figure on the right corresponds to a vacuum shell with standard orientation ($\eta_L\eta_R > 0$), connecting two Boulware-Deser solutions of different branches.

two "exterior regions" of a spherical solution, it is also feasible to construct a space by joining two "interior regions", instead. This corresponds to the case $\eta_L\eta_R < 0$ as well.

Definition 2. *The orientation defined by $\eta_L\eta_R > 0$ will be called the standard orientation. A shell with standard orientation will be called a standard shell. The orientation defined by $\eta_L\eta_R < 0$ will be called the wormhole orientation. [This makes actual sense when $\eta_R = +1$. When $\eta_R = -1$ the latter case represents a closed universe, containing singularities.]*

The components of the normal vector with respect to the basis $\boldsymbol{e}_A := (\partial_{t_L}, \partial_{r_L}, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ of $\mathcal{V}_\mathcal{L}$ and the basis $\boldsymbol{e}_{A'} := (\partial_{t_R}, \partial_{r_R}, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ of $\mathcal{V}_\mathcal{R}$ are respectively given by

$$n^A = \eta_L\left(\frac{\dot{a}}{f_L}, \sqrt{f_L + \dot{a}^2}, 0, 0, 0\right), \qquad n^{A'} = \eta_R\left(\frac{\dot{a}}{f_R}, \sqrt{f_R + \dot{a}^2}, 0, 0, 0\right).$$

where dot denotes differentiation with respect to τ. This formula for the normal vector extends the definition of the orientation factors to the situation where the shell is inside the horizon when r is a timelike coordinate.

We can introduce the basis $\boldsymbol{e}_a = (\partial_\tau, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ intrinsic to Σ. The extrinsic curvature is then defined as $K_{ab} := \boldsymbol{e}_a \cdot \nabla_{\boldsymbol{e}_b}\boldsymbol{n} = -\boldsymbol{n} \cdot \nabla_{\boldsymbol{e}_b}\boldsymbol{e}_a$. In terms of a coordinate basis we have $e_a^A = \frac{\partial X^A}{\partial\zeta^a}$ and the extrinsic curvature takes the explicit form

$$K_{ab} = -n_A\left(\frac{\partial^2 X^A}{\partial\zeta^a\partial\zeta^b} + \Gamma_{BC}^A\frac{\partial X^B}{\partial\zeta^a}\frac{\partial X^C}{\partial\zeta^b}\right),$$

and in our case the components read

$$K_\tau^\tau = \eta\frac{\ddot{a} + \frac{1}{2}f'}{\sqrt{\dot{a}^2 + f}}, \qquad K_\theta^\theta = K_\chi^\chi = K_\varphi^\varphi = \frac{\eta}{a}\sqrt{\dot{a}^2 + f}. \tag{10}$$

We denote the extrinsic curvature with respect to the embedding into $\mathcal{V}_L$ and $\mathcal{V}_R$ by $(K_L)_{ab}$ and $(K_R)_{ab}$ respectively. At a singular shell $(K_L)_{ab} \neq (K_R)_{ab}$, i.e. the extrinsic curvature jumps from

one side to the other. This is a covariant way of expressing the fact that the metric is not C^1 (i.e. there does not exist any coordinate system where the metric is C^1). In General Relativity this amounts to saying that (non-null) vacuum shells do not exist since Israel conditions cannot be satisfied without the introduction of a induced stress tensor on the spherical shell. Things are different in the case of the gravity theory defined by action (1). This is because the Gauss-Bonnet term induces additional terms in the junction conditions, which supplements the Israel equation. In section 2.3 we will show how both contributions can be combined to yield vacuum spherically symmetric thin shells. First we briefly discuss spacelike shells.

2.2.2   Spacelike shell

Solutions of a different sort are those constructed by joining two spaces through a spacelike juncture. Let us suppose now that Σ is now a spacelike hypersurface. The motion of the shell in the $r - t$ plane is parameterized by $(t, r) = (T(\tau), a(\tau))$, where it is necessary to remember that τ is now a spacelike coordinate on Σ. The induced metric on Σ is then given by

$$d\hat{s}^2 = +d\tau^2 + a(\tau)^2 d\Omega^2 \,. \tag{11}$$

The components of the normal vector with respect to the basis $\boldsymbol{e}_A := (\partial_{t_L}, \partial_{r_L}, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ of $\mathcal{V}_L$ and the basis $\boldsymbol{e}_{A'} := (\partial_{t_R}, \partial_{r_R}, \boldsymbol{e}_\theta, \boldsymbol{e}_\chi, \boldsymbol{e}_\varphi)$ of $\mathcal{V}_R$ are respectively:

$$n^A = \eta_L \left(\frac{\dot{a}}{f_L}, \sqrt{\dot{a}^2 - f_L}, 0, 0, 0 \right) \,, \qquad n^{A'} = \eta_R \left(\frac{\dot{a}}{f_R}, \sqrt{\dot{a}^2 - f_R}, 0, 0, 0 \right) \,.$$

This defines the orientation factors in the case of a spacelike shell. The components of the extrinsic curvature are:

$$K^\tau_\tau = \eta \frac{\ddot{a} - \frac{1}{2} f'}{\sqrt{\dot{a}^2 - f}} \,, \qquad K^\theta_\theta = K^\phi_\phi = K^\chi_\chi = \frac{\eta}{a} \sqrt{\dot{a}^2 - f} \,. \tag{12}$$

2.3   The junction condition for a vacuum shell

The Einstein-Gauss-Bonnet field equations are well-defined distributionally at Σ due to the property of quasi-linearity in second derivatives (see e.g. Refs [27, 13]). Thus, one can define a distributional stress tensor $T_{AB} = \delta(\Sigma) e^b_A e^b_B S_{ab}$, where S_{ab} is the intrinsic stress tensor induced on the shell and $\delta(\Sigma)$ denotes a Dirac delta function with support on the shell world-volume Σ.

Integrating the field equations from left to right in an infinitesimally thin region across Σ one obtains the junction condition. This relates the discontinuous change of spacetime geometry across Σ with the stress tensor S^b_a. For the Einstein-Gauss-Bonnet theory the general formulas can be found in the Refs. [10, 11, 13].

$$(\mathfrak{Q}_R)^b_a - (\mathfrak{Q}_L)^b_a = -\kappa^2 S^b_a \,, \tag{13}$$

where the symmetric tensor $\mathfrak{Q}^a_b$ is given by

$$\mathfrak{Q}^a_b = \mp \delta^{ac}_{bd} K^d_c + \alpha \, \delta^{acde}_{bfgh} \left(\mp K^f_c R^{gh}_{de} + \frac{2}{3} K^f_c K^g_d K^h_e \right) \,. \tag{14}$$

Above, the sign $\mp$ depends on the signature of the junction hypersurface: it is minus for the timelike case and plus for the spacelike case. In this expression, lower case Roman letters from the beginning of the alphabet a, b etc. represent four-dimensional tensor indices on the tangent space of the world-volume of the shell. The symbol K^a_b refers to the extrinsic curvature, while the symbol R^{ab}_{cd} appearing here corresponds to the four-dimensional intrinsic curvature (see the appendix for details).

Once applied to the spherically symmetric (or $k = -1, 0$) case the tensor $\mathfrak{Q}^b_a$ turns out to be diagonal with components

$$\mathfrak{Q}^\tau_\tau = -3\sigma \, a^{-3} \left(\eta \, a^2 \, \sqrt{\dot{a}^2 + f} + 4\alpha \, \eta \, \sqrt{\dot{a}^2 + f} \, \left(k + \frac{2}{3}\sigma \dot{a}^2 - \frac{1}{3} f \right) \right) \,, \tag{15}$$

$$\mathfrak{Q}^\theta_\theta = \mathfrak{Q}^\chi_\chi = \mathfrak{Q}^\varphi_\varphi \,. \tag{16}$$

The precise form of $\mathfrak{Q}_\theta^\theta$ will not be needed but is given in the appendix for completeness. The above formula was written in a way that is valid for both timelike and spacelike shells, where we have defined

$$\sigma = +1 \quad \text{(timelike shell)}\,, \qquad \sigma = -1 \quad \text{(spacelike shell)}\,.$$

Also, let us be reminded of the fact that η_L and η_R (with $\eta^2 = 1$) are the orientation factors in each region, which are independent one from each other. Above, the subscripts L, R signify the quantity evaluated on Σ induced by regions $\mathcal{V}_L$ and $\mathcal{V}_R$ respectively (e.g. $\mathfrak{Q}_L$ is a function of η_L and $f_L(a)$)[4].

One may verify that the following equation is satisfied

$$\frac{d}{d\tau}\big(a^3 \mathfrak{Q}_\tau^\tau\big) = \dot{a}\, 3a^2 \mathfrak{Q}_\theta^\theta \,, \tag{17}$$

which expresses the conservation of S_a^b. The reason why one obtains exact conservation, i.e. no energy flow to the bulk, is that the normal-tangential components of the energy tensor in the bulk is the same in both sides of the junction hypersurface [10, 13].

The main point here is that, unlike the Israel conditions in Einstein gravity, non-trivial solutions to (C) are possible even when $S_a^b = 0$. That is, the extrinsic curvature can be discontinuous across Σ with no matter on the shell to serve as a source. The discontinuity is then self-supported gravitationally and this is due to non-trivial cancelations between the terms of the junction conditions. Similar configurations are impossible in Einstein gravity. From now on we consider the vacuum case

$$S_a^b = 0 \,. \tag{18}$$

In the next section we will treat the static shell in detail. An exhaustive study of the space of solutions describing both static and dynamical shells is left until sections 5 and 6. Let us now first briefly introduce the basic features of the general solution for a dynamical shell.

Equation (17) tells us that when $\dot{a} \neq 0$, the components of the junction condition are not independent; namely

$$(\mathfrak{Q}_R)_\tau^\tau - (\mathfrak{Q}_L)_\tau^\tau = 0 \quad \Rightarrow \quad (\mathfrak{Q}_R)_\theta^\theta - (\mathfrak{Q}_L)_\theta^\theta = 0 \,.$$

Therefore, for time-dependent solutions it suffices to impose only the first condition. This can be factorized as follows,

$$\Big(\eta_R \sqrt{\dot{a}^2 + \sigma f_R} - \eta_L \sqrt{\dot{a}^2 + \sigma f_L}\Big) \times$$
$$\times \left\{ a^2 + 4\alpha(k + \sigma \dot{a}^2) - \sigma \frac{4\alpha}{3}\Big(f_R + f_L + 2\sigma \dot{a}^2 + \eta_R \eta_L \sqrt{\sigma f_R + \dot{a}^2}\sqrt{\sigma f_L + \dot{a}^2}\Big) \right\} = 0\,. \tag{19}$$

Equation (19) contains all the information about the spherically symmetric junctions in empty space, which we generically call vacuum shells. Certainly, there exist several cases to be explored. First of all, there are the parameters k, M and ξ, which characterize each of the two Boulware-Deser metrics to be joined. On the other hand, there are two possible orientations for each one of the spaces, and this is given by the sign of the respective η. The solutions to (19) include both wormhole-like and bubble-like geometries, depending on whether the orientation is $\eta_L \eta_R < 0$ or $\eta_L \eta_R > 0$ respectively. Furthermore, there is the sign of σ, what tells us whether the signature of the junction hypersurface is timelike ($\sigma = +1$) or spacelike ($\sigma = -1$). So, this permits a very interesting catalogue of geometries which we survey in section 4 and further explore in subsequent sections.

The vanishing of the first factor in (19) would imply that the metric is smooth across Σ. Rejecting this as the trivial solution, we demand that the second factor vanishes. From the second factor, squaring appropriately, we obtain

$$\dot{a}^2 = \sigma\, \frac{\big(f_R + f_L - 3(k + a^2/4\alpha)\big)^2 - f_R f_L}{3\big(f_R + f_L - 2(k + a^2/4\alpha)\big)} =: -V(a)\,, \tag{20}$$

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[4]From now on we shall be concerned with $f(a)$, i.e. the metric function evaluated at the shell. In an abuse of notation we shall just write f instead of $f(a)$.

along with two inequalities discussed below. The system, because of the symmetry, has reduced to an essentially one-dimensional problem, given by the ordinary differential equation (20). It is seemingly equivalent to the problem of a particle moving in a potential[5] $V(a)$. Nevertheless, it is worth pointing out that, unlike the equation for a single particle, here we find that the energy h is unavoidably fixed to zero instead of arising as a constant of motion. An important difference arises in the case where there is a minimum of $V(a)$ precisely at $V = 0$. The constraint $h = 0$, provided the fact that the minimum of $V(a)$ is precisely at zero energy, would lead to the conclusion that the shell can not move but it would be stacked at the bottom of the potential. Actually, this is the case if no external system acts as a perturbation. One such perturbation can be thought of as being an incoming particle which, after perturbing the shell, scatters back to infinity spending an energy δh through the process. This would provide energy for the vacuum shell to move. One can also think about a slight change in the parameters of the solution yielding a shifting $V(a) \to V(a) - \delta h$, see [31].

Now, let us notice that since we have squared the junction condition, we must substitute (20) back into (19) to check the consistency. When doing so, the solutions of equation (20) are solutions of the junction condition if and only if the following restrictions are obeyed

$$-\eta_R\eta_L \left(2f_R + f_L - 3(k + a^2/4\alpha)\right) \left(2f_L + f_R - 3(k + a^2/4\alpha)\right) \geq 0 \; ; \tag{21}$$

and

$$(f_R + f_L - 2(k + a^2/4\alpha)) > 0 \qquad \text{timelike shell} \tag{22}$$
$$(f_R + f_L - 2(k + a^2/4\alpha)) < 0 \qquad \text{spacelike shell} \,. \tag{23}$$

Furthermore, we also have an inequality which is not an extra condition but rather follows as a consequence of equation (20). The fact that $\dot{a}^2$ is positive in (20) implies that

$$\left(f_R + f_L - 3(k + a^2/4\alpha)\right)^2 - f_R f_L \geq 0 \,. \tag{24}$$

for both timelike and spacelike. This inequality provides further information about the space of solutions of (20).

Proposition 3. *For a dynamical vacuum shell with a timelike world-volume Σ, the scale factor of the metric (9) on Σ is governed by (20), under the inequalities (21) and (22).*
On the other hand, for a dynamical vacuum shell with a spacelike world-volume Σ, the scale factor of the metric (11) on Σ is governed by (20), under the inequalities (21) and (23).

Now, let us begin by studying the inequalities to give idea of what kinds of solutions exist. With this in mind, let us translate the restrictive inequalities (21-23) into simpler terms. The metric function evaluated on the hypersurface is

$$f_L(a) = k + \frac{a^2}{4\alpha}\left(1 + \xi_L Y_L(a)\right), \qquad Y_L(a) \equiv \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{16\alpha M_L}{a^4}} \,, \tag{25}$$

and similarly for f_R. Recall that ξ_L and ξ_R are independent of each other, with $\xi = +1$ being the exotic branch of the Boulware-Deser solution. It is convenient to write the inequalities in terms of the square roots $Y_L(a)$ and $Y_R(a)$; namely

$$-\eta_R\eta_L \left(2\xi_R Y_R + \xi_L Y_L\right) \left(2\xi_L Y_L + \xi_R Y_R\right) \geq 0 \; ; \tag{26}$$

and

$$\alpha(\xi_R Y_R + \xi_L Y_L) > 0 \qquad \text{timelike shell} \tag{27}$$
$$\alpha(\xi_R Y_R + \xi_L Y_L) < 0 \qquad \text{spacelike shell} \,. \tag{28}$$

These inequalities contain relevant information about the global structure of the solutions. Let us summarize this information in the following table

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[5]Notice that the effective potential for the spacelike shell is simply minus the potential for the timelike shell.

Timelike shells $(\sigma = +1)$	Product of orientation factors $(\eta_L \eta_R)$	Product of branch signs $(\xi_L \xi_R)$	Inequalities imposed on solutions
Standard orientation	+1	+1	No solution
	+1	-1	$\frac{1}{2}Y_L \leq Y_R(a) \leq 2Y_L(a)$; $\xi_R(M_R - M_L) > 0$
"Wormhole" orientation	-1	+1	$\alpha \xi_R > 0$; $Y_L, Y_R > 0$
	-1	-1	$Y_R \geq 2Y_L$ or $Y_R \leq \frac{1}{2}Y_L$; $\xi_R(M_R - M_L) > 0$
Spacelike shells $(\sigma = -1)$	Product of orientation factors $(\eta_L \eta_R)$	Product of branch signs $(\xi_L \xi_R)$	Inequalities imposed on solutions
Standard orientation	+1	+1	No solution
	+1	-1	$\frac{1}{2}Y_L \leq Y_R(a) \leq 2Y_L(a)$; $\xi_R(M_R - M_L) > 0$
"Wormhole" orientation	-1	+1	$\alpha \xi_R < 0$; $Y_L, Y_R > 0$
	-1	-1	$Y_R \geq 2Y_L$ or $Y_R \leq \frac{1}{2}Y_L$; $\xi_R(M_R - M_L) < 0$

From the conditions obtained here we conclude the following:

Remark 4. *Vacuum shells with the standard orientation always involve the gluing of a plus branch* $(\xi = +1)$ *metric with a minus branch* $(\xi = -1)$ *metric.*

Now the plus branch has a different effective cosmological constant to the minus branch. In this sense, standard shells are a kind of false vacuum bubble. This is discussed further in section 4.1.

Remark 5. *Vacuum shells which involve the gluing of two minus branch* $(\xi = -1)$ *metrics exist only when the Gauss-Bonnet coupling constant* α *satisfies* $\alpha < 0$. *They always have the wormhole orientation.*

In the analysis above it has been explicitly assumed that $\dot{a} \neq 0$. Nevertheless, the case $\dot{a} = 0$ is also of considerable interest. This describes static shells in the timelike case, and also an analogous situation for the spacelike case which we call instantaneous shells. In the next section, the case of constant a shells is considered in detail. It can be checked that, as expected, all the information about the constant a solutions can be obtained from the dynamical case by imposing both $V(a_0) = 0$ and $V'(a_0) = 0$. Thus, proposition 3 gives the general solution of all the vacuum shells, including the static ones.

Closing the general discussion of the dynamical vacuum we note the following. The potential $V(a)$ in (20) and the restrictive inequalities (21) and (22), (23) are symmetric in the exchange

$$\xi_L, M_L \leftrightarrow \xi_R, M_R . \tag{29}$$

That is, the same kinds of motion are possible for the two situations obtained if we swap the values of the parameters ξ, M in $\mathcal{V}_L$ and $\mathcal{V}_R$. In the constant a case, governed by $V(a_0) = 0 = V'(a_0)$, the symmetry means that the value of a_0 is left unchanged under the swapping.

3 Static vacuum shells

Now, let us discuss the solutions at constant a. That is, the static and instantaneous solutions, depending on whether the juncture corresponds to the timelike or spacelike case respectively.

The bulk metric in each of the two region is assumed to be of the Boulware-Deser form (5) with $(k = \pm 1, 0)$ and considering $a = a_0$ fixed. Although the main focus will be on the spherically symmetric case $k = +1$, the analysis can be straightforwardly extended to the cases $k = -1$ and $k = 0$. Then, there are two possibilities to be distinguished; namely,

- Static shell: For the timelike case the shell is located at fixed radius $r_L = r_R = a_0$. The proper time on the shell's world-volume is $\tau = t_L \sqrt{f_L(a)} = t_R \sqrt{f_R(a)}$ so that the induced metric on Σ turns out to be $d\hat{s}^2 = -d\tau^2 + a_0^2 d\Omega^2$. Then, the extrinsic curvature components are $K^\tau_\tau = \eta \frac{f'}{2\sqrt{f}}$, $K^\theta_\theta = K^\chi_\chi = K^\varphi_\varphi = \frac{\eta \sqrt{f}}{a}$ and the intrinsic curvature components are $R^{\theta\varphi}{}_{\theta\varphi} = k/a_0^2$, etc.

- Instantaneous shell: In the spacelike case there is an exotic kind of shell, which exists when f is negative. The metric function is negative inside of an event horizon or outside of a cosmological horizon, where r actually plays the role of a timelike coordinate. Matching two metrics at time $r_\pm = a_0$ therefore describes an instantaneous transition from one smooth metric to another. We can introduce $\tau = t_L \sqrt{-f_L(a)} = t_R \sqrt{-f_R(a)}$ which is a spacelike intrinsic coordinate on the shell, so that the induced metric on Σ is $ds^2 = +d\tau^2 + a_0^2 d\Omega^2$. The extrinsic curvature components are $K^\tau_\tau = -\eta \frac{f'}{2\sqrt{-f}}$, $K^\theta_\theta = K^\chi_\chi = K^\varphi_\varphi = \frac{\eta \sqrt{-f}}{a}$.

It is worth noticing that both the static and instantaneous shells can be analyzed together, provided the presence of σ in the equations. Recall that the sign of σ carries the information about the signature of the junction hypersurface. Then, by considering the quantities introduced above, and by substituting this in the junction conditions with $S^a_b = 0$, we get

$$S^\tau_\tau = 0 \ \Rightarrow \quad \left(\eta_R \sqrt{f_R} - \eta_L \sqrt{f_L} \right) \left(a_0^2 + \frac{4\alpha}{3} \left\{ 3k - f_R - f_L - \sigma \eta_L \eta_R \sqrt{f_L f_R} \right\} \right) = 0 \ . \tag{30}$$

$$S^\theta_\theta = 0 \ \Rightarrow \quad \left(\frac{\eta_R}{\sqrt{f_R}} - \frac{\eta_L}{\sqrt{f_L}} \right) \left(k - \frac{\Lambda a_0^2}{3} - \sigma \eta_L \eta_R \sqrt{f_L f_R} \right) = 0 \ , \tag{31}$$

where $\sigma = +1$ is the static shell and $\sigma = -1$ is the instantaneous shell. The l.h.s. of (4) conveniently appears in the $\theta - \theta$ component of the junction condition and we have used it to eliminate the derivative of f from the formula. This is why Λ appears explicitly in equation (31).

In both equations (30) and (31), the first factor vanishes if and only if the metric is smooth. Again, rejecting this as the trivial solution, we demand that the second factor vanishes in both equations. So, we have

Proposition 6. *A static vacuum shell is described by*

$$f_L + f_R = 2k + \frac{3a_0^2}{4\alpha} + \frac{\Lambda a_0^2}{3} \ , \tag{32}$$

$$\eta_L \eta_R \sqrt{f_L f_R} = k - \frac{\Lambda a_0^2}{3} \ , \tag{33}$$

under the condition $f_L, f_R > 0$. On the other hand, an instantaneous vacuum shell is described by

$$f_L + f_R = 2k + \frac{3a_0^2}{4\alpha} + \frac{\Lambda a_0^2}{3} \ , \tag{34}$$

$$-\eta_L \eta_R \sqrt{f_L f_R} = k - \frac{\Lambda a_0^2}{3} \ , \tag{35}$$

under the condition $f_L, f_R < 0$.

We have included for completeness the instantaneous shells. Now, let us consider some examples of the static case with more attention. As mentioned, a more complete analysis of the space of solutions will be given in sections 5 and 6.

3.1 The moduli space of solutions

Now, to continue the study of the different solutions we find it convenient to introduce some notation. For the rest of this section it is convenient to define the dimensionless parameters

$$x \equiv \frac{4\alpha\Lambda}{3} , \qquad y \equiv \frac{\Lambda}{3}a_0^2 , \qquad \bar{M} \equiv \frac{M}{\alpha} . \tag{36}$$

By x and y we measure the Gauss-Bonnet coupling and the vacuum shell radius respectively in units of Λ. The parameter y is useful for our purposes but it is meaningful only when $\Lambda \neq 0$. In terms of these parameters, the Boulware-Deser solution evaluated at $r = a_0$ has the form

$$f_{L,R}(a_0) \equiv 1 + \frac{y}{x}\left(k + \xi_{L,R}\sqrt{1 + x + \frac{x^2}{y^2}\bar{M}_{L,R}} \right) . \tag{37}$$

The general solution will be derived in the following way: We will solve the junction conditions for $\bar{M}_L$ and $\bar{M}_R$ in terms of (x,y). The range of admissible values of (x,y) turns out to be restricted by inequalities coming from demanding the metric to be real-valued. So there is a continuous space of solutions.

Definition 7. *The range of values of (x,y) for which solutions exist will be called the moduli space.*

The parameters x and y are coordinates of this moduli space. The complete description of the moduli space will be given in more appropriate parameters introduced in section 6. For the moment, let us consider x, y and $\bar{M}$.

Since the moduli space is two dimensional, it can be plotted. So by obtaining a formula for the masses and by plotting the moduli space, we obtain all the solutions. Let us now do this explicitly for the case of non-vanishing cosmological constant.

3.2 Static spherical shells with $\Lambda \neq 0$

Consider static spherically symmetric shells with $\Lambda \neq 0$. For definiteness, let us focus on the case of timelike shells with $k = 1$. From Proposition 6 we have the following pair of equations

$$f_L + f_R = \frac{y}{x}(3 + x) + 2 , \tag{38}$$

$$\sqrt{f_L f_R} = \eta_L \eta_R (1 - y) , \tag{39}$$

where $f_L, f_R > 0$. We can see immediately from (39) that solutions with the wormhole orientation, i.e. $\eta_L \eta_R = -1$, only exist for $y \equiv \Lambda a_0^2/3 > 1$.

Remark 8. *Static vacuum shell wormholes exist only when $\Lambda > 0$.*

Solving the equations above we see that f_L and f_R obey the same quadratic equation where one f has the $+$ root of the solution and the other has the $-$ root. So we define a solution $f_{(+)}$ which corresponds to the $+$ root of the solution and an $f_{(-)}$ which corresponds to the $-$ root. So there are two solutions to the problem:

$$f_L = f_{(-)} \;\; , \;\; f_R = f_{(+)} \qquad \text{or,} \qquad f_L = f_{(+)} \;\; , \;\; f_R = f_{(-)} . \tag{40}$$

Substituting the explicit expression (37) for $f_{L,R}(a_0)$ we have: In the first case of (40), $M_L = M_{(-)}$, $\xi_L = \xi_{(-)}$ and $M_R = M_{(+)}$, $\xi_R = \xi_{(+)}$, and in the second case $+ \leftrightarrow -$, for constants $\xi_{(\pm)}$ and $M_{(\pm)}$ satisfying

$$1 + x - \sqrt{3}\sqrt{x(1+x)\left(\frac{4}{y} + \frac{3}{x} - 1\right)} = 2\xi_{(-)}\sqrt{1 + x + \frac{x^2\bar{M}_{(-)}}{y^2}} , \tag{41}$$

$$1 + x + \sqrt{3}\sqrt{x(1+x)\left(\frac{4}{y} + \frac{3}{x} - 1\right)} = 2\xi_{(+)}\sqrt{1 + x + \frac{x^2\bar{M}_{(+)}}{y^2}} . \tag{42}$$

13

For a solution to exist, the square root in the l.h.s. of the above equations must be real, so that we demand

$$x(1+x)\left(\frac{4}{y}+\frac{3}{x}-1\right) \geq 0 \qquad \text{(Existence of solutions)}. \tag{43}$$

Since we have squared the equations we must substitute back to check the consistency. So we get the following inequalities:[6]

$$\frac{y}{x}(3+x)+2 > 0 \qquad \text{(Timelike shells)}; \tag{44}$$

$$y < 1 \quad \text{(Standard orientation)}, \qquad y > 1 \quad \text{(Wormhole orientation)}. \tag{45}$$

The above inequalities are plotted in figures 11 and 12. Also we find the regions of the moduli space corresponding to the allowed branch signs $(\xi_{(-)}, \xi_{(+)})$.

$\xi_{(-)}$	$\xi_{(+)}$	Inequality	
$+1$		$1+x > 0 \;\bigcap\; x\left(\frac{3}{y}+\frac{2}{x}-1\right) < 0$	
-1		$1+x < 0 \;\bigcup\; x\left(\frac{3}{y}+\frac{2}{x}-1\right) > 0$	(Branches).
	$+1$	$1+x > 0 \;\bigcup\; x\left(\frac{3}{y}+\frac{2}{x}-1\right) < 0$	
	-1	$1+x < 0 \;\bigcap\; x\left(\frac{3}{y}+\frac{2}{x}-1\right) > 0$	

The standard shells are always $(-, +)$. The regions $(+, +)$, $(-, +)$ and $(-, -)$ for the wormholes are shown in figure 13.

Remark 9. *Provided $\Lambda > 0$ and assuming the existence of two asymptotic regions we find $\alpha > 0$. Consequently, at least one of the two spherically symmetric spaces connected through the throat turns out to be asymptotically Anti-de Sitter.*

The next step is computing the masses. We can solve (41) and (42) to give the parameter M in each region, namely

$$\bar{M}_{(-)} = \frac{y(1+x)}{2x^2}\left\{6x+3y-xy-y\sqrt{3}\sqrt{x(1+x)\left(\frac{4}{y}+\frac{3}{x}-1\right)}\right\}, \tag{46}$$

$$\bar{M}_{(+)} = \frac{y(1+x)}{2x^2}\left\{6x+3y-xy+y\sqrt{3}\sqrt{x(1+x)\left(\frac{4}{y}+\frac{3}{x}-1\right)}\right\}. \tag{47}$$

As mentioned above, relations (40), the left-metric can be either a metric with parameters $(\xi_{(-)}, M_{(-)})$ or a metric with $(\xi_{(+)}, M_{(+)})$, and the other way around for the right-metric. For wormholes the two solutions (40) correspond to the same spacetime looked at from the opposite way around. In the case of standard shells, they correspond to swapping the mass and branch sign of the interior with those of the exterior region.

The metrics with parameters $(\xi_{(-)}, M_{(-)})$ and $(\xi_{(+)}, M_{(+)})$ as determined by the solutions we found above have different properties. We will call these metrics minus- and plus-metrics respectively.

Also we note the following useful expression: we can eliminate y to get an implicit equation for the masses and x. The solution lies on sections of the curves

$$1 + \frac{9}{4}\frac{x(x+1)}{3-x}\frac{1}{\bar{M}_{(+)}+\bar{M}_{(-)}}\left(1+(-1)^p\sqrt{1+\frac{4}{9}\frac{(3-x)}{x(x+1)}(\bar{M}_{(+)}+\bar{M}_{(-)})}\right)$$

$$= (-1)^q\sqrt{1+\frac{(3-x)}{(1+x)}\frac{(\bar{M}_{(+)}-\bar{M}_{(-)})^2}{(\bar{M}_{(+)}+\bar{M}_{(-)})^2}} \tag{48}$$

where the signs $(-1)^p$ and $(-1)^q$ are to be determined by consistency.

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[6]Note that the timelike condition (44), when combined with the reality condition (43) can be equivalently stated

$$xy(1+x) > 0 \qquad \text{(Timelike shell)}.$$

This is useful for plotting the graphs.

3.3 Instantaneous shells

Before concluding this section, let us briefly comment on spacelike junction conditions with $\dot{a} = 0$. For instance, consider the case $\Lambda \neq 0$. From Proposition 6 we have the following pair of equations:

$$f_L + f_R = \frac{y}{x}(3+x) + 2 \,, \tag{49}$$

$$\sqrt{f_L f_R} = -\eta_L \eta_R (1-y) \,, \tag{50}$$

The solution is exactly the same as the above except that the inequalities (44) and (45) are reversed. That means

$$\frac{y}{x}(3+x) + 2 < 0 \qquad \text{(Spacelike shells)} \,; \tag{51}$$

$$y > 1 \quad \text{(Standard orientation)} \,, \qquad y < 1 \qquad \text{(Wormhole orientation)} \,. \tag{52}$$

The inequality (43) and mass formulae are the same. The moduli space of these solutions is plotted in figure 14. They exist for $\alpha < 0$.

3.4 Static spherical shells with $\Lambda = 0$

Now, we will consider the case of static spherically symmetric shells with $\Lambda = 0$. This is an interesting special case. The analysis simplifies considerably and, besides, there are some qualitative differences between this and the case $\Lambda \neq 0$. In this case, the equations reduce to

$$f_L + f_R = 2 + \frac{3a_0^2}{4\alpha} \,, \tag{53}$$

$$\eta_L \eta_R \sqrt{f_L f_R} = 1 \,, \tag{54}$$

We see from the second equation that $\eta_L \eta_R$ must be $+1$, i.e. static wormholes do not exist for $\Lambda = 0$. Then, the solution is either $M_L = M_{(-)}, M_R = M_{(+)}$ or $M_L = M_{(+)}, M_R = M_{(-)}$ where

$$\frac{M_{(\pm)}}{\alpha} = \frac{1}{2}\frac{a_0^2}{4\alpha}\left(6 + \frac{3a_0^2}{4\alpha} \pm \sqrt{12\frac{a_0^2}{4\alpha} + 9\left(\frac{a_0^2}{4\alpha}\right)^2}\right) \,. \tag{55}$$

The consistency of the solution requires

$$\alpha > 0 \,, \qquad (\xi_{(-)}, \xi_{(+)}) = (-1, +1) \,, \tag{56}$$

so that $M_{(-)}$ and $M_{(+)}$ correspond to minus branch and exotic plus branch metrics respectively. There are solutions for all positive values of $M_{(-)}$ (the plus branch mass parameter is also positive but in that case the bulk spacetime asymptotically takes the form of a negative mass AdS-Schwarzschild solution). When the throat radius is small compared to the scale set by the Gauss-Bonnet coupling constant, $a_0^2 << \alpha$, the masses are also small compared to α, namely $M_{(-)}/\alpha \sim M_{(+)}/\alpha \sim 3a_0^2/4\alpha$. On the other hand, for large radius $a_0^2 >> \alpha$, the masses are large, $M_{(-)}/\alpha \sim a_0^2/2\alpha$, $M_{(+)}/\alpha \sim 3a_0^4/16\alpha^2$. Figure 10 shows a plot of the masses as a function of α and also an implicit plot of $M_{(+)}$ as a function of $M_{(-)}$.

4 Surveying static vacuum solutions

In the previous section we have shown the existence of static vacuum shells in the spherically symmetric case and found some basic qualitative features, as well as a formula for the mass parameters in each region. A more exhaustive treatment of the static shells will be left for section 6. Before going any further let us summarize the catalogue of vacuum solutions that arise through the geometric surgery we described above. The first cases of interest are those corresponding to the standard orientation $\eta_L \eta_R > 0$.
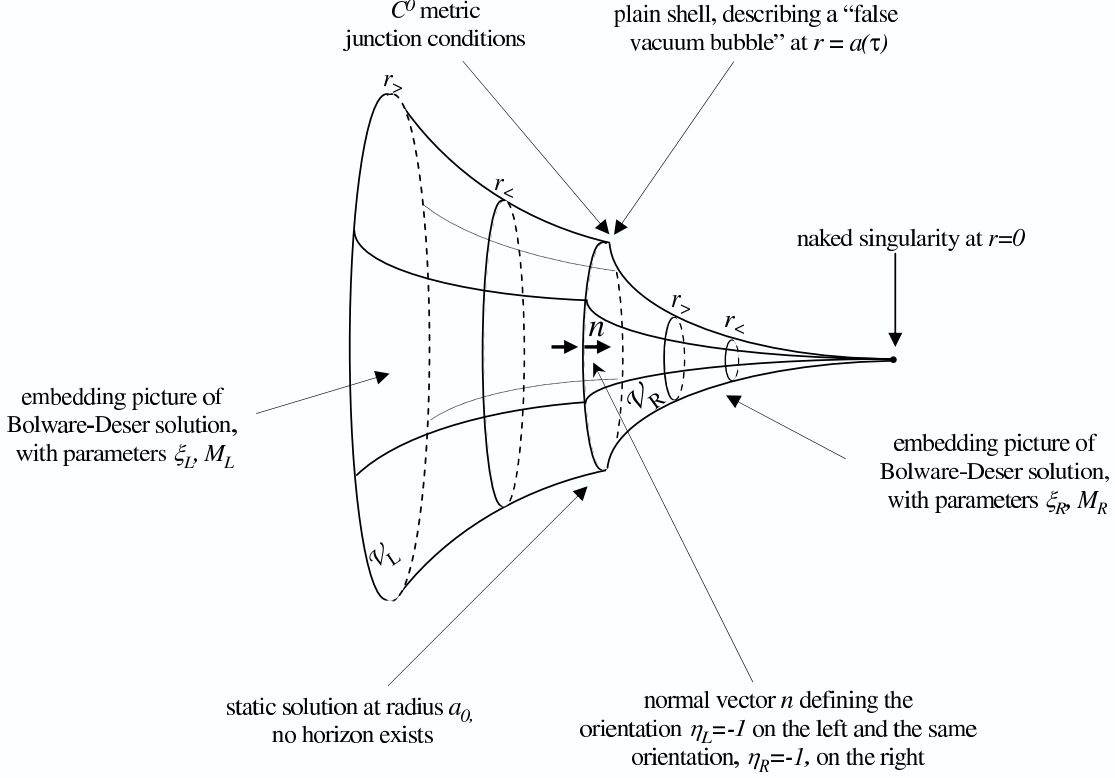
Figure 2: A spherically symmetric spacetime with metric of the class C^0. A vacuum shell with the standard orientation always connects two regions with different branch signs ξ (and generically with different mass parameters M). Each region has a different effective cosmological constant.

4.1   Standard shells and false vacuum bubbles

The vacuum shells with the standard orientation are always $(\xi_{(-)}, \xi_{(+)}) = (-1, +1)$ branch. So region $\mathcal{V}_L$ has a different effective cosmological constant to region $\mathcal{V}_R$, as can be seen from the expansion of the metric for large r. For example, when the bare cosmological constant $\Lambda = 0$ we have on one side of the shell the effective cosmological constant $\Lambda_d^{(+)} = -3/2\alpha$ and on the other $\Lambda_d^{(-)} = 0$. In the region with $\Lambda_d^{(+)}$ the graviton is expected to have ghost instability. In this sense the shell is like the false vacuum bubbles[7] studied in Refs. [33], but for a false vacuum which is of purely gravitational origin.

These kind of solutions might lead to curious implications. For instance, let us consider the following construction: Suppose we have a "well behaved" minus branch ($\xi_L = -1$) solution with positive mass M_L; where by "well behaved" we mean a solution in which the singularity is hidden behind an event horizon and for which we get a suitable GR limit for small α. Now, let us cut out the black hole at some radius $r = a(\tau) > r_H$ and then replace it with the interior of a plus branch ($\xi_R = +1$) solution, i.e. a naked singularity. By doing this we would be constructing a vacuum solution whose geometry, from the point of view of an external observer, would coincide with that of a black hole but, instead, would not possess a horizon. A particle in free fall would not find a horizon but rather a naked singularity as soon as it passes through the C^0 junction hypersurface located at $r = a > r_H$. The solutions with $\Lambda = 0$ and $\alpha > 0$ are a clear example of this. As can be seen from figure 10 there are solutions for all positive $M_{(-)}$; so that we can indeed cut out the event horizon and replace it with a naked singularity!

Also, for $\Lambda \neq 0$ "false vacuum bubble" solutions gluing a positive mass Boulware-Deser branch

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[7]Strictly speaking, this label of false vacuum bubble would be correct if the minus branch metric were lower total energy with respect to the plus branch metric and if the classical transition were impossible.

with a naked singularity do exist. This is seen by looking at the moduli space described in figure 12. One might expect that such cosmic-censorship-spoiling shells be unstable and in section 5.1 we will confirm that they are unstable with respect to small perturbations.

4.2 Vacuum wormhole-like geometries

So far, we have discussed different kinds of geometries constructed by a cut and paste procedure of two spaces that were initially provided with the Boulware-Deser metric on them. The strategy was to make use of the junction conditions holding in Einstein-Gauss-Bonnet theory and, in particular, we have shown that solutions with non-trivial topology, which have no analogues in Einstein gravity, do arise through this method. A remarkable example is the existence of vacuum wormhole-like geometries[8], corresponding to the case $\eta_L \eta_R < 0$. These "wormholes" can be thought of as belonging to two different classes: The first class describes actual wormholes, presenting two different asymptotic regions which are connected through a throat located at radius $r_L = r_R = a$; the radius of the throat being larger than the radius where the event horizons (or naked singularities) would be. The two asymptotic regions are $r_L \to \infty$ and $r_R \to \infty$ as measured by the radial coordinate in the respective sides of the junction. This type of geometry is an example of a vacuum spherically symmetric wormhole solution in Lovelock theory and its existence is a remarkable fact on its own. On the other hand, a second class of wormhole-like geometry with no asymptotic regions also exists. This second class is obtained also by considering the orientation $\eta_L \eta_R < 0$, this time cutting away the exterior region of both geometries and gluing the two interior regions together. We shall discuss this later; first let us discuss the static wormhole solutions with two asymptotic regions (actual wormholes).

4.2.1 Geometries presenting two asymptotic regions

Let us begin by emphasizing that such static wormhole solutions only exist if at least one of the two bulk regions corresponds to $\xi = +1$. That is, at least one of the two Boulware-Deser metrics has to correspond to what we have called the exotic branch. This could have deep implications in what regards semiclassical stability [14]. It is also remarkable that for these static wormholes to exist it is necessary that $\Lambda > 0$. Furthermore, the existence of two asymptotic regions demands $\alpha > 0$ (for values $\alpha < 0$ there are only solutions with "closed universe" geometry to be discussed below). Moreover, since the static wormholes only exist if at least one of the branches corresponds to $\xi = +1$, then at least one of the regions connected through the throat possesses a negative effective cosmological constant.

Another interesting feature concerns the stability under radial perturbations. This is seen in Fig 15. In particular, it can be shown that stable static wormholes only exist for the case $\xi_L = \xi_R = +1$; namely, the case where both Boulware-Deser metrics correspond to the exotic branch. Nevertheless, no stable wormholes exist for the case $M_L = M_R$, and thus, concisely, the static symmetric wormholes are unstable under perturbations that preserve the spherical symmetry.

An interesting possibility is that of having wormhole solutions whose Boulware-Deser metrics would correspond to negative mass parameters. For instance, one can construct a static wormhole with one side being of the "good branch" $\xi_L = -1$ and having a negative mass $M_L < 0$. In that case, from the point of view of a naive external observer, the vacuum solution would seem to correspond to a naked singularity. However, now we know that the inclusion of non-trivial junctures makes it possible to replace such a singularity by an exterior region on the other side of a non-smooth wormhole throat. This has a deep implication in what concerns the "cosmic censorship principle" since for the appropriate values of the coupling constants, and unlike what usually happens in pure gravitational theories, the spherically symmetric vacuum solutions presenting naked singularity cannot be unambiguously classified (and consequently systematically excluded) in terms of the mass parameter.

Another particular case that deserves to be mentioned as a special one is that of having a massless solution in one of the sides of the wormhole geometry. For instance, such a construction is achieved if the massless side corresponds to the exotic branch $\xi = +1$ and the massive side to the branch $\xi = -1$.

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[8]Smooth wormhole solutions in Lovelock theory have been found previously with matter source in refs. [34] and without matter for special choice of coupling constants in refs. [35, 26].
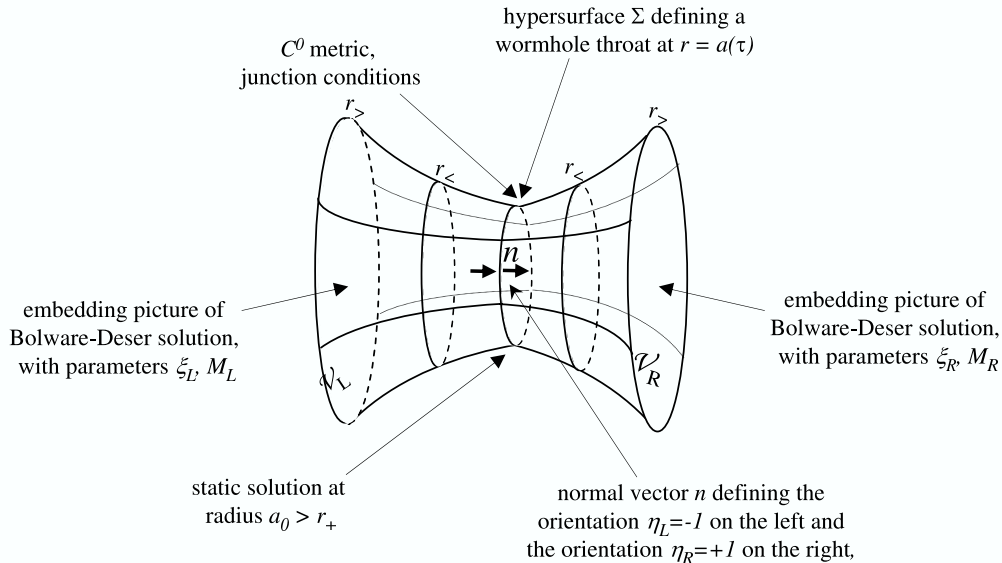
hypersurface Σ defining a
wormhole throat at $r = a(\tau)$

C^0 metric,
junction conditions

$r_>$ $r_>$

$r_<$ $r_<$

n

$\mathcal{V}_L$ $\mathcal{V}_R$

embedding picture of
Bolware-Deser solution,
with parameters ξ_L, M_L

embedding picture of
Bolware-Deser solution,
with parameters ξ_R, M_R

static solution at
radius $a_0 > r_+$

normal vector n defining the
orientation $\eta_L = -1$ on the left and
the orientation $\eta_R = +1$ on the right,

Figure 3: Static vacuum wormhole solution. The junction conditions are satisfied on the timelike hypersurface Σ for the case of the orientation $\eta_L \eta_R < 0$. This solution presents two disconnected asymptotically de-Sitter regions.

In these cases, the wormhole throat turns out to be a kind of puncture of the (A)dS spacetime, let us call it a "hole in the vacuum". Since (A)dS is homogeneously isotropic, a spherically symmetric matching can be done anywhere: remarkably, several of these "holes" could be located at different places in the spacetime and each "hole" would not influence the others. We shall discuss this kind of geometry in more detail in section 7.2. The massless side may then correspond to a microscopic de-Sitter geometry and, presumably, its cosmological horizon, yielding thermal radiation, could be seen from the massive sides. This is an intriguing possibility that deserves to be further explored.

4.2.2 Closed universe type geometries

Now, let us comment on the second class of wormholes; namely those with no asymptotic regions. As mentioned, these geometries are constructed by gluing the interior of the throat of both regions, instead of the exterior. One can perform the matching by keeping the region that is inside the throat but still outside the horizons. Consequently, one gets a geometry that resembles a "static closed universe" with horizons. This exotic geometry has no asymptotic regions at all, and, because of this, this second type of geometry does not represent what one would usually call a wormhole. Nevertheless, we shall abuse the notation and call "wormhole" any timelike junction with the orientation $\eta_L \eta_R < 0$.

Static solutions of this kind without naked singularities (i.e. with horizon) exist for negative values of the coupling α and $x \equiv 4\alpha\Lambda/3 < -1$. In this range of the coupling constants the Boulware-Deser metric develops a branch singularity at fixed radius $r_c^4 = \frac{16 M \alpha}{|x| - 1}$, where the curvature diverges. This branch singularity represents the maximum three-sphere radius: the metric becomes non-real for $r > r_c$. In addition there is a curvature singularity at $r = 0$. In this region of the space of parameters we would say that the Boulware-Deser geometry is somehow pathological. However, if junction conditions are appropriately applied, then a well-behaved C^0 vacuum geometry can be constructed by simply taking a pair of such pathological spaces, cutting out the naked singularities and joining them together. To see that this is possible, it is sufficient to consider the symmetrical case. It can be checked from equations (46) and (47) that two bulk regions with equal masses $M_L/\alpha = M_R/\alpha = \frac{4(1+x)}{(x-3)}$ can be matched at a throat radius $a^2 = \frac{16|\alpha|}{3-x}$. Consulting figure 13 (these solutions are located on the upper bounding curve of the left part of the moduli space) we see that wormhole solutions exist when the bulk regions have branch signs $(\xi_L, \xi_R) = (-1, -1)$. The bulk metric has a horizon r_H, which separates $r = 0$ (a timelike naked singularity) from $r = r_c$, which is a
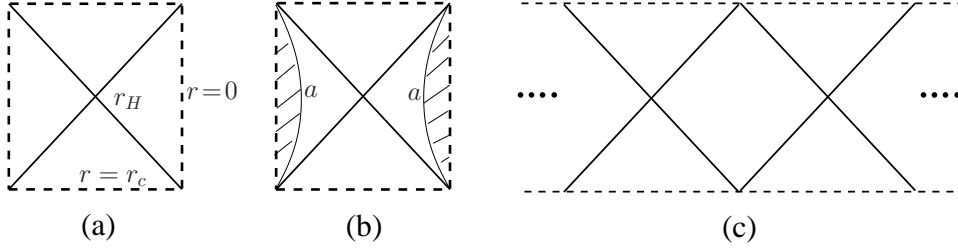
Figure 4: (a) The causal diagram of the smooth spherically symmetric solution for $\alpha < 0$, $1 + \frac{4\alpha\Lambda}{3} < 0$. (b) The vacuum shell is introduced at radius $r = a$ cutting out the $r = 0$ singularity. (c) Causal diagram of the resulting spacetime (a C^0, spherically symmetric vacuum solution).

spacelike singularity. The static shell is located at $a < r_H$. So by cutting out the regions $r < a$ and joining with the wormhole orientation the naked singularities can be removed. The causal diagram of the original pathological spacetimes and the extended causal diagram of the C^0 closed universe, which results from the matching, with horizons are shown in figure 4.2.2.

5   Dynamical vacuum shells

In general, vacuum shells will be dynamical objects. We discuss the dynamics here and also discuss the issue of radial stability of the static solutions.

5.1   General solution

Let us briefly recapitulate upon the equation (20), which governs the dynamics of the shells. We can treat both the timelike and spacelike together since, as we noticed in section 2, the analysis is completely analogous. A dynamical vacuum shell is governed by a differential equation of the form

$$\dot{a}^2 + V(a) = 0\,; \tag{57}$$

see (20) above. It is useful to express $V(a)$ in terms of the non-negative quantity $Y = \sqrt{1 + \frac{4\alpha\Lambda}{3} + \frac{16M\alpha}{a^4}}$, and the effective potential then reads

$$V(a) = \sigma\left(k + \frac{a^2}{4\alpha}\right) - \frac{\sigma a^2}{4\alpha}\left(\frac{3(\xi_R Y_R + \xi_L Y_L)^2 + (\xi_R Y_R - \xi_L Y_L)^2}{12(\xi_R Y_R + \xi_L Y_L)}\right)\,. \tag{58}$$

In addition to the differential equation, the solution must obey the inequalities (26)-(28). It is convenient to rewrite them as follows

$$-\eta_R\eta_L\left(9(\xi_R Y_R + \xi_L Y_L)^2 - (\xi_R Y_R - \xi_L Y_L)^2\right) \geq 0\,, \tag{59}$$

$$\sigma\alpha(\xi_R Y_R + \xi_L Y_L) > 0\,. \tag{60}$$

Note that the effective potential (58), $V(a) = \frac{\sigma a^2}{4\alpha} + \sigma k + \Delta V(a)$, consists of a quadratic piece, a constant determined by the three-dimensional curvature k of the shell, and another piece which, by inequality (60) obeys $\Delta V < 0$.

To analyze the motion of a shell we shall need to know the derivatives of the potential. This is worked out in appendix D. Differentiating the potential we get the following expression for the acceleration of a moving shell,

$$\ddot{a} = -\sigma\frac{a}{4\alpha}\left[1 - \frac{1 + 4\alpha\Lambda/3}{\xi_R Y_R + \xi_L Y_L}\right]\,. \tag{61}$$

Considering the sign of this acceleration and making use of inequality (60) we can make some general observations:

19

Remark 10. *For a timelike shell ($\sigma = +1$):*
When $1 + \frac{4\alpha\Lambda}{3} \geq 0$ and $\alpha < 0$ a vacuum shell always experiences a repulsive force away from $r = 0$;
When $1 + \frac{4\alpha\Lambda}{3} \leq 0$ and $\alpha > 0$ a vacuum shell always experiences an attractive force towards $r = 0$.

In the situations not covered by Remark 10 the potential may have an extremum. From (61) we deduce that there is an extremum at $r = a_e$ iff

$$\xi_R Y_R(a_e) + \xi_L Y_L(a_e) = 1 + \frac{4\alpha\Lambda}{3}. \tag{62}$$

Recalling inequality (60), we conclude that an extremum can exist only if

$$\sigma\alpha\left(1 + \frac{4\alpha\Lambda}{3}\right) > 0. \tag{63}$$

The extremum will be a minimum or maximum depending on the sign of the second derivative of the potential evaluated there,

$$V''(a_e) = \frac{\sigma}{\alpha}\left(\frac{1 + 4\alpha\Lambda/3}{\xi_R\xi_L Y_R(a_e)Y_L(a_e)} - 1\right). \tag{64}$$

There is a general result for vacuum shells separating different branch metrics. From (63) we see that $V''(a_e)$ in (64) must be negative for $1 + \frac{4\alpha\Lambda}{3} \geq 0$.

Proposition 11. *In the range[9] $1 + \frac{4\alpha\Lambda}{3} \geq 0$ for the product of Gauss-Bonnet coupling and cosmological constant: Let Σ be a vacuum shell such that $\xi_L\xi_R = -1$. Then the potential never has a minimum. If Σ is a timelike shell it will either be in an (unstable) static state, or, if it is moving, will either expand or collapse, it can not be bound.*

We have already remarked in section 2 that any shell with standard orientation *must* match two bulk metrics of opposite branch sign ($\xi_L\xi_R = -1$), except in the trivial case of a smooth matching. So we obtain the following general result about instability of standard shells:

Corollary 12. *Let $1 + \frac{4\alpha\Lambda}{3} \geq 0$. A timelike shell with standard orientation is either in an (unstable) static motion, or, if it is moving, will either expand or collapse, it can not be bound.*

We have already seen in section 3 that static shells with standard orientation are always in a state of unstable equilibrium in the (physical) regime $1 + \frac{4\alpha\Lambda}{3} \geq 0$. The proposition above strengthens this result to include dynamical shells. A dynamical shell with standard orientation can not be oscillatory. It must either disappear into a singularity or fly out towards spatial infinity.

There is not such a strong result for shells with the wormhole orientation. Indeed in section 3 we found stable static wormholes for $1 + \frac{4\alpha\Lambda}{3} \geq 0$ which matched two bulk metrics of the plus branch. We can however derive a strong result about instability concerning bulk metrics of the minus branch. When $\xi_L = \xi_R = -1$ we see from (62) that for $1 + \frac{4\alpha\Lambda}{3} \geq 0$ an extremum is not possible. Furthermore, combining the results of Remarks 5 and 10 we see that the shell is always expanding:

Proposition 13. *Let $1 + \frac{4\alpha\Lambda}{3} \geq 0$ and let Σ be a timelike vacuum shell with wormhole orientation, and $\mathcal{V}_L$ and $\mathcal{V}_R$ be minus branch bulk metrics $(\xi_L, \xi_R) = (-1, -1)$. Then the shell always experiences a repulsive force away from $r = 0$.*

So in summary, we have found some general results for the range of parameters $1 + \frac{4\alpha\Lambda}{3} \geq 0$. This range is important because it includes the case $|\alpha\Lambda| << 1$ and therefore applies when the Gauss-Bonnet term is a small correction. Combining these results, we conclude that, in this range of parameters, all timelike vacuum shells involving the minus branch are unstable. The only vacuum shell solutions which can be static or oscillatory are wormholes which match two regions of the exotic plus branch.

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[9]In the case of the wormhole orientation, by using the inequality (59), the result can be extended to apply to the range $1 + \frac{4\alpha\Lambda}{3} > -\frac{1}{2}$.

5.2 Comment on the stability of static shells

Dynamical equation (57) resembles the equation for a particle moving under the influence of an effective potential (58). Nevertheless, as pointed out in section 2, this is not strictly the case due to the presence of the vanishing energy constraint. This is important for the case when the extremum of the potential is at $a = a_0$ with $V(a_0) = 0$, i.e. when static solutions exist. When $V''(a_0) < 0$ we conclude that the shell is unstable with respect to the radial component of a perturbation- a slight shift $a \to a_0 + \delta a$ will cause the shell to accelerate away from the (unstable) equilibrium radius. When $V''(a_0) > 0$ we conclude that the shell is stable with respect to radial perturbations. There is however a slight subtlety: as mentioned previously the energy is unavoidably fixed instead of arising as a constant of motion. So for a fixed potential, there is no real solution for a when $V > 0$. We can consider spherically symmetric solutions which are close-by in the space of the solutions, i.e. with slightly different parameters $M_{L,R}$ and w such that the value of the potential at a_e is slightly negative: let us say $V(a_e) = 0^-$. This means that such a solution oscillates between two radii around a_e at which the potential vanishes. This is certainly a stable solution though not static, a 'bounded excursion' [31]. Now if we let a_e coincide with the a_0 of the original static solution, this means that for slightly different parameters than those for which a_0 is a static solution, there exists an oscillating solution around a_0. Therefore a static solution a_0 which is a minimum of the potential gives information about when infinitesimal bounded excursions can happen. More generally, the dynamics of the perturbed shell can be thought of as corresponding to a perturbation of the above equation $V(a) \to V(a) - \delta h$, provided energy δh from an external excitation. The stable regions of the moduli space of static solutions are plotted in figures 15 and 16. The graph will take an elegant form in terms of the change of variables to be introduced in section 6 (see fig 8).

In the rest of this section we present some illustrative examples of dynamical vacuum shells, first in symmetrical wormhole solutions and then in the context of Chern-Simons gravity.

5.3 Symmetric dynamical wormholes

Now let us consider the case where the masses in each bulk region are the same, being $M_L = M_R = M$. The inequalities (59) and (60) are equivalent to:

Remark 14. *If Σ is a vacuum shell joining two bulk regions with the same mass $M_R = M_L$ then:*
i) The bulk solutions must have the same branch sign $\xi_L = \xi_R = sign(\sigma\alpha)$;
ii) The shell must have wormhole orientation.

So the spacetime is completely left-right mirror-symmetric. The equation of motion reads

$$\sigma\dot{a}^2 + \frac{a^2}{4\alpha}\left[1 - \xi\frac{Y(a)}{2}\right] + k = 0 \ . \tag{65}$$

The general solution is rather complicated. Next we proceed to consider a simple case where both masses vanish.

The case where $M_{L,R} = 0$ is an interesting special case of the symmetric wormholes, which exists for $1 + \frac{4\alpha\Lambda}{3} > 0$. The equation of motion reduces to

$$\sigma\dot{a}^2 + \frac{a^2}{4\alpha}\left(1 - \xi\frac{\sqrt{1 + \frac{4\alpha\Lambda}{3}}}{2}\right) + k = 0 \ . \tag{66}$$

Remark 15. *Consider a timelike shell ($\sigma = +1$), that is $sign(\alpha) = \xi$.*
Bounded motions: $\xi = +1$, $\frac{4\alpha\Lambda}{3} < 3$, $k = +1$; $\xi = +1$, $\frac{4\alpha\Lambda}{3} = 3$, $k = 0$.
Unbounded motions: $\xi = +1$, $\frac{4\alpha\Lambda}{3} = 3$, $k = -1$; $\xi = +1$, $\frac{4\alpha\Lambda}{3} > 3$, any k ; $\xi = -1$, $\frac{4\alpha\Lambda}{3} > -1$, any k.
The same bounded or unbounded configurations exist in the spacelike case provided one replaces k with $-k$, for the opposite sign of α.

The hyperbolic shell, $k = -1$, admits a stationary vacuum wormhole solution: for $sign(\alpha) = \xi = +1$ and $4\alpha\Lambda/3 = 3$ we have that $\ddot{a} = 0$ and $\dot{a}^2 = 1$.

When $\Lambda = 0$ and $\xi = -1$ the two bulk regions have flat Minkowski metrics. When the throat of the symmetric wormhole is a sphere the world-volume of the shell is described by $t^2 - r^2 = 8\alpha/3$. When $\alpha > 0$ the shell is spacelike; when $\alpha < 0$ it is timelike.

5.4   Chern-Simons dynamical vacuum junctions

When $1 + \frac{4\alpha\Lambda}{3} = 0$ some very special things happen. For this choice of coupling constants the Einstein-Gauss-Bonnet theory (in first order formalism) is equivalent to a Chern-Simons theory for the deSitter ($\alpha < 0$) or Anti de Sitter ($\alpha > 0$) group[10]. In this case the metric function takes the very simple form

$$f(r) = 1 + \frac{r^2}{4\alpha} - \mu \ , \tag{67}$$

where μ is a constant and the mass is proportional to $\mu^2 - 1$[18]. When $\mu > 0$ the bulk solution is a black hole. When $\mu < 0$ the bulk metric would have a naked singularity at the origin.

The dynamics of vacuum shells takes a very simple form. The quantity $a^2 \xi Y / 4\alpha = -\mu$ for each bulk region is a constant and therefore the non-harmonic part of the potential ΔV is a constant. The equation of motion takes the form

$$\dot{a}^2 + \frac{\sigma}{4\alpha}a^2 = \mathcal{E} \quad , \qquad \mathcal{E} = -\sigma \left(k + \frac{3(\mu_R + \mu_L)^2 + (\mu_R - \mu_L)^2}{12(\mu_R + \mu_L)} \right) \ . \tag{68}$$

The potential is like that of a harmonic oscillator potential (or an upside-down harmonic potential if $\sigma\alpha$ is negative), although it should be remembered that the origin $r = 0$ of the bulk spacetimes is singular so the shell can not really oscillate. The solution is constrained according to the two inequalities (59) and (60), which now read

$$- \eta_L \eta_R (9(\mu_R + \mu_L)^2 - (\mu_R - \mu_L)^2) \geq 0 \ , \tag{69}$$

$$- \sigma (\mu_R + \mu_L) > 0 \ . \tag{70}$$

The last inequality tells that $\mathcal{E} > -\sigma k$. These inequalities are generally consistent with $\mathcal{E} > 0$ so that solutions do indeed exist.

For instance, consider the timelike shells in this theory. Note that, from inequality (70), at least one out of μ_R or μ_L must be negative. So it is not possible to match two black hole spacetimes. From inequality (69) we see that shells with the standard orientation must obey $\mu_R \mu_L < 0$.

Remark 16. *For the Chern-Simons combination $1 + \frac{4\alpha\Lambda}{3} = 0$, timelike vacuum shells always represent either:*
i) a matching between a bulk region of a black hole spacetime with bulk region of a naked singularity spacetime; or
ii) a matching, with wormhole orientation, between two bulk regions of naked singularity spacetimes.

Now, let us analyze the de-Sitter invariant Chern-Simons gravity, which corresponds to $\alpha\Lambda = -3/4$ with $\alpha < 0$. In this case, the potential is like an inverted harmonic oscillator centered at the origin. There are solutions for $\mathcal{E}$ positive, negative and zero.

Let us just focus on the case $\mathcal{E} > 0$. The trajectory of a timelike shell is then given by

$$a(\tau) = 2\sqrt{|\alpha\mathcal{E}|} \ \sinh \left(\pm \frac{\tau}{2\sqrt{|\alpha|}} + \text{const.} \right) \ , \tag{71}$$

which is a shell either emerging from the past white hole or falling into the future black hole, depending on the sign $\pm$ in the argument.

For $\mathcal{E} < 0$ the hyperbolic sine is replaced by the cosine. $\mathcal{E} = 0$ gives an increasing and a decreasing exponential.

On the other hand, for $\mathcal{E} > 0$ and $k = 1$, one could consider Euclideanization of the problem. Presumably, this could be relevant in describing the decay of the exotic negative μ spacetime. Define an angle χ by

$$\chi = \frac{\tau_E}{2\sqrt{|\alpha|}} \tag{72}$$

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[10] The case of Poincaré Chern-Simons theory was discussed in Ref. [13].

up to a constant, where τ_E is the Euclidean proper time of the shell. The metric on the Euclidean world sheet of the shell reads

$$ds_\Sigma^2 = 4|\alpha|\left(d\chi^2 + \mathcal{E}\sin^2\chi d\Omega^2\right) . \tag{73}$$

When $\mathcal{E} < 1$ we have a deficit solid angle, and, when $\mathcal{E} > 1$, an excess. In both cases the space has a curvature singularity at the poles $\chi = 0$ and $\chi = \pi$. Therefore, the smoothness of the Euclidean shell requires $\mathcal{E} = 1$. This metric is spherically symmetric in the five dimensional sense in this case, whence it describes a 4-sphere. The 4-sphere separates a ball of Euclidean black-hole solution with mass parameter μ_R from another solution with μ_L, obeying the relation

$$\mu_L^2 + \mu_R^2 + \mu_L\mu_R + 6(\mu_L + \mu_R) = 0 . \tag{74}$$

It is interesting to note that the size of the Euclidean world sheet depends essentially only on α and not on the $\mu_{L,R}$; the latter change its shape, which is fixed to spherical by the above relation.

The curve (74) is an ellipse. It is symmetrical around the line $\mu_L = \mu_R$ and tangential with the line $\mu_L + \mu_R = 0$ at $\mu_L = \mu_R = 0$. It exists completely in the region $\mu_L + \mu_R \leq 0$. In view of the inequality (70) all points of the curve are included except $\mu_L = \mu_R = 0$. Therefore the 4-sphere Euclidean world sheet does exists for certain values of the parameters. Whether this interesting configuration is a mere curiosity or it is related to semiclassical transitions between the μ_L and μ_R spacetime is an open question.

We can also consider the Anti-de Sitter invariant Chern-Simons theory, corresponding to $\alpha > 0$. In this case, the effective potential V turns out to be a quadratic potential centered at the singularity at the origin. The analysis is similar to that of the dS case except that there are solutions only with $\mathcal{E} > 0$.

On the other hand, we can also think about the spacelike shells for this case of Chern-Simons couplings $\Lambda\alpha = -3/4$. These shells represent a sudden classical transition from a spacetime with some mass parameter $\mu_{\rm in}$ to another with a different mass parameter $\mu_{\rm out}$. Such transitions occur for quite general values of $\mu_{\rm in}$, and this is a concise manifestation of the extreme degeneracy of the field equations of the Chern-Simons theories.

6 Constant solutions revisited

In this section we will perform an exhaustive analysis of the space of constant solutions, what we have called the moduli space.

6.1 Static and instantaneous spherical vacuum shells

To begin, it will be convenient to introduce new dimensionless parameters, defined as follows

$$u \equiv \sqrt{3}\sqrt{x(x+1)\left(\frac{4}{y} + \frac{3}{x} - 1\right)} \quad , \quad w \equiv x + 1 . \tag{75}$$

On should think of u and w as functions of α, Λ and a_0^2, via the definitions (36). Here,

$$u \geq 0 .$$

The inverse transformation is given by

$$y = \frac{12w(w-1)}{u^2 + 3(w^2 - 4w)} \quad , \quad x = w - 1 . \tag{76}$$

Each point on the (w, u) plane such that $u^2 + 3(w^2 - 4w) \neq 0$ uniquely determines the values of the basic dimensionless ratios x and y and therefore the solution[11].

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[11]It is useful to remember that the radius a_0 of the vacuum shell is given in terms of these variables by

$$a_0^2 = 4\alpha \cdot \frac{12w}{u^2 + 3(w^2 - 4w)} . \tag{77}$$

Definition 17. *We will call the allowed domain on the (u, w) plane as the (u, w) parameter space representing the moduli space of the vacuum shell. Similarly the allowed domain on the plane of x and y is the (x, y) parameter space for the vacuum shell for non-zero Λ. The various possible pairs of parameters that uniquely represent all possible points of the moduli space can be thought of its coordinates.*

In terms of these new variables we have that the vacuum shells are described by equations (see Proposition 6):

$$f_L + f_R = \frac{2(u^2 + 9w^2)}{u^2 + 3(w^2 - 4w)} \ , \qquad \sqrt{f_L f_R} = \sigma \eta_R \eta_L \, \frac{u^2 - 9w^2}{u^2 + 3(w^2 - 4w)} \ . \tag{78}$$

Solving this (for details see appendix E) we find:

Proposition 18. *The masses in the two bulk regions are $M_{(+)}$ and $M_{(-)}$:*

$$\bar{M}_{(\pm)} = \frac{36w^2((w \pm u)^2 - 4w)}{(u^2 + 3(w^2 - 4w))^2} \ . \tag{79}$$

The moduli space is divided into: timelike shell or spacelike shell solutions by the inequality $u^2 + 3(w^2 - 4w) > 0$ or < 0 respectively; standard orientation and wormhole orientation by $u^2 - 9w^2 > 0$ or < 0 respectively. (see Fig. 5). Furthermore the branch sign of the metric in each region is given by $\xi_{(\pm)} = sign(w \pm u)$.

The points along the curves $\pm u = 3w$ and $u^2 + 3(w^2 - 4w)$ are not regarded as part of the moduli space. Along the curve $u^2 - w^2 = 0$, which we will call the *branch curve*, one of the branch signs is undetermined. The line $w = 0$ is peculiar as it implies that either $a_0 = 0$, or $|\alpha| = \infty$ and $\Lambda = 0$. The latter is the case of pure Gauss-Bonnet gravity, whose solutions were found in Ref. [13]. For $|\alpha| < \infty$ the line $w = 0$ is excluded from the moduli space as it corresponds to smooth geometries.

We further note that the signs of α and Λ on the moduli space are given by $\text{sign}(\alpha) = \text{sign}\big(w\,(u^2 + 3(w^2 - 4w))\big)$ and $\text{sign}(\Lambda) = \text{sign}(\alpha)\,\text{sign}(w - 1)$. The region $w \approx 1$ is where the Gauss-Bonnet coupling is small with respect to the cosmological scale.

In what follows we further categorize the solutions according to other physical properties. The entire information we will get is given in the Fig.6.

We first note that the formulae (79) for $M_{(+)}$ and $M_{(-)}$ are related simply by $u \leftrightarrow -u$. Although the true moduli space is the upper half plane $u \geq 0$, it is useful to formally extend to $u < 0$ (see Lemma 42). In this way we may plot inequalities for the region with mass $M_{(-)}$ in the lower half plane and for the region corresponding to $M_{(+)}$ in the upper half plane.

6.2 The masses over the moduli space

Important physical properties of the solutions have to do with what values the masses $M_{(\pm)}$ take, w.r.t. sign, magnitude and relative magnitude, over the moduli space. Let us comment on it below.

6.2.1 Equal mass solutions

A question with a very simple answer is where on the moduli space we could have $M_{(+)} = M_{(-)}$. We have seen that this happens at $u = 0$. Explicitly, from (79) we have

$$\bar{M}_{(+)} - \bar{M}_{(-)} = \frac{144uw^3}{\big(u^2 + 3(w^2 - 4w)\big)^2} \ . \tag{80}$$

Proposition 19. $M_{(+)} = M_{(-)}$ *only at the boundary $u = 0$. Therefore such solutions exist only for wormholes.*

From Proposition 41 we have that at the points where $M_{(+)} = M_{(-)}$ we have also that $\xi_{(+)} = \xi_{(-)}$.

Lemma 20. *Symmetric configuration are such $M_L = M_R$ and $\xi_L = \xi_R$. They exist only at the boundary $u = 0$ of the moduli space and they can be either time- or space-like shell wormholes.*
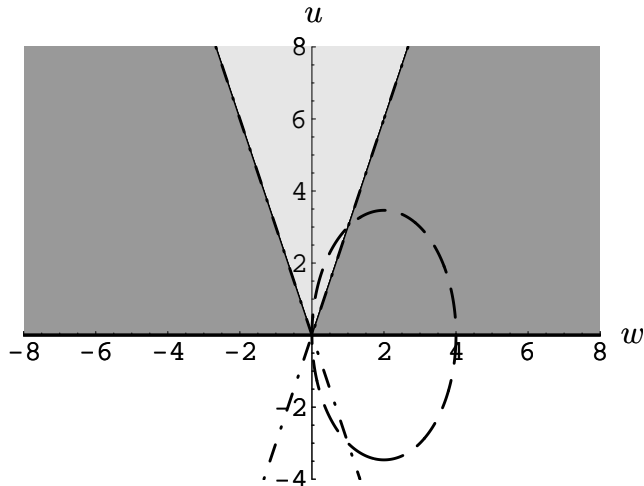
Figure 5: The space of *constant radius* solutions, which we have called the moduli space, is depicted here. The dimensionless variables w and u are defined at the beginning of section 6.1. The ellipse divides solutions into spacelike (inside) and timelike (outside); The diagonal lines divide solutions into standard orientation (light grey) which have well-defined inner and outer region of the shell, and wormhole orientation (dark grey), where both regions can be thought of as exterior or interior depending on whether a non-compact or compact region is maintained. Solutions exist for $u \geq 0$. The line $w = 0$, $u > 0$ for finite α does not actually belong to the moduli space as being trivial: the junction condition require the metric across the shell must be continuous in this case. In terms of the couplings α and Λ, w is given simply by $w \equiv 1 + \frac{4\alpha\Lambda}{3}$. The combination of the couplings $w = 0$ corresponds to the case where Einstein-Gauss-Bonnet gravity can be written as a Chern-Simons theory with (A)dS gauge group. It for this combination that the smooth C^2 metrics fail to be unique [23][25]. Note that the pure Gauss-Bonnet case, which formally corresponds to $w = 0$, $\Lambda = 0$ in the limit that $\alpha \to \infty$ but $M\alpha$ is finite, does have nontrivial solutions, which were considered separately in Ref. [13].

The equal mass $\bar{M} = \bar{M}_{(\pm)}$ of the symmetric case reads

$$\bar{M} = \frac{4w}{w - 4} \ . \tag{81}$$

So, symmetric configurations exist for all $w \neq 4$ and $\bar{M}$ can take all real values except 4.

6.2.2 Zero mass solutions

The masses $\bar{M}_{(\pm)}$ change sign crossing the curves where they vanish, and of course these curves are where $M_{(\pm)}$ vanish too. From the formula (79) and Proposition 41 we have that $\bar{M}_{(\pm)} = 0$ along the curves

$$(u \pm w)^2 = 4w \ , \tag{82}$$

respectively. They exist only for $w > 0$. The masses cannot vanish for $w < 0$.

Using Lemma 42 it is sufficient to look only at one of the curves e.g. $(u - w)^2 = 4w$, which also reads $u = \pm 2\sqrt{w} + w$, extended over the whole plane. The other mass is

$$\bar{M}_0^{(\pm)} = \frac{9w\sqrt{w}}{(w - 3)\sqrt{w} \pm 2} \ . \tag{83}$$

The curve $(u - w)^2 - 4w = 0$ goes to negative values of u for $0 < w < 4$. On the $u > 0$ side it appears disconnected emerging into two pieces at $w = 0$ and $w = 4$, fig. 6. Therefore
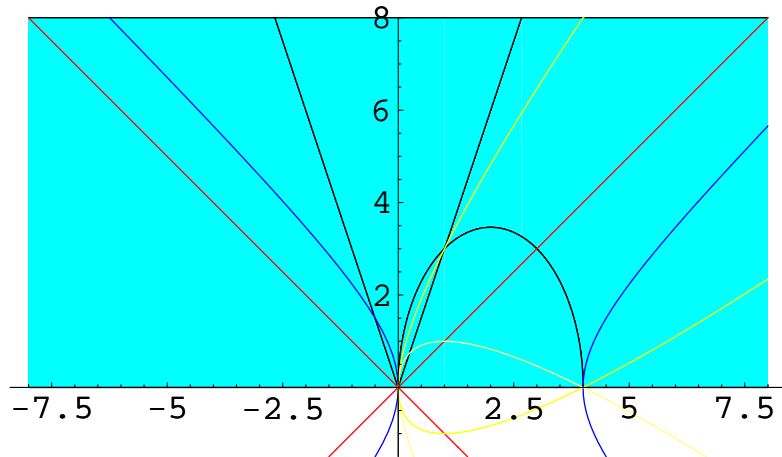
Figure 6: The moduli space showing the various curves listed in section 6.3. The basic division according to the time- or space-likeness of the world-volume of the shell and the orientation of the matching were described in the figure 5 and are shown in black lines here: the points along them they do not belong to the moduli space.

The diagonal (red) lines divide the space according to the branches of the bulk metrics which we classify by the pair of signs $(\xi_{(-)}, \xi_{(+)})$ explained in section 3.2 and 6.1: in the region on the left the branch signs are $(+, +)$ i.e. both the bulk metrics on each side of the shell belong to the "exotic" Boulware-Deser branch which does not have a well-defined limit $\alpha \to 0$; the region in between the diagonal lines is $(-, +)$; in the region on the left the branches are $(-, -)$ i.e. both metrics belong to the branch with a well-defined $\alpha \to 0$ limit (however the vacuum juncture requires that these solutions only exist for $w \equiv 1 + \frac{4\alpha\Lambda}{3} < 0$, i.e. they have no asymptotics: certain curvature singularities appear at finite radius [26]).

The hyperbola that exists on the outside of and touches the elliptic region of spacelike solutions only at the border of the ellipse at the points $w = 0$ and $w = 4$ (blue line), is what we have called the stability curve: crossing this curve the second derivative $V''(a_0)$ of the potential (20) or (57) evaluated at the constant solutions $a = a_0$ changes sign, which is a measure of (in)stability under perturbations. The constants solutions for which $V''(a_0) > 0$ are depicted in figure 8.

The remaining two lines (yellow lines) are symmetric around the horizontal line $u = 0$. Each curve corresponds to solutions such that one of the mass parameters vanishes i.e. one of the bulk regions is pure vacuum. Note that they exist only for $w = 1 + \frac{4\alpha\Lambda}{3} > 0$. These configurations are discussed in sections 7.2 and 7.3 as an interesting example of certain non-trivial features C^0 metrics acquire when Einstein gravity is supplemented by the Gauss-Bonnet term is five dimensions.
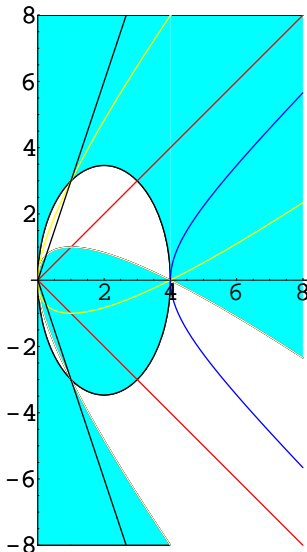
Figure 7: In the upper half plain, the shaded region is where $M_{(+)} > 0$. The inequality $M_{(-)} > 0$ has been plotted in the lower half plain, making use of Lemma 42 (it should be remembered that in the lower half plane we have actually plotted $M_{(-)}(-u)$)). We note that $M_{(+)}$ and $M_{(-)}$ are both negative for all $w < 0$, which is the left half of the diagram.

Proposition 21. $\bar{M}_{(-)} = 0$ for $u = \pm 2\sqrt{w} + w > 0$ where $\bar{M}_{(+)} = \bar{M}_0^{(\pm)}$ respectively. $\bar{M}_{(+)} = 0$ for $u = 2\sqrt{w} - w > 0$ where $\bar{M}_{(-)} = \bar{M}_0^{(-)}$.

Independently of whether the mass that vanishes is an $M_{(+)}$ or an $M_{(-)}$ note also the following

Remark 22. *When the zero mass is of branch ξ the massive side has mass $\bar{M}_0^{(-\xi)}$ and the matching happens according to $u = |w - 2\xi\sqrt{w}| > 0$. The branch of the massive side depends, as always, on which side of the line $u = w$ we are.*

6.2.3 Sign of the mass parameters

Now, let us discuss the positivity of the mass parameters. The signs of $\bar{M}_{(\pm)}$ behave quite simply. From formula (79) we have:

Proposition 23. *$\bar{M}_{(\pm)} < 0$ at the convex region defined by the curves (82) i.e. where $(u \pm w)^2 - 4w < 0$ respectively. They have an overlap for $0 \leq u < 2\sqrt{w} - w$, inside the spacelike shell wormhole region. $\bar{M}_{(+)} < 0$ only in this overlap.*

Remark 24. *The entire curve $u - w = 0$ exists within the region where $\bar{M}_{(-)} < 0$. This is also seen by the fact that the r.h.s. of (E.3) vanishes there. $\bar{M}_{(+)} > 0$ along $u - w = 0$.*

The above mean that $\bar{M}_{(-)} < 0$ in a very large part of the moduli space for $w > 0$. Therefore the metrics $f_{(-)}$ will have inner branch singularities, discussed in appendix B. The signs of $M_{(\pm)} = \alpha \bar{M}_{(\pm)}$ themselves are depicted in the figure 7 using also formula (E.6).

6.2.4 Mass as a function of the radius of the shell

We see from (79) that given a mass $M_{(\pm)}$, for given α and Λ, the radius a_0 of the shell where the matching takes place is determined by a fourth order polynomial of u, namely

$$(u^2 + 3(w^2 - 4w))^2 \bar{M}_* - 36w^2 (u^2 + 2wu + w^2 - 4w) = 0 ; \tag{84}$$

given a u we can obtain a_0^2 by (77). As discussed in Lemma 42, $\bar{M}_*$ is an $\bar{M}_{(+)}$ when u is non-negative and an $\bar{M}_{(-)}$ when u is non-positive.

The equation above does not seem to be very enlightening. However, we can combine it with some pieces of information we have: First we know that u takes values on the entire real line. Secondly, there is at least one real solution u, since $\bar{M}_*$ is *defined* by (79) to correspond to some real u. Besides, $\bar{M}_*$ takes all real values itself as one may verify.

So, one may ask the following: For a given $\bar{M}_*$, and a given w, how many *different* real solutions u exist and what is their sign? Now, the l.h.s. of (84) is an even order polynomial. Then we know that there must be at least a second u producing the same M_*. What we a priori do not know is whether the second u is of the same sign or of the opposite.

There is one case where the second solution coincides with the first, and therefore has the same sign. This is when the root u is also an extremum of the polynomial. It is easy to verify when this happens. We simply differentiate the polynomial w.r.t. u and use (79) to find the following answer

$$(u + 3w)(u^2 - w^2 + 4w) = 0 \ . \tag{85}$$

The points on the orientation curve $u + 3w = 0$ are not included in the moduli space. Therefore we have that there is a single u when $\bar{M}_*$ and w are such that $u^2 - w^2 + 4w = 0$. We will see below that this is the *stability curve* i.e. the curve which separates the radially stable from the unstable solutions on the moduli space as we will see below (see also figure 8).

A related fact is given in the following

Remark 25. *For fixed α and Λ we can think of the masses as functions of the radius of the shell a_0: $M_{(\pm)} = M_{(\pm)}(a_0)$. The function $M_{(+)}(a_0)$ has a global minimum and the function $M_{(-)}(a_0)$ has a global maximum for radii a_0 given by $u^2 - w^2 + 4w = 0$.*

Thus, there is simpler question one may ask: Given *pair* of masses $M_{(+)}$ and $M_{(-)}$, when can the matching happen at more than one shell radii a_0?

The answer is that this can never happen:

Proposition 26. *For any w, any u such that (w, u) belongs to the moduli space gives a pair of mass parameters $\bar{M}_{(+)}$ and $\bar{M}_{(-)}$. Then, this is the unique u that gives these mass parameters.*

The proof is given in appendix E. Therefore, remembering that u is single valued in terms of the shell radius a_0, the junction conditions define a single-valued function $a_0 = a_0(M_{(-)}, M_{(+)})$, in fact one-to-one on the space of the allowed values of $M_{(\pm)}$. As we know from section 2.3 a_0 is a symmetric function of M_L and M_R and it is given by $a_0 = a_0(M_{(-)}, M_{(+)})$, via the correspondence implied in (40). Thus given the bulk metrics, the a =constant vacuum shell is *unique*. So we see that a weakened version of uniqueness of solutions does survive. Note that for shells with standard orientation there are exactly two inequivalent configurations corresponding to the same shell radius, depending on whether $M_{(+)}$ is the mass of the inner or the outer region.

6.3 The spectrum of curves

We notice that, throughout the computations, the quantity

$$W \equiv w^2 - 4w = w(w - 4) \ , \tag{86}$$

appears often. Now we comment on how it turns out to be convenient to extract information on the moduli space. First, notice that W clearly vanishes at $w = 0$ and $w = 4$. We also encounter the curves

$$u^2 = \pm W, \quad u^2 = \pm 3W, \qquad u = \pm w, \quad u = \pm 3w, \quad u = 0 \ ; \tag{87}$$

which in detail correspond to

$$
\begin{aligned}
&u^2 = 3(w^2 - 4w): \quad \bar{M}_{(+)} + \bar{M}_{(-)} = 2\bar{M} \text{ curve,}\\
&u^2 = -3(w^2 - 4w): \quad \text{causality curve,}\\
&u^2 = (w^2 - 4w): \quad \text{stability curve,}\\
&u^2 = -(w^2 - 4w): \quad \bar{M}_{(+)} + \bar{M}_{(-)} = 0 \text{ curve,}\\
&u = \pm 3w: \quad \text{orientation curve,}\\
&u = \pm w: \quad \text{branch curve,}\\
&u = 0: \quad \text{boundary curve (where } \bar{M}_{(\pm)} = \bar{M}) \ .
\end{aligned}
\tag{88}
$$

We also found the curve where $\bar{M}_{(\pm)} = 0$ to be

$$
\begin{aligned}
&(u - w)^2 - 4w = 0 \ , \quad \text{i.e.} \quad u = u^{(\pm)} = \pm 2\sqrt{w} + w: \quad \text{zero minus-mass curve ,}\\
&(u + w)^2 - 4w = 0 \ , \quad \text{i.e.} \quad u = -u^{(-)} = 2\sqrt{w} - w: \quad \text{zero plus-mass curve ,}
\end{aligned}
$$

respectively. And notice that in terms of W this simply reads

$$
u^{(+)} u^{(-)} = W \ .
\tag{89}
$$

The first four curves in our list, which involve W, are conic sections with symmetry axes the lines $w = 2$ and $u = 0$. The orientation and branch curves on the other hand have symmetry axes the lines $w = 0$ and $u = 0$. The conic sections and especially the causality curve which is an ellipse, $u^2 + 3(w - 2)^2 = 12$, break the symmetry between positive and negative values of w. The image of the causality curve around $w = 0$ would be centered at $w = -2$ i.e. $x = -3$.

The above analysis manifestly shows that the quantity $W \equiv w^2 - 4w$ captures much important information about the moduli space.

Actually, the parameterization of the space of solutions in terms of variables (u, w) had shown to present advantages in order to classify the whole set of solutions. To emphasize this, and for completeness, let us also express the regions of radial stability over the moduli space in terms of these variables. Such regions are known to be characterized by the second derivative of the effective potential, which in terms of u and w is seen to be

$$
V''(a_0) = -\frac{1}{a_0^2} \frac{w \left(u^2 - w^2 + 4w\right)}{\left(u^2 - w^2\right)\left(u^2 + 3(w^2 - 4w)\right)} \ .
\tag{90}
$$

The regions where $V''(a_0) > 0$ are shown in figure 8. $V''(a_0) = 0$ along the curve $u^2 - w^2 + 4w = 0$ which we have already called the stability curve, for reasons that become clear now. According to remark 25 this is where the mass $\bar{M}_{(\pm)}(a_0)$ have extrema.

7 Nontrivial features of C^0 metrics

7.1 Topology

In Einstein gravity in four dimensions there is a variety of smooth, everywhere non-singular vacuum configurations in general characterized by some non-trivial topological property, e.g. Euler number. Any topological property they may have is an intrinsic feature of the smooth solution.

In five dimensions and in Einstein-Gauss-Bonnet gravity similar configurations may exist as well. The equations of motions though are such that one can manufacture, by cut and paste along the world-volume of vacuum shells, similar kind of solutions with the difference that they are not smooth, i.e. not C^1. In this case there is no intrinsic property in the vacuum solution, we are simply building objects which are much simpler locally. For that reason one may call these objects non-topological, though they certainly have non-trivial topological features. An analogy for this would be the difference existing between an object with an exactly given smooth metric which has the topology, e.g. of the sphere, and tetrahedra built out of flat pieces.
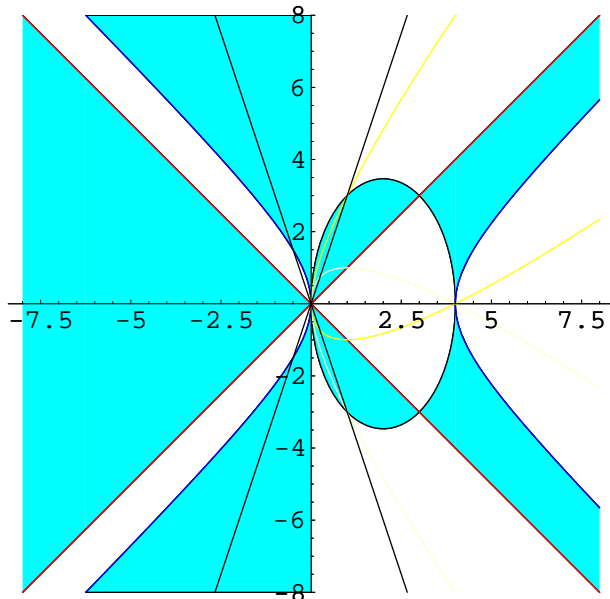
Figure 8: The shaded regions are where $V''(a_0) > 0$. For the timelike shells (outside of the ellipse), the unshaded regions correspond to solutions unstable with respect to radial perturbations.

This digression leads us to recognize a great difference with respect to four dimensions. Unlike in four dimensions, in five-dimensional Lovelock theory, spacetimes which are defined by some simple property locally, for example being vacuum and spherically symmetric, are by no means well defined globally, if smoothness is given up. For each such metric, which may itself have non-trivial topological features, one can construct infinitely many other spacetimes by cut and paste which locally are given by the same simple property. That is, the theory allows for many different topologies where one would expect it to allow only for different coordinates.

A general analysis of the objects obtained by geometric surgery along vacuum shells is an interesting problem and contains much of the actual physics of five-dimensional Lovelock gravity (that is, Einstein plus the Gauss-Bonnet term). In this work we mainly focus on the direct implications of their existence illustrated by appropriate examples. A systematic analysis is left for future work. Below we analyze how a constant curvature vacuum is modified by wormholes (and related configurations). It turns out that, the smaller such constructions with wormholes are with respect to the scale set by α, the more complicated the topology can be.

7.2   Holes in the vacuum

An interesting special case of a wormhole is when on one side we have pure vacuum, as mentioned already in section 4. Starting from a constant curvature background, by introducing the vacuum wormhole we cut a hole in the constant curvature manifold, replacing it with an "outgoing" spacetime region of mass parameter M. Of course the topology of the vacuum is not the same anymore; there are now non-contractible 3-spheres. Nevertheless, it turns out that these configurations are everywhere non-singular in the following sense: *the only singularities that exist in spacetime are integrable*[12].

These wormholes belong to a more general class of constructions: Depending on whether the vacuum shell is time- or space-like and the orientation of the matching (i.e. the different combinations of the orientation signs η_L and η_R), one obtains distinct types of configurations some of which contain only integrable singularities. Configurations we call "instantons" mentioned below belong to this more general class.

---

[12]The curvature and Lovelock tensor are singular at the shell but only in the sense of delta functions. Local integrals of these quantities are finite and the physical laws defined by the field equations do not break down there. In this sense the solutions are not singular.

The analysis gets simplified and clarified if we express everything in terms of the constant curvature. When the mass parameter is zero the metric is defined as

$$f(r) = 1 - Kr^2 . \tag{91}$$

This means that for the metric of branch ξ we have:

$$4\alpha K = -(1 + \xi\sqrt{w}) . \tag{92}$$

These configurations exist for $w > 0$.

We consult Remark 22 and also Proposition 21. The relevant points on the moduli space are on the curve $u = |w - 2\xi\sqrt{w}| > 0$. According to Proposition 38 the points on this curve such that $u > 3w$ correspond to standard shell configurations. In detail, standard shells are the configurations corresponding to: $u = 2\sqrt{w} - w$ for $w \in (0, 1/4)$, and $u = 2\sqrt{w} + w$ for $w \in (0, 1)$.

$u = |w - 2\xi\sqrt{w}|$ is a continuous curve. The points with $w = 0$ and $w = 4$ do not belong in the moduli space. The same for the points with $w = 1/4$ and $w = 1$. So in all, from (92) we have that $4\alpha K \neq -3, -3/2, -1, 0$.

Now from Remark 22 and Proposition 21 one finds that the mass in all cases is

$$M = 9\alpha \frac{(4\alpha K + 1)^3}{(4\alpha K)^2 (4\alpha K + 3)} . \tag{93}$$

Also

$$\Lambda = 6K + 12\alpha K^3.$$

We have that this construction is possible when $w > 0$. Therefore, from Remark 44, we have that the sign of α depends solely on the causal character of the vacuum shell. Namely, it is $\alpha > 0$ when the shell is timelike, and $\alpha < 0$ when the shell is spacelike. We have the following

Remark 27. *All standard orientation shell configurations with zero mass in one of the bulk regions are spacelike.*

Remark 28. *The variable $4\alpha K$ takes values on the entire real line with the exception of the points $-3, -3/2, -1, 0$. With these exceptions in mind we have:*

Spacelike shells i.e. $\alpha < 0$: $4\alpha K \in (-3, 0)$. In the interval $(-3/2, 0)$ exist all the standard shell configurations.

Timelike shells i.e. $\alpha > 0$: $4\alpha K \in (-\infty, -3) \cup (0, \infty)$.

The mass M has poles at the boundary of the spacelike shell region. One may note that thought of as a function of α both poles are of first order.

From the formula $u = |w - 2\xi\sqrt{w}| > 0$ we find for the radius of the shell in all cases is

$$a_0^2 = K^{-1} \left(1 + \frac{4\alpha K}{3} \right)^{-1} . \tag{94}$$

The vacuum of constant curvature K is a locally homogeneous spacetime and in particular is locally spatially homogeneous. Having placed one vacuum shell around some arbitrarily chosen origin, we have seen that outside of the shell the homogeneity is everywhere maintained. As long as it does not cross the first, we may place a second vacuum shell and in fact an arbitrary number of them modifying the manifold in a way depicted in Fig. 9.

Let $K > 0$ and $\alpha > 0$. It is useful to rewrite this as

$$\alpha K = \frac{3}{8} \left\{ \sqrt{\frac{16}{3} \frac{\alpha}{a_0^2} + 1} - 1 \right\} . \tag{95}$$

It is clear from both the last formula that in units of α the radius of the shell is an increasing function of the radius of universe $1/\sqrt{K}$. When the shell is microscopic, i.e. small in units of α, we have that $Ka_0^2 \ll 1$. When the shell is macroscopic we have that $Ka_0^2 \simeq 1$. A microscopic universe could fit roughly

$$(Ka_0^2)^{-2} = \frac{1}{4} \left(\sqrt{\frac{16}{3} \frac{\alpha}{a_0^2} + 1} + 1 \right)^2 \tag{96}$$

vacuum shells of radius a_0. So the more microscopic the universe the more complicated its topology can be.
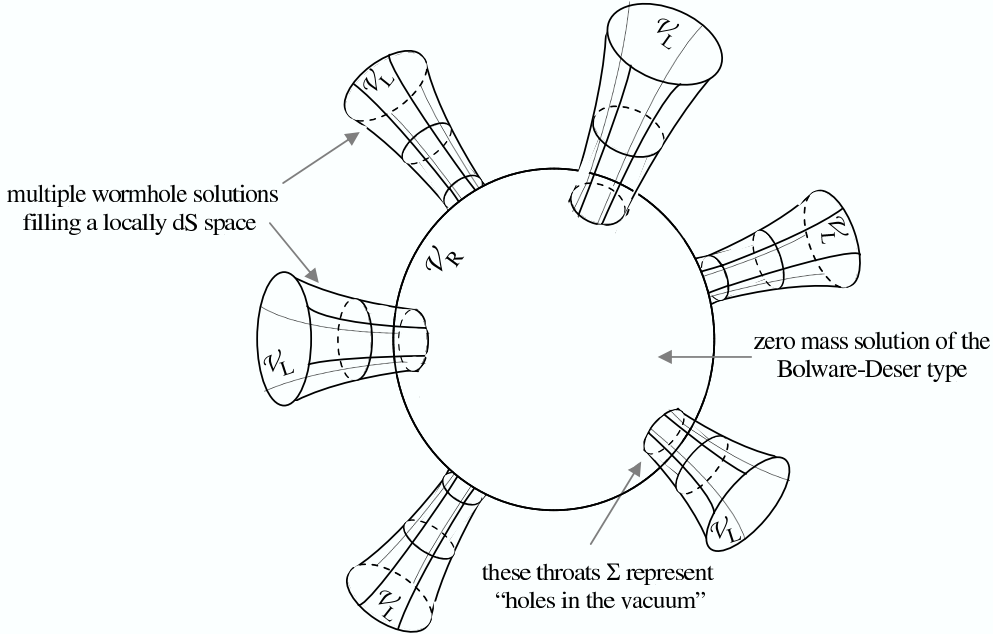
Figure 9: $K > 0$. When $\alpha > 0$ the solution is a multi-soliton, with multiple asymptotic massive regions joined to a de Sitter space. A spatial slice of a multi-soliton is sketched above. When $\alpha < 0$ and $0 < K < 3/8|\alpha|$ the solutions are multi-instantons. The radii of the instantons have a lower bound: $a_0^2 > 16|\alpha|/3$.

7.3 Black hole spectrum and degeneracy

Reversing in a sense our viewpoint from the previous section, we may think of the matching of a massive metric with one of constant curvature along a vacuum shell, as a way to eliminate the singularity at the origin, or better to replace it with an integrable singularity along the vacuum shell.

Consider for example configurations along the curve $u = 2\sqrt{w} + w$ and $\alpha > 0$ i.e. $w > 1$ (and the vacuum shell is static). Then $K > 0$, that is $M > 0$ and $\Lambda > 0$, and the massive branch is an exotic branch ($\xi = +1$). This metric alone has a naked singularity at the origin. By constructing the vacuum wormhole we have managed to replace a region around the origin with a region of a de Sitter spacetime which contains the horizon. That is, the spacetime which asymptotically looks like an exotic branch, massive, Boulware-Deser spacetime is actually everywhere non-singular and has horizons. We might reasonably expect that thermal effects of the horizons will be felt in this would-be singular spacetime.

The mass parameter M in the massive region is determined by the curvature K of the de Sitter region. So then, the de Sitter space mimics a particle, or some fairly localized mass, as viewed from sufficiently far away. The entropy S related to the existence of the de Sitter horizon depends on K and therefore on M. We expect $\partial S/\partial K > 0$. We know that $\partial M/\partial K < 0$. Therefore that entropy will decrease with M. This is not surprising since a positive mass in the exotic branch behaves effectively like a negative gravitational mass.

The previous example shows that the spectrum of black holes in Einstein gravity modified by the Gauss-Bonnet term is not the same when C^0 metrics are allowed, compared to the smooth Boulware-Deser metrics. A space which by an asymptotic observer who thinks in terms of smooth metrics would not be recognized as a black hole might actually be one. Conversely, a spacetime which asymptotically would be a recognized as a Boulware-Deser black hole, could actually be a spacetime with a naked singularity, or a black hole different to the one expected.

Consider, for example, the case $\Lambda = 0$ and $\alpha < 0$. From the analysis in appendix B we see that this spacetime is a black hole for $M > |\alpha|$ (and $\xi = -1$). The horizon hides an inner branch singularity. Our analysis in section 6 shows that we can cut this spacetime and match it with

32

wormhole orientation along a spacelike shell, i.e. within the horizon, with another spacetime of exotic branch metric. In that spacetime r is everywhere a timelike variable. Thus although outside the horizon spacetime looks like a specific Boulware-Deser black hole spacetime it can actually be a different one. The two different states have the same energy as measured at spatial infinity and horizons with the same properties: as black holes they must be degenerate. Whether the usual entropy calculations take into account the effects of this degeneracy in the number of states is not clear to us. It is amusing to think that the modifications to the usual Bekenstein-Hawking formula in the presence of the Gauss-Bonnet term, see e.g. [16], are essentially due to such degeneracies.

7.4 Other types of shells

The analysis has focused on the spherically symmetric case ($k = 1$). This can readily be extended to the case of $k = -1$ (where the bulk metrics are taken to be topological black holes, with horizons some compactified hyperbolic space, or the corresponding naked singularity spacetime). Similar features are expected to occur (wormholes and shells of standard orientation exist, typically involving the exotic plus branch.) Also the case of $k = 0$ for toroidal black holes or naked singularity spacetimes, can be investigated.

We have seen that spacelike shells exist, representing a sudden transition from one solution to another. These present problems in terms of the predictability of the field equations. It would be useful to know whether the shells are generic or if they only occur for a certain range of the coupling constants and mass parameters.

The Euclidean version of the C^0 wormholes may be important for estimating the transition rate between the (unstable) plus branch and the (stable) minus branch solutions.

These are left for future work.

7.5 On uniqueness and staticity of solutions

In this work we construct and analyze solutions of Einstein-Gauss-Bonnet gravity whose metric is class C^0, piecewise analytic in the coordinates. The solutions are made by joining together two spherically symmetric pieces. Since the shell itself admits $SO(4)$ isometry group, the resulting global spacetime is spherically symmetric. To put things into this context and discuss the special implications of low differentiability we start by reviewing the existing relevant theorems in Einstein and Lovelock gravity.

We start with a uniqueness and staticity theorem, applying to Lovelock gravity in general, which imposes the stronger conditions on differentiability.

Theorem 29 (Ref. [25]). *For generic values of the couplings (including the cosmological constant), class C^2 solutions of the Lovelock gravity field equations with spherical, planar or hyperbolic symmetry are isometric to the corresponding static solutions.*

In particular, in Einstein-Gauss-Bonnet gravity in five dimensions C^2 solutions with spherical symmetry are isometric to the Boulware-Deser solutions when $\Lambda \neq -3/4\alpha$.

When we let the metric become merely continuous at hypersurfaces, we have seen already that one can construct many different time-independent solutions of the vacuum field equations: for example, when $\Lambda = 0$ with $\alpha > 0$, one can construct multiple concentric vacuum discontinuities separating Boulware-Deser solutions. So uniqueness of black hole solutions does not hold for C^0 metrics in Lovelock gravity. In fact neither does staticity. We return to discuss this below, after we revisit the corresponding theorems in Einstein gravity.

Theorem 30 (Ref. [38][40]). *A differentiability class C^0 and spherically symmetric vacuum solution of Einstein gravity is: i) static, ii) equivalent to the Schwarzschild solution.*

That a spherically symmetric vacuum solution is static can be shown by finding a timelike Killing vector, which also happens to be hypersurface orthogonal, even when the solution is given in forms that don't look very much like the usual Schwarzschild metric and which assume lower differentiability [39], see [40].

Theorem 31 (Ref. [38]). *A C^0 solution of the Einstein gravity field equations is well defined as the limit of a sequence of (at least) C^2 solutions. The metric is assumed to become C^0 only at smooth hypersurfaces.*

Fields of low differentiability, e.g. with a discontinuous first derivative, can be understood as solutions of field equations in the weak sense, as limits of sequences of smoother fields. The fact that this limit is well defined makes the junction conditions of Israel well defined (the above work appeared earlier than Israel's famous work). Now based on the junction conditions one may conclude: any hypersurfaces where the metric is not smooth must be a null hypersurfaces (we may call them shock waves). Then one may show that there are no spherically symmetric shock waves in Einstein gravity, see e.g. [40].

The result regarding limits of smooth metrics holds in Einstein-Gauss-Bonnet and in fact in Lovelock gravity in general (see the appendix of Ref. [13]).

Theorem 32. *A C^0 solution of Lovelock gravity field equations is well defined as the limit of a sequence of (at least) C^2 solutions. The metric is assumed to become C^0 only at smooth hypersurfaces and their intersections.*

So considerations related to uniqueness or non-uniqueness similar to the above are meaningful in Lovelock gravity as well. In this paper we have demonstrated:

Theorem 33. *There exist spherically symmetric C^0 solutions of Einstein-Gauss-Bonnet gravity in five dimensions which are not given by the Boulware-Deser metric, but rather they are piecewise of the Boulware-Deser form. There exist solutions which are not static.*

In section 6 we found that for any value of the couplings α and Λ such that[13] $\Lambda > -3/4\alpha$, there exist static (time-independent) vacuum shells: spherically symmetric C^0 vacuum metrics are not unique for a wide range of couplings α and Λ in Einstein-Gauss-Bonnet gravity in five dimensions. One can in fact construct arbitrarily complicated spherically symmetric configurations by having an infinity of concentric discontinuities. The exotic branch ($\xi = +1$) is typically involved. Though the radius of a static vacuum shell is uniquely fixed by the metrics in the bulk, C^0 static metrics are to a high degree non-unique as one does not a priori know how many vacuum shells there may be in spacetime.

Now recall section 5.1. The time-dependent solutions, i.e. non-static ones, exist *always*: For any non-zero value of α and any value of Λ there exists[14] a time-dependent vacuum shell solution $a(\tau)$. The shell radius function $a(\tau)$ and the orientation signs η_L and η_R, completely define the world-volume of the shell intrinsically as well as its embedding in spacetime (section 2.2). That is, they define a C^0 metric in spacetime. Therefore a non-static C^0 metric which respects everywhere spherical symmetry can always be constructed in Einstein-Gauss-Bonnet gravity with cosmological constant (which can be also zero). However in section 5.1 we have obtained a general result concerning shells with bulk metrics which have a well defined General Relativistic limit as $\alpha \to 0$ (the $\xi = -1$ branch). In the range of parameters $1 + \frac{4\alpha\Lambda}{3} > 0$ all shell solutions involving the minus branch are unstable in the sense that they can not be in a stable static state, neither can they perform bounded oscillations.

Theorem 33 shows that uniqueness does not apply to C^0 metrics. How is this to be interpreted? One could simply reject non-smooth metrics as unphysical. However, according to Theorem 32 these C^0 solutions are well defined as the limit of a family of smooth geometries. As such, they approximate arbitrarily closely to some smooth solution of the theory. Now suppose $g_{\mu\nu}^{(n)}$ is a family of smooth metrics which converge to a spherically symmetric vacuum shell solution as $n \to \infty$. For finite n, $g_{\mu\nu}^{(n)}$ can not be a spherically symmetric vacuum solution, because the uniqueness theorem holds for smooth metrics. So it must either deviate slightly from spherical symmetry or have some

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[13]According to that section, $y(w-1)w = yx(x+1) > 0$ for timelike i.e. static shells and $yx(x+1) < 0$ for spacelike i.e. instantaneous shells. Via the simple relations of x and y to the couplings these read for non-zero Λ: $3/4\alpha + \Lambda > 0$ and $3/4\alpha + \Lambda < 0$ respectively. As can be seen from the results of section 6 they actually hold for $\Lambda = 0$ as well.

[14]In fact, it exists for a wide range of the bulk metric masses M_L and M_R, possibly for all values of the masses for which the metrics are real. What is more important, for given values of the couplings α and Λ, for any given Boulware-Deser metric one can construct a time-dependent vacuum shell for some other Boulware-Deser metric on the other side.

small amount of matter as source. Assuming that suitable $g_{\mu\nu}^{(n)}$ can be constructed which obey the energy conditions, our results can be taken as evidence for the generic existence of such exotic features as smooth wormholes in this theory.

8   Conclusions

In this paper we have presented a method for generating new exact vacuum solutions of five-dimensional Lovelock theory of gravity. The solutions we obtained are spherically symmetric space-times whose metrics are C^0 functions, and are composed by patches of different five-dimensional Boulware-Deser spacetimes.

The proof of the Birkhoff's theorem for this theory (see e.g. Refs. [36], [22]) involves an assumption of differentiability [25]. We have seen that if this assumption is relaxed then there are C^0 metrics (which satisfy the field equations in the distributional sense). The Birkhoff's theorem still holds, but merely in a (weaker) piece-wise form. Uniqueness and staticity of the metric turns out to be valid only locally in regions where the metric is differentiable.

We have used a geometric surgery procedure, employing the junction conditions of Einstein-Gauss-Bonnet theory to join spherically symmetric pieces of spacetime. This lead us to find different geometries with quite interesting global structure. In particular, we have shown that vacuum wormholes do exist in this theory. The wormholes connect two different asymptotically (Anti)de-Sitter spaces and, in certain sense, they represent gravitational solitons in five-dimensions. Although their metrics are not C^1 functions, they are globally static vacuum solutions of gravity equations of motion and have finite mass. These metrics, being non differentiable where the wormhole throat is located, still represent exact solutions defined everywhere, provided the junction conditions are obeyed. This is ultimately due to cancelations among different terms in the junction conditions.

We have analyzed both static and dynamical solutions and, related to this, we have pointed out a new type of classical instability that arises in Einstein-Gauss-Bonnet gravity for certain range of the Gauss-Bonnet coupling. This concerns fundamental aspects such as predictability and uniqueness.

Appendix

A   Spherically symmetric solutions in Einstein-Gauss-Bonnet gravity

The spherically symmetric static solution of Einstein-Gauss-Bonnet theory of gravity was obtained by Boulware and Deser in 1985, [Ref]. In five dimensions, and in terms of a suitable Schwarzschild-like ansatz (3), the metric is given by (5). On the other hand, if the higher dimensional theory with the dimensionally extended quadratic Gauss-Bonnet term is considered, then the black hole solutions take a similar form, namely

$$f_{\pm}(r) = k + \frac{r^2}{4\alpha} \pm \frac{r^2}{4\alpha}\sqrt{1 + \frac{\alpha M_D}{r^{D-1}} + \alpha\Lambda_D} \qquad (A.1)$$

where M_D is an integration constant, and Λ_D is a numerical factor proportional to Λ and which only depends on the dimension D. The ambiguity in expressing the sign $\pm$ reflects the existence of two branches, and corresponds to the parameter ξ introduced in section 2. The parameter k takes the value $k = 1$ in the case where the base manifold corresponds to the unitary $(D-2)$-sphere $d\Omega^2_{D-2}$; besides, k may take the values -1 or 0, if the base manifold is of negative or vanishing constant curvature, respectively.

Now, let us analyse the large distance limit of the solution (5). Asymptotically, this solution tends to the five-dimensional Schwarzschild solution when $\alpha \to 0$, as it is naturally expected. Namely

$$f_-(r) \simeq 1 - \frac{2M}{r^2} - \frac{\Lambda}{6}r^2, \tag{A.2}$$

which represents a (A)dS-Schwarzschild black hole in five dimensions. Notice that the large r^2/α limit of the solution $f_+(r)$ acquires a large additional cosmological constant term $\sim \frac{r^2}{2\alpha}$. In particular, this implies that (A)dS space-time is a solution of the theory even for the case $\Lambda = 0$.

It is also worth noting that in the case of non-vanishing cosmological constant, besides the leading term in the expansion (A.2), we find finite-α corrections to the black hole parameters [37]. Namely

$$f(r) = 1 - \frac{2m_d}{\pi r^2} - \frac{\Lambda_d}{6}r^2 + \mathcal{O}(\alpha r^{-6}) \tag{A.3}$$

where the *dressed* parameters m_d and Λ_d are given by

$$\Lambda_d = \Lambda \left(1 + \sum_{n=2}^{\infty} c_n\, x^{n-1}\right) = 1 - \sqrt{1+x}, \quad m_d = m\left(1 + \sum_{n=2}^{\infty} n\, c_n\, (-x)^{n-1}\right),$$

with

$$c_n = \frac{(2n-3)!!}{2^{n-1}n!}, \quad x := \frac{4}{3}\Lambda\alpha.$$

It is important to emphasize the difference existing between (A.2) and (A.3): While the first corresponds to the actual limit $\alpha \to 0$, the second represents the large r^2/α regime which takes into account finite-α contributions. For instance, the finite-α corrections to the mass are found by simply collecting the coefficients of the Newtonian term $\sim r^2$. The parameter x controls the *dressing* of the whole set of black hole parameters. The above power expansion converges for values such that $x < 1$. On the other hand, for $x > 1$ we find a different expansion, leading to the following *dressed* parameters in the large r regime

$$m_d = \frac{m}{\sqrt{|x|}}\left(1 + \sum_{n=2}^{\infty} n\, c_n\, (-x)^{1-n}\right)$$

Thus, we note that the Newtonian term $\sim m_d r^{-2}$ vanishes in the limit $|\Lambda\alpha| \to \infty$. The particular case $x = 1$ is discussed below. Moreover, it is possible to see that, if one considers the case $\alpha\Lambda > 0$, the effective cosmological constant in the large x limit turns out to be

$$\Lambda_d = \sqrt{\frac{3\Lambda}{\alpha}} - \frac{3}{2\alpha} + \mathcal{O}(1/\sqrt{|x|}) \ .$$

One of the relevant differences existing between the black hole solutions in Einstein theory and in Einstein-Gauss-Bonnet theory is the fact that, in the latter, the metric does not diverge at the origin of Schwarzschild coordinates, $r = 0$, though its curvature is still singular. From (5), we easily observe

$$f_\pm(r=0) = 1 \pm \sqrt{\frac{M}{\alpha}}.$$

In particular, this implies that the metric presents a angular deficit around the origin, and, also, that massive objects with no even horizon exist; thus, these correspond to naked singularities.

Another interesting feature of the presence of the Gauss-Bonnet term is that, for the particular choice of the parameters $\alpha\Lambda = -\frac{3}{4}$, the solution takes the form

$$f_\pm(r) = \frac{r^2}{4\alpha} - \mathcal{M} \tag{A.4}$$

36

where we have considered $\Lambda < 0$ and $\alpha > 0$, and where $\mathcal{M} + 1 = \sqrt{\frac{M}{\alpha}}$. This solution resembles the Bañados-Teitelboim-Zanelli black hole [42, 41]. Actually, the solution (A.4) shares several properties with the three-dimensional black hole geometry, as it is the case of its thermodynamics properties. Parameter $\mathcal{M}$ in Eq. (A.4) plays the role of the mass M in the BTZ solution. For instance, just like AdS_3 space-time is obtained as a particular case of the BTZ geometry by setting the negative mass $M = -(8G)^{-1}$, the five-dimensional Anti-de Sitter space corresponds to setting $\mathcal{M} = -1$ in Eq. (A.4). Moreover, notice that in the large $\mathcal{M}^{-1}$ limit the solution becomes the metric to which AdS_5 tends in the near boundary limit. Similarly, the massless BTZ corresponds to the boundary of AdS_3. Besides, as it was already mentioned, a conical singularity is found in the range $0 < M < \alpha$ (corresponding to $-1 < \mathcal{M} < 0$), and this completes the parallelism with the three-dimensional black hole.

B   Properties of the Boulware-Deser metric

The Boulware-Deser(-Cai) metric is given by (3) with metric function (5).[15] The metric has two branches for given cosmological constant Λ and energy M: $\xi = \pm 1$. [These we call as the plus- and minus- branch respectively; also, more descriptively, as the "exotic" and the "good" branch.] Therefore solving the vacuum field equations for spherically symmetric metrics we obtain *two* solutions. Asymptotically they read

$$f = \frac{1 + \xi\sqrt{w}}{4\alpha}\, r^2 + 1 + \frac{2\xi M}{\sqrt{w}\, r^2} + \mathcal{O}(r^{-4})\ , \tag{B.1}$$

where we use our variable $w \equiv 1 + \frac{4\alpha\Lambda}{3} > 0$.

For $\Lambda = 0$, the $\xi = +1$ branch depends on α asymptotically, while the asymptotically flat branch $\xi = -1$ does not. Also, the sign of the Schwarzschild type of term depends on the branch: the two branches view the energy M differently, i.e. the exotic metric of the Boulware-Deser solution does not reduce to Einstein solution in the "infrared" limit.

The sign ξ is in some sense a charge which determines how a certain energy M enters a metric and thus if the field will be attractive or repulsive. As noted in [32], the graviton is a ghost on the asymptotic $\xi = +1$ branch, because the linear Einstein tensor appears to have the opposite overall sign (that is, this metric is classically unstable). This wrong sign is reflected in the inverted sign of the Schwarzschild term.

An interesting issue about the Boulware-Deser solutions is that it contains a square root, whose reality imposes constraints. From (5) we see that: when $w < 0$ there is a *maximum* radius; when $M/\alpha < 0$ there is a *minimum* radius in spacetime. At those finite radii there exists curvature singularities, known as branch singularities [26]. We call them outer and inner branch singularities, respectively to the cases above. These unusual spacetimes can also have horizons behind which the singularities are hidden.

We turn now to discuss the horizon structure of the Boulware-Deser spacetimes. The following does not intend to be an exhaustive analysis, it is rather a list of general formulas in our notation useful for our purposes. We will use the dimensionless parameters w and $\bar{M}$. Recall the Boulware-Deser metric function $f(r)$ given in (5) and define r_H by $f(r_H) = 0$. One finds that if $w \neq 1$

$$r_{H\pm}^2 = 4\alpha\, \frac{1 \pm \sqrt{M(w)}}{w - 1}\ . \tag{B.2}$$

We have defined the useful quantity

$$\bar{M}(w) = w + (1 - w)\bar{M}\ , \tag{B.3}$$

which looks an interpolation between $\bar{M}$ and 1.

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[15] Only the the spherically symmetric case $k = 1$ was discussed by Boulware and Deser. The cases $k = 0, -1$ were analyzed later by Cai in [16]. As we are mainly interested in the spherically symmetric case we will call this metric Boulware-Deser.

From the definition of r_{H+} we see that $r_{H+} > 0$ if:

$$\frac{3}{\Lambda} = \frac{4\alpha}{w-1} > 0 \ . \tag{B.4}$$

That is $\Lambda > 0$. Also $r_{H-} > 0$ one finds that it is equivalent to $M > \alpha$. Therefore we have:

Remark 34. *Elementary conditions for the existence of r_{H+} is $\Lambda > 0$ and for the existence of r_{H-} the condition $M > \alpha$.*

When $0 < |\alpha| < \infty$, $w = 1 \Leftrightarrow \Lambda = 0$. So the previous formula holds for non-zero Λ. When $\Lambda = 0$, the correct result can be obtained as the limit $w \to 1$ of the previous formula for r_{H-}. It reads

$$r_{H-}^2 = 2\alpha (\bar{M} - 1) \ . \tag{B.5}$$

We must substitute (B.2) back to $f(r_H) = 0$ to solve for the signs. We have:

$$-\xi = \text{sign} \left(\frac{w \pm \sqrt{\bar{M}(w)}}{1 \pm \sqrt{\bar{M}(w)}} \right) \ , \tag{B.6}$$

for $r_{H\pm}$ respectively. Again the case $w = 1$ i.e. $\Lambda = 0$ can be correctly obtained from the limit $w \to 1$ for r_{H-}. Explicitly it reads

$$-\xi = \text{sign} \left(\frac{\bar{M} + 1}{\bar{M} - 1} \right) \ . \tag{B.7}$$

We have used the sign function defined by $\text{sign}(x) = x/|x|$. When $x = 0$ it is ambiguous.

Before continuing note the following. One implicit inequality that should be respected for horizons to exist is

$$\bar{M}(w) \geq 0 \ . \tag{B.8}$$

This is related to the reality of the square root of the Boulware-Deser metric function (5). $\bar{M}$ and w cannot be both negative. That is, if $w \bar{M} \geq 0$ then it must be $w + \bar{M} \geq 0$. This is precisely what is guarantied by (B.8).

From remark 34 we have

Remark 35. $r_{H+} > 0$ *is equivalent to* $\text{sign}(\alpha) = \text{sign}(w - 1)$. $r_{H-} > 0$ *is equivalent to* $\text{sign}(\alpha) = \text{sign}(\bar{M} - 1)$.

Note that, as we will solve the problem of existence for the real numbers $r_{H\pm}^2/4\alpha$ the positivity conditions above essentially restrict the sign of α.

Now

$$r_{H+}^2 - r_{H-}^2 = \frac{8\alpha}{w-1} \sqrt{\bar{M}(w)} \ , \tag{B.9}$$

and (B.4) tell us that

Remark 36. *If r_{H+} exists then $r_{H+} \geq r_{H-}$.*

A solution r_H corresponds to a horizon if $r_H > 0$ and there exist $r_1 < r_H < r_2$ such that $f(r_1)f(r_2) < 0$.

Recall (B.6). We have some *Special cases*:

i). $w \pm \sqrt{\bar{M}(w)} = 0 \Leftrightarrow \bar{M} = -w$. (Note again that the correct result for $w = 1$ is obtained as the limit). Then one of the two solutions $r_{H\pm}$ coincides with a *branch singularity*. I.e. in this case the branch singularity is *null*. [This is possible for $\alpha < 0$ otherwise this solution doesn't exist.]

There is a single horizon solution given by $r_H^2 = 4\alpha \frac{w+1}{w-1}$. It is a horizon of the branch ξ according to $-\xi = \text{sign}(w(w+1))$. [Of course the case $w = 0 = \bar{M}$ does not have two branches.] The case

$w = -1$ i.e. $\bar{M} = 1$ does not have a horizon as r_H vanishes (if $\xi = -\text{sign}(\alpha)$), or the metric function f, which reads

$$f = 1 + \frac{r^2}{4\alpha} + \xi \text{sign}(\alpha)\sqrt{1 - \left(\frac{r^2}{4\alpha}\right)^2} \; , \tag{B.10}$$

can vanish only at the branch singularity when $\alpha < 0$.

Finally one should bear in mind that $r_H = r_{H-}$ when $\bar{M} = -w > 0$ and $r_H = r_{H+}$ when $\bar{M} = -w < 0$.

$ii)$. $1 - \sqrt{\bar{M}(w)} = 0 \Leftrightarrow \bar{M}(w) = \bar{M} = 1$. (Note again that the correct result for $w = 1$ is obtained as the limit). We just learned that when we also we have $w = -1$ there are no horizons. So we assume that $w \neq -1$. We observe that $r_{H-} = 0$. This actually happens if $\xi = -\text{sign}(\alpha)$ otherwise this solution doesn't exist.

The single horizon solution is $r_{H+}^2 = 4\alpha \frac{2}{w-1} = \frac{6}{\Lambda}$. It is a horizon of the branch ξ according to $-\xi = \text{sign}(w+1)$.

$iii)$. $\bar{M}(w) = 0$. This is the saturated case where the two radii coincide: $r_{H\pm}^2 = 4\alpha/(w-1) = 3/\Lambda$. Condition (B.6) works well in this case: $\xi = -\text{sign}(w)$. Also from $r_H^2 > 0$ we have $\text{sign}(\alpha) = \text{sign}(w-1)$.

In this case r_H is not a horizon radius. It is the (single) zero of f which has the same sign everywhere else. There are three non-trivial cases. $w < 0$: Then there is an outer branch singularity and $f(r) \geq 0$. $0 < w < 1$: Then there is an inner branch singularity and $f(r) \leq 0$. $w > 1$: Then $0 < r < \infty$ and $f(r) \leq 0$. $\square$

Recall (B.6).

Proposition 37. *With the exception of cases covered in i) and ii) we have: The radius r_{H+} is a horizon of the branch ξ if*

$$-\xi = \text{sign}\left(w + \sqrt{\bar{M}(w)}\right) \; ; \tag{B.11}$$

the radius r_{H-} is a horizon of the branch ξ if

$$-\xi = \text{sign}((\bar{M} + w)(\bar{M} - 1))\,\text{sign}\left(w + \sqrt{\bar{M}(w)}\right) \; . \tag{B.12}$$

The type of the horizon, i.e. whether it is black hole, inner or cosmological horizon, can be determined by the sign of the first derivative of $f(r)$ (combined with Remark 36). We have

$$r_{H\pm}f'(r_{H\pm}) = \mp 2\,\sqrt{\bar{M}(w)}\,\cdot\frac{1 \pm \sqrt{\bar{M}(w)}}{w \pm \sqrt{\bar{M}(w)}} \; . \tag{B.13}$$

Therefore for $\bar{M}(w) > 0$, when r_{H-} or r_{H+} does correspond to a horizon, the type is determined by

$$\text{sign}(f'(r_{H\pm})) = \pm\xi \; . \tag{B.14}$$

Remarks 35 and 36, Proposition 37, and formula (B.14) provide criteria for the existence and the type of horizons for each branch ξ of the Boulware-Deser metric.

For the exotic branch ($\xi = +1$) a black hole horizon must be an r_{H+}. This is not possible by (B.11). Thus there no black holes in the exotic branch. For the good branch ($\xi = -1$) a black hole horizon must be an r_{H-}. From (34), this is possible only for $M > \alpha$.

C   The Junction conditions

For our purposes, a singular shell Σ is a submanifold of codimension one at which the metric is continuous but the extrinsic curvature has a finite discontinuity. The field equations of Einstein-Gauss-Bonnet theory are given by (2). Integrating the field equations across Σ, one obtains the junction conditions

$$(\mathfrak{Q}_R)_a^b - (\mathfrak{Q}_L)_a^b = -\kappa^2 S_a^b \; ,$$

with $\mathfrak{Q}_a^b$ given by [16] (14). Lower case Roman letters from the beginning of the alphabet a, b etc. represent four-dimensional tensor indices on the tangent space of the world-volume of the shell. The $R^{ab}{}_{cd}$ appearing in the junction condition is the four-dimensional intrinsic curvature. The antisymmetrized Kronecker delta is defined as $\delta^{a_1\ldots a_p}_{b_1\ldots b_p} \equiv p!\, \delta^{a_1}_{[b_1} \cdots \delta^{a_p}_{b_p]}$.

Now we calculate the intrinsic curvature of the world-volume of a spherical shell of radius $a(\tau)$ and the extrinsic curvature (which takes a diagonal form). There are two cases: For the timelike case the components are

$$R^{\tau\phi}{}_{\tau\phi} = \frac{\ddot{a}}{a}\,, \qquad R^{\phi\theta}{}_{\phi\theta} = R^{\theta\chi}{}_{\theta\chi} = R^{\chi\phi}{}_{\chi\phi} = \frac{(k+\dot{a}^2)}{a^2}\,,$$

$$K^{\tau}_{\tau} = \eta\frac{\ddot{a}+\frac{1}{2}f'}{\sqrt{\dot{a}^2+f}}\,, \qquad K^{\theta}_{\theta} = K^{\phi}_{\phi} = K^{\chi}_{\chi} = \frac{\eta}{a}\sqrt{\dot{a}^2+f}\,;$$

while for the spacelike case these are

$$R^{\tau\phi}{}_{\tau\phi} = -\frac{\ddot{a}}{a}\,, \qquad R^{\phi\theta}{}_{\phi\theta} = \frac{(k-\dot{a}^2)}{a^2}\,,$$

$$K^{\tau}_{\tau} = \eta\frac{\ddot{a}-\frac{1}{2}f'}{\sqrt{\dot{a}^2-f}}\,, \qquad K^{\theta}_{\theta} = K^{\phi}_{\phi} = K^{\chi}_{\chi} = \frac{\eta}{a}\sqrt{\dot{a}^2-f}\,.$$

In this paper we are interested in pure vacuum shells, i.e. when $S^a_b = 0$. It is clear that in this case one can pull out a factor of $\Delta K^d_c \equiv (K_+ - K_-)^d_c$, which is the jump in the extrinsic curvature across the shell.

$$\Delta K^d_c\,(\cdots) = S^a_b = 0\,. \tag{C.2}$$

In the case of interest in this paper, the extrinsic curvature is diagonal. Thus, one expects each component of the junction conditions to factorize conveniently.

Using the above formulae, we derive $\mathfrak{Q}^\tau_\tau$ given in (15). The angular components are, for the timelike case:

$$\mathfrak{Q}^\theta_\theta = -2!\, a^{-2} \left\{ \eta\, \frac{\frac{1}{2}f'\{a^2 + 4\alpha(k-f)\}}{\sqrt{\dot{a}^2+f}} + \eta\, 2a\sqrt{\dot{a}^2+f} + \eta\, 4\alpha\, \frac{\ddot{a}}{\sqrt{\dot{a}^2+f}} \left(k+f+2\dot{a}^2+\frac{a^2}{4\alpha}\right) \right\}. \tag{C.3}$$

D   The derivatives of the potential

As before, let us denote the derivative with respect to a by a prime. In analysing dynamical shells and the stability of static shells it is useful to calculate the derivatives of $V(a)$ with respect to a, V', V'' etc. First we recall the definition of $Y(a)$; namely

$$f(a) \equiv k + \frac{a^2}{4\alpha}\left(1 + \xi Y(a)\right), \qquad Y := \sqrt{w + \frac{16M\alpha}{a^4}}\,. \tag{D.1}$$

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[16] The notation of Ref. [13] has been used. However in that reference there was an unconventional sign convention used (in equation A3) for the definition of extrinsic curvature. Although none of the results of that paper were affected by this, unfortunately the formulae B13-B17 for the Einstein-Gauss-Bonnet in the appendix were a mixture of inconsistent sign conventions. Here we correct this sign error by choosing the standard sign convention as in Refs. [28] and [10]. The developed expression is:

$$\left[\sigma(K^a_b - \delta^a_b K) + 2\alpha\left(3J^a_b - \delta^a_b J - 2\varsigma P^{ac}{}_{bd}K^d_c\right)\right]^+_- = -\kappa^2 S^a_b\,, \tag{C.1}$$

where σ is ± 1 for a timelike/spacelike shell, $J_{ab} := (2KK_{ac}K^c_b + K_{cd}K^d K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2 K_{ab})/3$ and $P_{abcd} := R_{abcd} + 2R_{b[c}g_{d]a} - 2R_{a[c}g_{d]b} + Rg_{a[c}g_{d]b}$ is the trace-free part of the intrinsic curvature. In the case of a timelike shell ($\sigma = +1$), this expression agrees with that given in Ref. [10, 11].

Note that Y obeys the simple differential equation:

$$(Ya^2)' = \frac{2wa}{Y}, \tag{D.2}$$

where we recall that $w := 1 + \frac{4\alpha\Lambda}{3}$.

In terms of Y_R and Y_L, the effective potential defined in (20) takes the form:

$$\sigma V = \left(k + \frac{a^2}{4\alpha}\right) - \frac{a^2}{12\alpha}\left(\xi_R Y_R + \xi_L Y_L - \frac{\xi_R \xi_L Y_R Y_L}{\xi_R Y_R + \xi_L Y_L}\right). \tag{D.3}$$

This can be also written as

$$V(a) = \sigma\left(k + \frac{a^2}{4\alpha}\right) - \frac{\sigma a^2}{4\alpha}\left(\frac{3(\xi_R Y_R + \xi_L Y_L)^2 + (\xi_R Y_R - \xi_L Y_L)^2}{12(\xi_R Y_R + \xi_L Y_L)}\right). \tag{D.4}$$

By repeated application of the differential equation (D.2) we obtain:

$$\sigma V' = \frac{a}{2\alpha}\left(1 - \frac{w}{\xi_R Y_R + \xi_L Y_L}\right), \tag{D.5}$$

$$\sigma V'' = \frac{1}{2\alpha}\left(1 - \frac{3w}{\xi_R Y_R + \xi_L Y_L} + \frac{2w^2}{\xi_R \xi_L Y_R Y_L(\xi_R Y_R + \xi_L Y_L)}\right), \tag{D.6}$$

Note that the second derivative of V depends on a only implicitly through $Y(a)$.

Let a_e be the radius at which V is an extremum, $V'(a_e) = 0$. From (D.5) we have

$$\xi_R Y_R(a_e) + \xi_L Y_L(a_e) = w. \tag{D.7}$$

It is of interest to know whether the extremum is minimum or maximum. The second derivative evaluated at the extremum is:

$$V''(a_e) = \frac{\sigma}{\alpha}\left(\frac{w}{\xi_R \xi_L Y_R(a_e) Y_L(a_e)} - 1\right), \tag{D.8}$$

If the right hand side of (D.8) is positive, the extremum is a minimum.

Let us look for a solution where the minimum of the potential coincides with $V = 0$. Imposing at some radius a_0 that $V(a_0) = V'(a_0) = 0$ implies:

$$\xi_R Y_R + \xi_L Y_L = w, \tag{D.9}$$

$$\xi_R \xi_L Y_R Y_L = w^2 - \left(3 + \frac{12k\alpha}{a_0^2}\right)w. \tag{D.10}$$

One can verify as a consistency check that the static and instantaneous shell solutions of section 3 are recovered. In terms of the metric functions f the above two equations are:

$$f_R + f_L = \left(\frac{3}{4\alpha} + \frac{\Lambda}{3}\right)a_0^2 + 2k, \qquad f_R f_L = \left(\frac{\Lambda a_0^2}{3} - k\right)^2,$$

c.f. the junction conditions for static and instantaneous shells in proposition 6. Upon imposing the inequalities (21-23) we recover exactly the solutions of that section.

It is important in analyzing the stability of the static ($\sigma = +1$) vacuum shells to know the sign of V'' evaluated at the static radius a_0.

$$V''(a_0) = \frac{1}{\alpha}\left(\frac{w}{w^2 - \left(3 + \frac{12k\alpha}{a_0^2}\right)} - 1\right), \tag{D.11}$$

Note that this can also be written

$$V''(a_0) = -\frac{1}{\alpha}\left(1 + \frac{\frac{ka_0^2}{4\alpha}}{3 + (2 - \frac{4\alpha\Lambda}{3})\frac{ka_0^2}{4\alpha}}\right) = -\frac{1}{\alpha}\frac{xy - 3kx - 3y}{xy - 3kx - 2y}$$

in terms of the original variables and of the variables of section 3 respectively.

E Some details of the space of constant solutions

In terms of the variables w and u introduced in section 6, the junction conditions for static or instantaneous shells are given by equation (78) with $f_{L,R} > 0$ for the timelike vacuum shell (which corresponds to $\sigma = +1$), and $f_{L,R} < 0$ for the spacelike vacuum shell ($\sigma = -1$). After squaring the equations above, we obtain

$$f_{(\pm)} = \frac{(3w \pm u)^2}{u^2 + 3(w^2 - 4w)} \ . \tag{E.1}$$

which turns out to be always real. The solution for $f_{L,R}$ is given by (40) as discussed there. Then from (78) we first have

Proposition 38. *Let the total moduli space be described in the (w, u) parameter space. Then it necessarily is a subset of the upper half plane $u \geq 0$ from which the points on the curves $\pm u = 3w$ and $u^2 + 3(w^2 - 4w) = 0$ are excluded. The four disconnected regions are divided according to the type of the matching by combinations of the following. Timelike: $u^2 + 3(w^2 - 4w) > 0$. Spacelike: $u^2 + 3(w^2 - 4w) < 0$. Same orientation i.e. $\eta_L \eta_R > 0 : u^2 - 9w^2 > 0$. Opposite orientation, i.e. $\eta_L \eta_R < 0 : u^2 - 9w^2 < 0$.*

It is good to remember

Remark 39. *The points $(0,0)$, $(1,3)$ and $(0,4)$ in the (w,u) plain do not belong to the moduli space. The point $(1,3)$ corresponds to the line $x = 0$ and $y \neq 0$.*

We have already used the fact that $f_{L,R}(r)$ are the Boulware-Deser metric functions. In order to completely solve our problem we must substitute for $f_{L,R}$ using the Boulware-Deser expression evaluated at $r = a_0$ given in equation (37),

$$f_L = f_L(a_0) \quad , \quad f_R = f_R(a_0) \ . \tag{E.2}$$

Recall (40). Similarly to equations (41) and (42) we have that, within the space of Proposition 38, (E.2) amount to

$$w\,(w \pm u) = 2w\,\xi_{(\pm)}\,\sqrt{w + \frac{\left(u^2 + 3(w^2 - 4w)\right)^2}{144 w^2}}\,\bar{M}_{(\pm)} \ . \tag{E.3}$$

The solution $w = 0$ is possible only if $\bar{M}_{L,R} = 0$. Then for $|\alpha| < \infty$ we have that $M_{L,R} = 0$ and the bulk metrics are simply $f_{L,R}(r) = 1 + r^2/(4\alpha)$. We have

Remark 40. *The line $w = 0$, which lies in the "cone" $u^2 - 9w^2 > 0$ and entirely within the timelike standard shell region, is excluded from the moduli space as it merely corresponds to smooth geometries.*

Therefore we work with non-zero w. Squaring the previous relation we find the mass parameters of $f_{(\pm)}(r)$ which are consistent with the vacuum shell solution; they are given by equation (79).

Substituting back into (E.3) we have the condition

$$\xi_{(\pm)}\,|w \pm u| = w \pm u \ . \tag{E.4}$$

The sign of $\xi_{(+)}$ is completely determined over the moduli space if $u + w \neq 0$ by $\xi_{(+)}(w + u) > 0$. Similarly, the sign of $\xi_{(-)}$ is determined if $w - u \neq 0$ by $\xi_{(-)}(w - u) > 0$. Now for $w + u = 0$ we find that $\xi_{(-)}|w| = w = -u < 0$. Similarly for $w - u = 0$ we find that $\xi_{(+)} > 0$, and this happens for $w > 0$.

We see that the signs $\xi_{\pm}$ are specified for each point on the moduli space, i.e. a solution of the vacuum shell. We will say that this is a solution of the vacuum shell of *type* $(\xi_{(-)}, \xi_{(+)})$. The exception is along the *branch curve* $u^2 - w^2 = 0$ where one of the signs is undetermined. We can summarize

Proposition 41. *The moduli space consists of the regions of the parameter space (w, u) given in Proposition 38 such that: i) the line $w = 0$ is excluded, ii) according to the branch signs $(\xi_{(-)}, \xi_{(+)})$ of*

the bulk regions the parameter space is divided as follows: $(+,+)$ for $u < w$; $(-,+)$ for $-u < w < u$, $(-,-)$ for $w < -u$.

The points along the branch curve $u^2 - w^2 = 0$ satisfy: if $w > 0$ then $\xi_{(+)} > 0$ and $\xi_{(-)}$ arbitrary, if $w < 0$ then $\xi_{(-)} < 0$ and $\xi_{(+)}$ arbitrary. The mass parameters $M_{(\pm)}$ are well defined and given over the moduli space by formula (79).

Propositions 38 and 41 categorize the allowed spherically symmetric vacuum shell solution at constant r in terms of spacelike/timelike and branch signs. This is plotted in figure 5.

Note also the following: Formula (79) says that we can define a function

$$\bar{M}_*(w,u) := \frac{36w^2((w+u)^2 - 4w)}{(u^2 + 3(w^2 - 4w))^2} \ , \tag{E.5}$$

defined on the whole of the (w,u) plain (minus the curve $u^2 + 3(w^2 - 4w) = 0$) and not only on the upper half. Then for $u > 0$, $\bar{M}_{(+)} = \bar{M}_*(w,u)$ and $\bar{M}_{(-)} = \bar{M}_*(w,-u)$. More generally, recalling also equations (E.1), and (E.4), one may extend also $f(a_0)$ and ξ, regarded as functions of w and u, over the whole of the (w,u) plane.

Lemma 42. *Let X denote any of the quantities $\bar{M}$, f, ξ, or combinations of them. One may define a function $X_*(w,u)$ such that $X_{(+)} = X_*(w,u)$ for $u \geq 0$. Then, $X_{(-)} = X_*(w,-u)$. At $u = 0$ we have $X_{(+)} = X_{(-)}$ i.e. $X_R = X_L$.*

The parameter space can be extended over the whole of the (u,w) plane. The mirror transformation $u \to -u$ has the effect of sending $(+) \leftrightarrow (-)$. So one may specify one type of quantities $\xi_{(+)}$, $\bar{M}_{(+)}$ on the whole plane and mirror image the results to obtain the values of $\xi_{(-)}$ and $\bar{M}_{(-)}$.

Now, we will also return to discuss in a more detailed manner the two most basic distinct types of constructions here (recall Definition 2): matching with the same orientation, i.e. standard shell solutions, and matching with opposite orientation, which we call collectively wormholes. Though the following definition has been already in use in our work, it is useful to formalize the following

Definition 43. *A plus-metric, corresponding to the metric function $f_{(+)}(r)$, is one whose mass parameters is given by $\bar{M}(w,u) = M_{(+)}$ and branch by $(w+u)/|w+u| = \xi_{(+)}$ over the moduli space. A minus-metric, corresponding to the metric function $f_{(-)}(r)$), is one whose mass parameters is given by $\bar{M}(w,-u) = \bar{M}_{(-)}$ and branch by $(w-u)/|w-u| = \xi_{(-)}$ over the moduli space.*

Now, let us make a remark on the sign of α. As $a_0^2 > 0$ it can be determined by the sign of y/x and is given by

$$\operatorname{sign}(\alpha) = \operatorname{sign}\big(w\,(u^2 + 3(w^2 - 4w))\big) \ . \tag{E.6}$$

Therefore we have

Remark 44. *$\alpha > 0$ only for timelike vacuum shells and in the region $w > 0$ (for standard or wormhole orientation). Inside the ellipse of the spacelike vacuum shells (see fig. 5), or for $w < 0$, we have: $\alpha < 0$.*

From the definition of w, the sign of the cosmological constant Λ is determined according to

$$\operatorname{sign}(\Lambda) = \operatorname{sign}(\alpha)\operatorname{sign}(w-1). \tag{E.7}$$

When $\Lambda = 0$ i.e. $w = 1 > 0$, the sign of α depends on the whether the shell is time- or space-like as we mentioned just above.

Proof of Proposition 26: For a given w, let u_0 be such that the corresponding point (u_0, w) belongs to the moduli space. We have

$$\bar{M}_{(+)} = 36w^2\,\frac{(u_0+w)^2 - 4w}{(u_0^2 + 3(w^2 - 4w))^2} \quad , \quad \bar{M}_{(-)} = 36w^2\,\frac{(u_0-w)^2 - 4w}{(u_0^2 + 3(w^2 - 4w))^2}. \tag{E.8}$$

Of course $u_0 \geq 0$.

There are two special cases to deal with before proceeding. First, consider $\bar{M}_{(+)} = \bar{M}_{(-)}$. We know that this is possible if and only if $u_0 = 0$. So for non-unique solutions we may restrict ourselves

to $u_0 > 0$. The second case is when $\bar{M}_{(+)} + \bar{M}_{(-)} = 0$. This happens in the moduli space along the circle: $u_0^2 + w^2 - 4w = 0$. Clearly there is a unique positive u_0 solving this equation.

Therefore it is adequate to consider $u_0 > 0$ and masses such that $\bar{M}_{(+)} \pm \bar{M}_{(-)} \neq 0$. The proof is by contradiction. Let us suppose that u_0 is not unique in the sense that there exists some $u_1 > 0$ in the moduli space such that $u_1 \neq u_0$ and which gives the same masses

$$\bar{M}_{(+)} = 36w^2 \, \frac{(u_1 + w)^2 - 4w}{(u_1^2 + 3(w^2 - 4w))^2} \quad , \quad \bar{M}_{(-)} = 36w^2 \, \frac{(u_1 - w)^2 - 4w}{(u_1^2 + 3(w^2 - 4w))^2} \, . \tag{E.9}$$

With a little rearranging subtracting the respective equations we have

$$(u_1 - u_0)\Big\{(u_1 + u_0)\big(u_1^2 + u_0^2 + 6(w^2 - 4w)\big)\bar{M}_{(+)} - 36w^2\,(u_1 + u_0 + 2w)\Big\} = 0 \, ,$$

$$(u_1 - u_0)\Big\{(u_1 + u_0)\big(u_1^2 + u_0^2 + 6(w^2 - 4w)\big)\bar{M}_{(-)} - 36w^2\,(u_1 + u_0 - 2w)\Big\} = 0 \, .$$

$u_1 \neq u_0$ so the quantities in the big brackets vanish. Adding and subtracting them we obtain the equations

$$u_1^2 + u_0^2 + 6(w^2 - 4w) = \frac{72w^2}{\bar{M}_{(+)} + \bar{M}_{(-)}} \quad , \quad u_1 + u_0 = 2w \, \frac{\bar{M}_{(+)} + \bar{M}_{(-)}}{\bar{M}_{(+)} - \bar{M}_{(+)}} \, . \tag{E.10}$$

Via (E.8) these equations express u_1 in terms of u_0 and w. The second of these tells us that

$$u_1 = -\frac{3w^2}{u_0} < 0 \, . \tag{E.11}$$

So we conclude that u_1 is negative, contradicting the assumption. $\square$

F   Diagrams of the moduli space

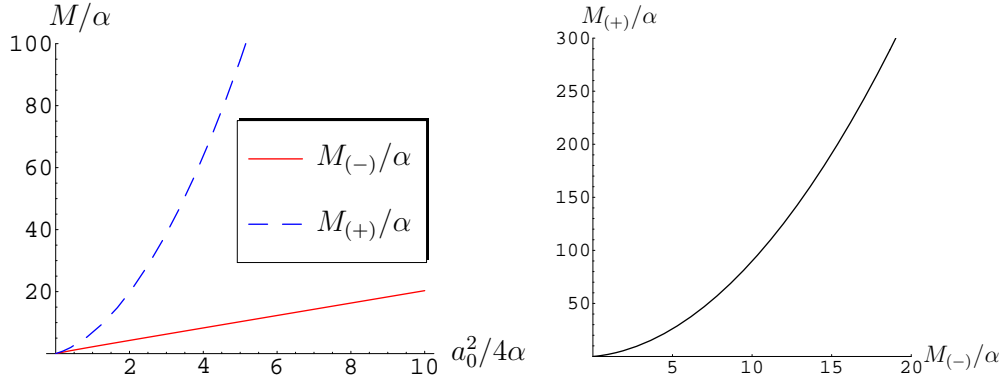Here we collect the diagrams referred to in section 3.

Figure 10:   For $\Lambda = 0$, spherically symmetric shells exist only with standard orientation and for $\alpha > 0$. Masses $M_{(-)}$, $M_{(+)}$ and shell radius a_0 are measured in units of the Gauss-Bonnet coupling, α.
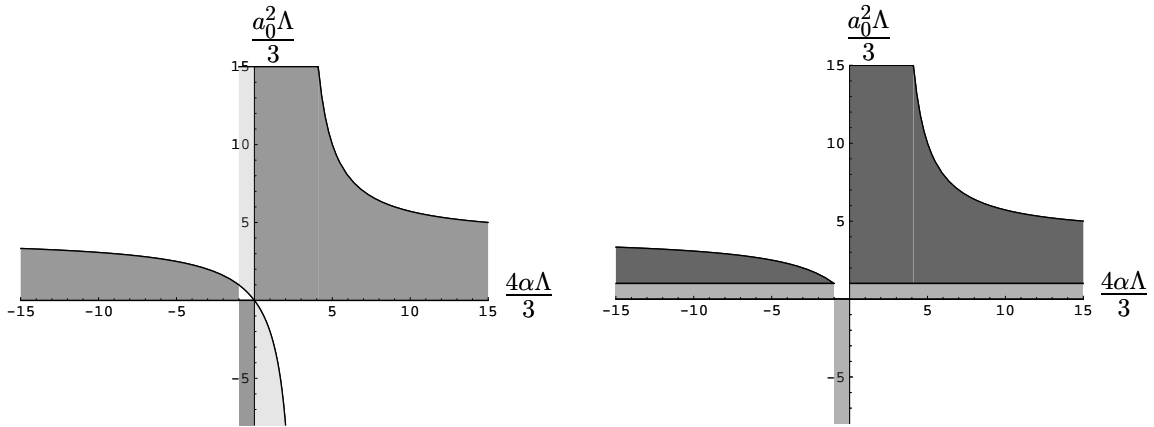

Figure 11: Static vacuum shells exist in the dark grey region. Instantaneous vacuum shells with $a = a_0$ exist in the light grey region.

Figure 12: The static vacuum shells can have the standard orientation $\eta_L \eta_R > 0$ (light grey) or wormhole orientation $\eta_L \eta_R < 0$ (dark grey).
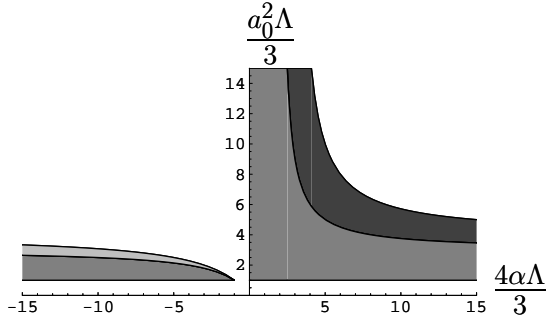
Figure 13: There are three types of static worm-holes according to the branch signs (ξ_L, ξ_R) in each bulk region: $(-, -)$ lightest grey; $(-, +)$ medium grey; $(+, +)$ dark grey.
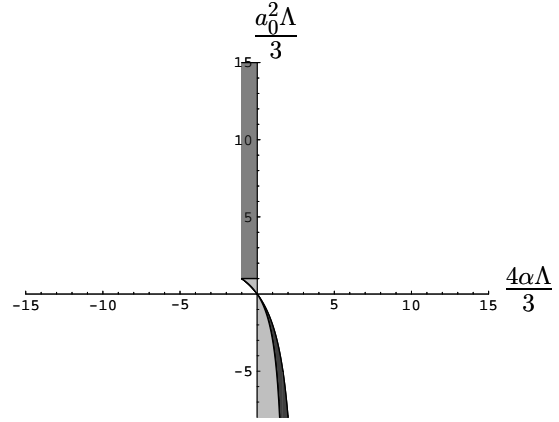
Figure 14: The different types of constant a instantaneous shells are: $(-, +)$ branch standard orientation (light grey); $(-, +)$ branch wormhole orientation (medium grey); $(+, +)$ branchstandard orientation (dark grey).
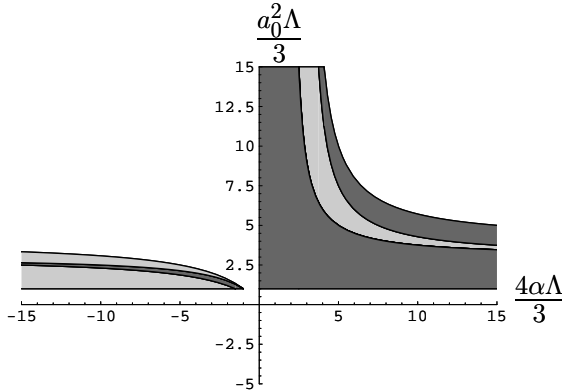

Figure 15: The stable region $V''(a_0) > 0$ for wormholes is shown in light grey. For positive α this is a region of the $(+,+)$ branch soluions. For negative α it includes all except a small region of the $(-, +)$ branch.
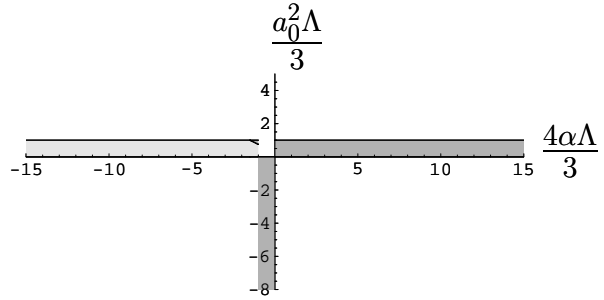
Figure 16: The stability of the standard shells. The stable regions are shown in light grey and the unstable regions in dark grey. All standard shells are $(-, +)$.

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