

Kolmogorov-Sinai and Bekenstein-Hawking entropies

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Abstract. It is shown that instability of stringy matter near the event horizon of a black hole (the spreading effect) can be characterized by the Lyapunov exponents. The Kolmogorov-Sinai entropy is the sum of all the positive Lyapunov exponents and equals to the inverse gravitational radius. Due to a replacement of the configuration space of a string by its phase space at distance of order of the string scale, the relation between the Kolmogorov-Sinai and Bekenstein-Hawking entropies is established. The KS entropy of a black hole measures the rate at which information about the state of a string collapsing into the black hole is lost with time as it spreads over the horizon.

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1. Introduction

It is well known that some general relativistic systems described by the Einstein equations can exhibit chaotic behavior [1, 2]. One of the most important quantities characterizing the chaotic behavior of a dynamical system is the Kolmogorov-Sinai (KS) entropy, which describes the rate of change of information about the phase space trajectories as a system evolves (a more formal definition will be given below). On the other hand, some general relativistic systems possess thermal properties and can be characterized by the Boltzmann, or the thermodynamical entropy. In particular, black holes being the solutions of the Einstein equations are characterized by the Bekenstein-Hawking (BH) entropy of just the same kind. In this connection, an important question arises: is there a relation between the KS and BH entropies?

In this paper I propose a possible relation between the KS and BH entropies. In the following sections we will introduce the main conceptions of chaotic dynamics and black hole thermodynamics, demonstrate chaotic behavior of stringy matter near the event horizon of a black hole, and establish the relation between the KS and BH entropies.

2. The main conceptions of chaotic dynamics and black hole thermodynamics

We begin with definitions. Suppose that phase space of a dynamical system is finite, then the KS entropy h_{KS} is the sum of all the positive Lyapunov exponents of the system, where the Lyapunov exponents h_i characterize the rate of exponential separation of the nearby system's trajectories in phase space as a result of a local instability [3]

$$d(t) = d(0) e^{h_i t}. \quad (1)$$

If this leads to an increase in the phase space volume occupied by the system with time

$$\Delta\Gamma(t) = \Delta\Gamma(0) e^{h_{KS} t}, \quad (2)$$

the Boltzmann entropy increases [4]

$$S(t) = h_{KS} t + \ln(\Delta\Gamma(0)). \quad (3)$$

As is easily seen, the KS entropy h_{KS} is not really an entropy but an entropy per unit time, or entropy rate, dS/dt . Note that the linear relation between the KS and thermodynamical entropies is not a general case [5, 6].

The BH entropy of a black hole, on the other hand,

$$S_{BH} = \frac{A}{4G} = \frac{\pi R_g^2}{l_P^2}, \quad (4)$$

is obtained from the thermodynamical relation $dE = TdS$, where the energy of the black hole is its mass M , the temperature is given by $T = 1/8\pi GM$, and the area of the event horizon A is related with the gravitational radius R_g , $R_g = 2GM$, in the usual way $A = 4\pi R_g^2$. The BH entropy is defined in the reference frame of an external distant observer at fixed static position above the horizon (an external observer).

Our purpose is to find a kinematic effect caused by the black hole geometry with respect to which a system evolves as in Eqns. (1),(2) in the reference frame of an external observer. For this purpose we repeat, for completeness, some well-known facts from [7] concerning the behavior of matter near the horizon without proofs, thus making our exposition self-contained.

3. A classical particle near the event horizon of a black hole

First consider a classical particle falling toward the horizon. The main fact is that the proper time in the frame of the particle τ and the Schwarzschild time of an external observer t are related through $\tau \sim e^{-t/2R_g}$ due to the redshift factor. Therefore in order to observe the particle before it crosses the horizon, we have to do it in a time which is exponentially small as $t \rightarrow \infty$. In other words, an external observer sees the particle as being slowed down with increasing powers of resolution. Moreover in the frame of the observer, the momentum of the particle increases like $\sim e^{t/2R_g}$, so that the observer sees the particle as being flattened in the direction of motion due to Lorentz longitudinal contraction.

The other important fact is that if the particle behaves as a conventional classical object it will appear to have fixed transverse size on the horizon. In so doing, the phase space volume holds its shape and remains the same; all information is conserved:

$$\Delta\Gamma_{clas}(t) = \Delta\Gamma_{clas}(0). \quad (5)$$

4. Chaotic behavior of a relativistic string collapsing into a black hole

My proposal rests on stringy matter having unusual kinematic properties near the event horizon of a black hole. According to string theory, the most promising candidate for a fundamental theory of matter, all particles are excitations of a one-dimensional object - a string. String theory is characterized by two fundamental parameters: the string scale, l_s , and the string coupling constant g ; if l_P is the Planck length then $l_P = g l_s$. An important fact is that strings behave very differently from ordinary particles. The crucial difference is that the size and shape of a string are sensitive to the time resolution. Susskind has shown [8] that the mean squared radius of a string, $\langle R_s \rangle^2$ depends on the time resolution, τ_r as $\langle R_s \rangle^2 \sim \ln(1/\tau_r)$ for $\tau_r \ll 1$.

Consider a string falling toward a black hole. As mentioned above, an external observer has a time resolution that decreases like $e^{-t/2R_g}$. This means that the string approaching the event horizon spreads in the transverse directions in the reference frame of the observer like $t/2R_g$ (there is also a longitudinal spreading but it is rapid to balance the Lorentz longitudinal contraction). Thus the string, in contrast to the classical particle, will not appear to have fixed transverse size on the horizon. As we have seen, the growth of the string is linear. But as noted by Susskind himself [7], this result was obtained in the framework of free string theory. It doesn't take into account such a

nonperturbative phenomenon as string interactions; there are indications [7]-[10] that a true growth must be exponential

$$\langle R_s \rangle^2 \sim e^{t/R_g}. \quad (6)$$

This also means that close trajectories of bits of the string diverge exponentially

$$d_{bit}(t) = d_{bit}(0) e^{t/2R_g}. \quad (7)$$

In addition, Susskind [9] and Mezhlumian, Peet and Thorlacius [10] have found that string configuration becomes chaotic and very complicated like a fractal during the spreading process. They have shown that as the correlation length of a string decreases exponentially with time the number of bits of the string increases exponentially

$$N_{bit} \sim e^{t/2R_g}. \quad (8)$$

They have interpreted this as a branching diffusion process, where every bit diffuses independently of others over the whole horizon and bifurcates into two bits and so on. According to the authors the diffusion process should provide necessary thermalization as the string spreads over the horizon.

But this picture also permits another interpretation. First the diffusion is a distinctive random process. But in our case there are no real random forces. The behavior of a string near the horizon is very well described by the Hamilton dynamics. If there are exact equations of motion no true randomness is possible. Second the string is a fundamental object. It is not a dissipative system. In the spreading process no points of a string are lost and also no points are gained: the number of bits of a string is conserved. Kinetics of the diffusion process is based on the random phase approximation, which implies rapid decay of correlations in the system. Chaotic dynamics of a string, on the other hand, gives the finite mixing time (see below Eqn. (13)), which just means a finite decay time of the correlations. So chaotic dynamics ensures the important condition of randomness that is crucial for deriving of diffusion kinetics. Therefore we can give the following interpretation of Eqn. (8). Initially bits of a string occupy one cell in phase space of a string. In the course of time, all bits will move to different phase-space points, mapping the cell at time $t = 0$ to another cell at time t . Hence we can interpret Eqn. (8) as an increase in the number of occupied cells

$$N_{cell}(t) = N_{cell}(0) e^{t/2R_g}. \quad (9)$$

But this number is proportional to the distance between the trajectories of bits that all initially occupy one cell (7), as required.

The spreading process begins to occur when the string reaches the horizon at distance of order of the string scale l_s from the horizon in a thin layer $\sim l_s$. But in string theory at such scales the mirror symmetry should takes place [11, 12]. In general it relates the complex and Kähler structures of some manifolds. In the simplest case for closed strings it exchanges the winding number around some circle with the corresponding momentum quantum number (T-duality) or, roughly speaking, coordinates with momenta. At the scales $\gg l_s$ we can always single out the configuration

space and the phase space is its cotangent bundle. At the scales $\sim l_s$ this is not the case: at such scales there is a replacement of the configuration space of a string by its phase space [12]. A similar phenomenon in quantum mechanics - a particle in magnetic field [13]: on the distances of order of the magnetic length $l_{mag} \sim \sqrt{\hbar c/eH}$ a replacement of the configuration plane transversal to the direction of the magnetic field by the phase plane takes place so that the number of states is A/l_{mag}^2 , where A is the area of the transversal plane.

5. The KS entropy of a black hole and its relation with the BH entropy

Hence instead Eqn. (5) we obtain

$$\Delta\Gamma_s(t) = \Delta\Gamma_s(0) e^{t/R_g}. \quad (10)$$

Then, taking into account Eqns. (6)-(10), we conclude that the spreading effect realizes a two-dimensional flow (or map) on the horizon by means of the positive Lyapunov exponents, $h_i = 1/2R_g; i = 1, 2$. Thus string matter collapsing into a black hole exhibit chaotic behavior which can be characterized by the KS entropy

$$h_{KS} = \frac{1}{R_g}. \quad (11)$$

Note that h_{KS} is infinite in purely random systems [3].

Finally we can obtain the relationship between the KS and BH entropies. Since $\Delta\Gamma_s(t) = 4\pi R_g^2$ and in the strong coupling regime ($g \sim 1, l_s \sim l_P$) $\Delta\Gamma_s(0) = l_P^2$ (or the same $\langle d_{bit}(t) \rangle^2 = 4\pi R_g^2$ and $\langle d_{bit}(0) \rangle^2 = l_P^2$), we have

$$h_{KS} = \frac{d(\ln S_{BH})}{dt}, \quad (12)$$

where S_{BH} is identified with the string entropy and expressed in terms of the characteristic time of the black hole R_g/c . Susskind has shown [14] that all black hole states are in one-to-one correspondence with single string states. This agrees with our identification.

The KS entropy h_{KS} of a dynamical system measures the rate at which information about the state of the system is lost with time. We can determine the average time over which the state of a string (or any body made of strings) can be predicted. Since the entire accessible phase space of the string is bounded by the horizon area, the trajectories of bits (7) mix together. This occurs when

$$t_{mix} \sim R_g. \quad (13)$$

At this time the string spreads over the entire horizon and can no longer expand due to the nonperturbative effects [7, 8, 9]. The result is crucial for the relaxation of the string to statistical equilibrium: to reach a statistical equilibrium in a finite time we should have the finite time of mixing (13). After the time t_{mix} all information contained in the string will be lost and we will be able only to make statistical predictions. This time is comparable to the characteristic time of a black hole R_g but is smaller than the black

hole lifetime $\sim R_g^3$. Thus the KS entropy h_{KS} of a black hole measures the rate at which information about the state of a string (or any body made of strings) collapsing into the black hole is lost with time as it spreads over the horizon.

We have demonstrated a relation between the KS and BH entropies for a string spreading over the event horizon of a black hole. It is widely believed, however, that the spreading effect is not a peculiar feature of a special (still hypothetical) kind of matter. In the framework of the so-called infrared/ultraviolet connection [7] it is a general property of all matter at energies above the Planck scale. If we want to study progressively smaller and smaller objects we must, according to conventional physics, to use higher and higher energy probes. But once gravity is involved that rule is changed radically. Since at energies above the Planck scale black holes are created, it follows [7] that as we raise the energy we probe larger and larger distances. In other words very high frequency is related to large size scale, $\Delta x \Delta \tau \sim l_p^2$. Then, taking into account the redshift factor, we can obtain the exponential growth of the transverse size of matter similar to Eqn. (6), as required.

In conclusion, let us turn to the form of the relation between the KS and BH entropies (12). It is interesting, to what extent it is special and can one obtain a similar relation from the general reasoning? For this purpose let us express the Boltzmann entropy not in terms of phase volume (2) but in terms of a distribution function $f(x, t)$

$$S = - \int f \ln f d\Gamma. \quad (14)$$

Now suppose that near equilibrium $f(x, t)$ can be presented in the form [1, 15]

$$f(x, t) = f_{eq} + (f_0(x) - f_{eq})e^{-h(t)t}. \quad (15)$$

By differentiating Eqn. (14) with respect to t and using Eqn. (15) we obtain

$$\frac{\partial S}{\partial t} = \int h (f(x, t) - f_{eq}) \ln f d\Gamma. \quad (16)$$

For short times $f \ll f_{eq}$ and Eqn. (16) reduces to

$$\frac{\partial S}{\partial t} = S_{eq} \int h(x) d\Gamma. \quad (17)$$

Then, since $h_{KS} = \int h(x) d\Gamma$ we have

$$h_{KS} = \frac{1}{S_{eq}} \frac{\partial S}{\partial t} \approx \left(\frac{\partial \ln S}{\partial t} \right)_{S \simeq S_{eq}}, \quad (18)$$

as required.

6. The KS entropy of other spaces with the event horizon

Of course, besides the black holes there are other general relativistic systems, which possess thermal properties, and de Sitter space is the most known of them. As is well

known, it is a thermodynamical system with the Gibbons-Hawking (GH) entropy given by

$$S_{GH} = \frac{A}{4l_P^2} = \frac{\pi}{H^2 l_P^2}, \quad (19)$$

where H is the Hubble constant, and the area of the event horizon A is related with the radius of de Sitter space R_{dS} , $R_{dS} = H^{-1}$, in the usual way $A = 4\pi R_{dS}^2$. We can repeat our experiment with a string by throwing it toward the event horizon of de Sitter space. Obviously, the result will be the same: the string spreads over the horizon. Thus, repeating the previous arguments, we can obtain the KS entropy of de Sitter space

$$h_{KS} = H, \quad (20)$$

and the relationship between the KS and GH entropies

$$h_{KS} = \frac{d(\ln S_{GH})}{dt}. \quad (21)$$

7. Conclusions

In this paper we have shown that stringy matter near the event horizon of a black hole with the gravitational radius R_g exhibits instability (the spreading effect), which can be characterized by the Lyapunov exponents. The Kolmogorov-Sinai entropy is the sum of all the positive Lyapunov exponents, $h_{KS} = 1/R_g$. Due to a replacement of the configuration space of a string by its phase space at distance of order of the string scale, the relation between the Kolmogorov-Sinai and Bekenstein-Hawking entropies is established, $h_{KS} = \partial(\ln S_{BH})/\partial t$, where the black hole entropy is identified with the string entropy and expressed in terms of the characteristic time of the black hole R_g/c . The KS entropy of a black hole measures the rate at which information about a string (or any body made of strings) collapsing into a black hole is lost as the string (the body) spreads over the horizon. Since the mixing time is finite $\sim R_g$, the system reach a statistical equilibrium in a finite time.

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