

The physical interpretation of the spectrum of black hole quasinormal modes

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When a classical black hole is perturbed, its relaxation is governed by a set of quasinormal modes with complex frequencies $\omega = \omega_R + i\omega_I$. We show that this behavior is the same as that of damped harmonic oscillators whose real frequencies are $(\omega_R^2 + \omega_I^2)^{1/2}$, rather than simply ω_R . Since, for highly excited modes, $\omega_I \gg \omega_R$, this observation changes drastically the physical understanding of the black hole spectrum, and forces a reexamination of various results in the literature. In particular, adapting a derivation by Hod, we find that the area of the horizon of a Schwarzschild black hole is quantized in units $\Delta A = 8\pi l_{\text{P}_1}^2$, in contrast with the original result $\Delta A = 4 \log(3) l_{\text{P}_1}^2$.

Perturbations of black holes (BHs) vanish in time as a superposition of damped oscillations, of the form

$$e^{-\omega_I t} [a \sin(\omega_R t) + b \cos(\omega_R t)], \quad (1)$$

with a spectrum of complex frequencies $\omega = \omega_R + i\omega_I$. These quasinormal modes are of great importance in gravitational-wave astrophysics, and might be observed in existing or advanced gravitational-wave detectors. Furthermore, BHs are often used as a testing ground for ideas in quantum gravity, and their quasinormal modes are obvious candidates for an interpretation in terms of quantum levels.

For Schwarzschild BHs, the quasinormal mode frequencies are labeled as ω_{nl} , where l is the angular momentum quantum number. For each l (with $l \geq 2$ for gravitational perturbation), there is a countable infinity of quasinormal modes, labeled by the ‘‘overtone’’ number n , with $n = 1, 2, \dots$. In Fig. 1 we show the frequencies of the $l = 2$ gravitational perturbations of a Schwarzschild BH of mass M : $\text{Im} \omega_n$ grows monotonically with n , so the least damped mode corresponds to $n = 1$, and has $2M \text{Im} \omega \simeq 0.1779$ (we use units $G = c = 1$). This is the mode that dominates the relaxation process. The next least damped mode is $n = 2$, with $2M \text{Im} \omega \simeq 0.5478$, and so on. In contrast, the real part of ω is not monotonic with n . It rather decreases at first, until it becomes exactly zero for $n = 9$, and then starts growing again, reaching a constant asymptotic value. For large n the asymptotic behavior of the frequencies of gravitational perturbations is independent of l , and is given by [2, 3, 4, 5, 6, 7, 8, 9, 10]

$$8\pi M \omega_n = \ln 3 + 2\pi i \left(n + \frac{1}{2} \right) + O\left(n^{-1/2}\right). \quad (2)$$

The pattern shown in Fig. 1 repeats for higher l . There is always a value \bar{n}_l of n such that, for $n < \bar{n}_l$, $\text{Re}(\omega_{n,l})$ decreases with n , while above this critical value it raises again, up to the asymptotic value $\ln 3 / (8\pi M)$ given by eq. (2).

If we compare with the normal mode structure of familiar classical systems, such as a vibrating rod, we have to admit that the structure displayed in Fig. 1, and particularly the ‘‘inverted branch’’ formed by the modes with

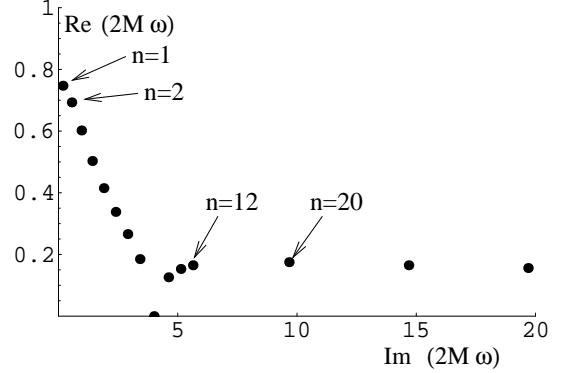


FIG. 1: $\text{Re}(2M\omega_{nl})$ against $\text{Im}(2M\omega_{nl})$ for $l = 2$ and $n = 1, 2, \dots, 12$, and for $n = 20, 30, 40$. Data taken from [1].

$n \leq \bar{n}_l$, is quite peculiar. In classical systems, the least damped mode is in general also the one with the *lowest* value of $\text{Re} \omega$, and typically $\text{Re} \omega$ and $\text{Im} \omega$ both increase with n . In contrast, in Fig. 1 the least damped mode is the one with the *highest* possible value of $\text{Re} \omega$ and, for $n < \bar{n}_l$, $\text{Re} \omega$ is a decreasing function of $\text{Im} \omega$! Even the ‘‘normal’’ branch $n > \bar{n}_l$ is somewhat puzzling. Now $\text{Re} \omega$ increases with n , which is more consistent with physical intuition, but still the fact that it saturates to a finite value is difficult to understand. In a normal macroscopic system, the underlying reason why, for large n , $\text{Im} \omega_n$ goes to infinity (and therefore these modes decay very fast), is that also $\text{Re} \omega_n$ diverges, so increasing n the wavelength $\lambda_n = 2\pi / \text{Re} \omega_n$ gets smaller and smaller, and finally becomes of the same order as the lattice spacing of the underlying atomic structure. At this point the perturbation can no longer be sustained as a wave by the medium, and quickly disappears in the thermal agitation of the lattice nuclei.

The quasinormal mode structure of Fig. 1 is no less puzzling if we attempt a semiclassical description and we interpret it as the structure of excited levels of a quantum BH. In normal quantum systems, the levels with high excitation energy, $E_n = \hbar \text{Re} \omega_n$, are those that decay fast, first of all because, in a multipole expansion, the decay width Γ grows with ω (e.g. $\Gamma \sim \omega^3$ for a dipole transition and $\Gamma \sim \omega^5$ for a quadrupole transition) and,

second, because they can decay into many different channels, i.e. into all the levels with lower excitation energy not forbidden by selection rules. So, again it is very surprising that, for $n < \bar{n}_l$, we have an inverted structure, where the lifetime of the state *increases* with its excitation energy. Quite puzzling is also the presence of a state with $\text{Re}\omega_n = 0$, and $\text{Im}\omega_n \neq 0$ (which exists for all l). So, the motivation of this work was to try to obtain a physical understanding of this level structure.

To this purpose, we describe a quasinormal mode as a damped harmonic oscillator $\xi(t)$, governed by the equation

$$\ddot{\xi} + \gamma_0 \dot{\xi} + \omega_0^2 \xi = f(t), \quad (3)$$

where γ_0 is the damping constant, ω_0 the proper frequency of the harmonic oscillator, and $f(t)$ an external force per unit mass. Solving this equation in Fourier transform we get

$$\xi(t) = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\tilde{f}(\omega)}{(\omega - \omega_+)(\omega - \omega_-)} e^{i\omega t}, \quad (4)$$

where ω_{\pm} are the two roots of the equation $\omega^2 - i\gamma_0\omega - \omega_0^2 = 0$, i.e.

$$\omega_{\pm} = \pm \sqrt{\omega_0^2 - (\gamma_0/2)^2} + i\frac{\gamma_0}{2}. \quad (5)$$

Consider the response to a Dirac delta perturbation, $f(t) \propto \delta(t)$, so $\tilde{f}(\omega) \propto 1$. For $t < 0$ we can close the integration contour in eq. (4) in the lower half-plane and, since ω_{\pm} both lie above the real axis, we get zero, as required by causality. For $t > 0$ we close the contour in the upper half-plane and we pick the residue of the two poles. So the result for $\xi(t)$ is a superposition of a term oscillating as $e^{i\omega_+ t}$ and of a term oscillating as $e^{i\omega_- t}$. Therefore, the behavior (1) is reproduced by a damped harmonic oscillator, with the identifications

$$\frac{\gamma_0}{2} = \omega_I, \quad \sqrt{\omega_0^2 - (\gamma_0/2)^2} = \omega_R, \quad (6)$$

which can be inverted to give

$$\omega_0 = \sqrt{\omega_R^2 + \omega_I^2}. \quad (7)$$

We see that the seemingly obvious identification $\omega_0 = \omega_R$ only holds when $\gamma_0/2 \ll \omega_0$, i.e. for very long-lived modes. For most BH quasinormal modes we are in the opposite limit; in particular, for highly excited modes, we have $\omega_I \gg \omega_R$, see Fig. 1, so $\omega_0 \simeq \omega_I$ rather than $\omega_0 \simeq \omega_R$! If we model the BH perturbations in terms of a collection of damped harmonic degrees of freedom (which can be useful both classically, to have an intuitive physical picture of a BH as a whole, and in semiclassical quantum gravity, to get hints about the quantum structure of spacetime) the correct identification for the frequency of the equivalent harmonic oscillator is given by eq. (7), together with $\gamma_0/2 = \omega_I$.

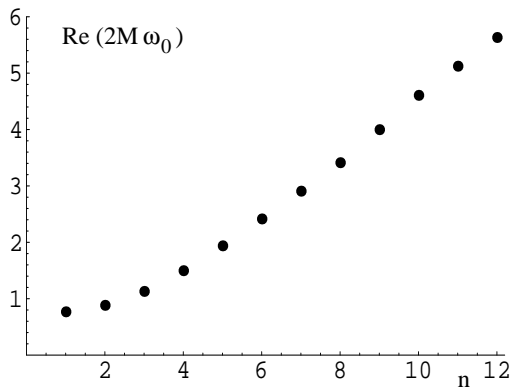


FIG. 2: $\text{Re}[2M(\omega_0)_n]$ for $l = 2$, against n .

In terms of ω_0 the energy level structure of a BH becomes physically very reasonable, and for $l = 2$ it is shown in Fig. 2 (a similar result holds for higher l). We see that the frequency $(\omega_0)_n$ increases monotonically with the overtone number n . Recall that the damping coefficient $\gamma_0/2$ is equal to ω_I , so also $\gamma_0/2$ increases monotonically with n . Thus, in terms of the equivalent harmonic oscillators, the least damped mode, which is still the $n = 1$ mode, is also the one with the *lowest* value of ω_0 , and the larger is $(\omega_0)_n$, the shorter is the lifetime, as we expected from physical intuition.

For large n , using eq. (2) and introducing the Hawking temperature $T_H = \hbar/(8\pi M)$, eq. (7) can be written in a very suggestive form,

$$\hbar(\omega_0)_n = \sqrt{m_0^2 + p_n^2}, \quad (8)$$

where

$$m_0 = T_H \ln 3, \quad p_n = 2\pi T_H \left(n + \frac{1}{2} \right). \quad (9)$$

The expression for p_n is especially intriguing, since it corresponds to a particle quantized with antiperiodic boundary conditions on a circle of length $L = \hbar/T_H = 8\pi M$. It is also interesting to observe that the equal spacing of the levels for large n is just what would be expected from a description of the horizon in terms of an effective membrane [11]. We can now reexamine some aspects of quantum BH physics, that have been previously discussed assuming that the relevant frequencies were $(\omega_R)_n$, using $(\omega_0)_n$ instead.

Area quantization. The idea that in quantum gravity the area of the BH horizon is quantized in units of l_{Pl}^2 (where l_{Pl} is the Planck length) has a long history, that goes back to Bekenstein [12]. His result was that the area quantum of a Schwarzschild BH is $\Delta A = 8\pi l_{\text{Pl}}^2$. Hod [5] found a similar quantization, but with a different numerical coefficient, using the properties of quasinormal modes of Schwarzschild BHs. Since for a Schwarzschild BH the horizon area A is related to the mass M by $A = 16\pi M^2$, a change ΔM in the BH mass produces

a change $\Delta A = 32\pi M \Delta M$ in the area. Hod considered a transition from an unexcited BH to a BH in a mode with n very large. He argued that for large n Bohr's correspondence principle should hold, so a semiclassical description should be adequate even in the absence of a full theory of quantum gravity, and concluded from eq. (2) that the minimum quantum that can be absorbed in this transition has $\Delta M = \hbar\omega = \hbar \ln(3)/(8\pi M)$. This gives $\Delta A = 4 \ln(3) l_{\text{Pl}}^2$ (recall that in units $G = c = 1$ we have $l_{\text{Pl}}^2 = \hbar$). The numerical factor $4 \ln(3)$ generated some excitement because of possible connections with loop quantum gravity [13] (see however [14]).

This conjecture is stimulating, but suffers from a number of difficulties. First of all, further studies showed that the factor $4 \ln 3$, that has its origin in ω_R , see eq. (2), is not universal. For instance, we can consider a rotating BH with angular momentum per unit mass a . Computing the asymptotic behavior of the quasinormal mode frequencies of gravitational perturbations, one finds that the large n limit and the limit $a \rightarrow 0$ do not commute. If we first consider the asymptotic value for a Kerr BH and then we let $a \rightarrow 0$, ω_R does not reduce to $\ln 3/(8\pi M)$, but rather vanishes as $a^{1/3}$ [7, 15, 16, 17]. This means that the area quantum becomes arbitrarily small if we give to the BH an infinitesimal rotation. Similarly, studying charged BHs, one finds that ω_R changes discontinuously if we interchange the limits $Q \rightarrow 0$ and $n \rightarrow \infty$. Furthermore, the study of generic spin- j perturbations revealed that the leading asymptotic value of the quasinormal mode frequencies is given by [7]

$$e^{8\pi M \omega} = -(1 + 2 \cos \pi j). \quad (10)$$

For gravitational perturbations ($j = 2$) and for scalar perturbations ($j = 0$) the right-hand side of eq. (10) is equal to -3 , and we recover eq. (2). For vector perturbation ($j = 1$), the right-hand side of eq. (10) is equal to $+1$, and we get $8\pi M \omega_n = 2\pi i n$, so the real part is now zero rather than $\ln 3$, and the corresponding quantum of area would also be zero. Equation (10) holds also for half-integer perturbations [18, 19]; in this case the right-hand side of eq. (10) is equal to -1 , and $8\pi M \omega_n = 2\pi i(n + 1/2)$, so again $\text{Re} \omega_n$ vanishes asymptotically. The conclusion is that the asymptotic value of $\text{Re} \omega_n$ (and also whether p_n in eq. (9) is proportional to n or to $n + 1/2$) depends on the spin of the perturbation, and is not an intrinsic property of the BH. A similar non-universal behavior was discussed in [20] in a large class of BH models, that in the (r_*, t) plane effectively reduce to 2-d dilaton gravity. In conclusion, the area quantization determined by Hod's conjecture does not reflect any intrinsic property of the BH, and the would-be area quantum vanishes in various instances.

Another criticism that can be raised to the above derivation is that one has considered only transitions from the ground state (i.e. a BH which is not excited) to a state with large n (or viceversa). However, it is also legitimate to consider transitions $n \rightarrow n'$ where n and n' are both large. Bohr correspondence principle, that

was advocated above, actually only holds for transitions where both $n, n' \gg 1$, so these are in fact the only transitions that should be considered within the above logic. Now, if we use eq. (2), we see that in a transition $n \rightarrow n'$ with $n, n' \gg 1$, $\text{Re} \omega_n$ changes by $O(1/n^{1/2})$. This means that in these transitions the area changes by arbitrarily small amounts. So, even restricting to $j = 2$ perturbation of Schwarzschild BHs, the area quantization holds only for a transition from (or to) a BH in its fundamental state, while transitions among excited levels do not obey it.

All the above difficulties are removed when, in Hod's conjecture, we use $(\omega_0)_n$ rather than $(\omega_R)_n$. We consider a transition $n \rightarrow n-1$, with n large. Then $(\omega_0)_n \simeq (\omega_I)_n$ and eq. (2) gives an absorbed energy $\Delta M = \hbar[(\omega_0)_n - (\omega_0)_{n-1}] = \hbar/(4M)$, so

$$\Delta A = 32\pi M \Delta M = 8\pi l_{\text{Pl}}^2, \quad (11)$$

which coincides with the old Bekenstein result. At large n all other transitions require a larger energy; e.g. $n \rightarrow n-2$ takes away about twice the energy, since for large n the $(\omega_0)_n$ are equally spaced. Even if we dare to extrapolate to low n , where semiclassical reasoning might go wrong, we still remain with a non-vanishing area quantum, of the order of $8\pi l_{\text{Pl}}^2$. As it is clear from Fig. 2, the transition from $n = 2$ to $n = 1$ is the one with the smallest possible jump. Using the values of ω_R and ω_I given in [1], we find $(\omega_0)_{n=2} - (\omega_0)_{n=1} \simeq 0.2/(4M)$, corresponding to $\Delta A \simeq 0.2 (8\pi l_{\text{Pl}}^2)$, while the transition from $n = 1$ to an unexcited BH has $\Delta A \simeq 1.5 (8\pi l_{\text{Pl}}^2)$.

Contrary to what happens for ω_R , the quantum of area obtained from the asymptotics of $(\omega_0)_n$ is an intrinsic property of Schwarzschild BHs: for large n the leading asymptotic behavior of ω_0 is given by the $O(n)$ term in ω_I , and it does not depend on the spin content of the perturbation, as we see from eq. (10). Furthermore, in contrast to what happens to ω_R , for ω_I the limits $a \rightarrow 0$ and $n \rightarrow \infty$ commute, and similarly for the limits $Q \rightarrow 0$ and $n \rightarrow \infty$ [7, 15, 16, 17]. The result (11) can therefore be consistently taken as an indication of a quantization of the area of the horizon of a Schwarzschild BH. (The generalization of these results to other spacetimes might however be non-trivial, see e.g. [21].) In this context, it is useful to remark that a gedanken experiment with black holes reveals the existence of a generalized uncertainty principle, which implies a minimum length of order l_{Pl} [22], and which fits very nicely with the above result.

Entropy and microstates. If, for large n , the horizon area is quantized, with a quantum $\Delta A = \alpha l_{\text{Pl}}^2$ (where for us $\alpha = 8\pi$ while for Hod $\alpha = 4 \ln 3$), the total horizon area A must be of the form $A = N \Delta A = N \alpha l_{\text{Pl}}^2$, where N is an integer. Observe that N is *not* the same as the integer n that we used to label the quasinormal mode. Even for a BH in its ground state, $n = 0$, N is very large since it must account for the area of the unexcited BH, $N = A/\Delta A = 16\pi M^2/(\alpha l_{\text{Pl}}^2)$. The famous Bekenstein formula associates to the BH an entropy $S = A/(4l_{\text{Pl}}^2)$, and therefore at level N we expect that a BH should

have a number of microstates $g(N)$ given by $g(N) \propto \exp\{(\Delta A)N/(4l_{\text{P}1}^2)\} = \exp\{\alpha N/4\}$. One might try to restrict the possible values of α as follows [23, 24]. One admits the presence of a subleading term in the Bekenstein formula, $S = A/(4l_{\text{P}1}^2) + \text{const.}$, and fixes the constant requiring that, for $N = 1$, there is only one microstate, $g(N) = 1$. This gives $g(N) = \exp\{(\alpha/4)(N - 1)\}$. One then requires that $g(N)$ be an integer. This restricts α to be of the form $\alpha = 4 \ln k$, with k an integer. The value $\alpha = 4 \ln 3$ is of this form, which is not the case for $\alpha = 8\pi$.

However, a number of objections can be raised to this attempt to restrict α . First of all, in the semiclassical regime where our results could be trusted, N is very large, of order of $A/l_{\text{P}1}^2$, so $g(N)$ is the exponential of a very large number. Even if the number of microstates must be an integer, there is no hope that a semiclassical computation can reproduce this number with a precision of order one, necessary to distinguish an integer from a non-integer value. In fact, this does not happen even in classical textbook computations in statistical mechanics. Furthermore, the above expression for $g(N)$ assumes that the same area quantum ΔA is valid from large N down to $N = 1$, where our semiclassical approximation is certainly unjustified. Indeed, we see from eqs. (8) and (9) that the levels are equally spaced only in the limit of highly excited modes, otherwise there are deviations.

Using our value of α in $S = \alpha N/4$ we find, to leading order in the large N limit,

$$S = 2\pi N + O(1), \quad (12)$$

and $g(N) \propto \exp\{2\pi N\}$. It is quite interesting to observe that eq. (12) agrees with the result found in Refs. [25, 26], with apparently very different arguments. In these works, using the periodicity of the *Euclidean* BH solutions, it was found that the entropy is an adiabatic invariant, with a spectrum given, through Bohr's correspondence principle, precisely by eq. (12). This argument required only standard rules of quantum mechanics, but it was somewhat speculative in that the rules were applied in Euclidean time.

On the other hand, the periodicity of the Euclidean solution also entered implicitly our arguments, since it is at the basis of the analytic computation of the asymptotic quasinormal modes frequencies, eq. (10). So it appears that the periodicity of the BH solutions in Euclidean time, beside providing a quick derivation of the value of Hawking temperature, is also at the origin of the area quantization law.

Acknowledgments. This work is supported by the Fond National Suisse. I thank a referee for useful comments.

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