# A QUASI-LOCAL MASS FOR 2-SPHERES WITH NEGATIVE GAUSS CURVATURE

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Abstract. We extend our previous definition of quasi-local mass to 2-spheres whose Gauss curvature is negative and prove its positivity.

## 1. Introduction

In [7], Liu and Yau propose a definition of quasi-local mass for any smooth spacelike, topological 2-sphere with positive Gauss curvature. In particular, Liu and Yau [7, 8] are able to use Shi-Tam's result [10] to prove its positivity. When the Gauss curvature of a 2-sphere is allowed to be negative, Wang and Yau [14] use Pogorelov's result [9] to embed the 2-sphere into the hyperbolic space to generalize Liu-Yau's definition, and prove its positivity by using a spinor argument of the positive mass theorem for asymptotically hyperbolic manifolds [15, 4, 16]. Wang-Yau's result is improved in certain sense by Shi and Tam [11].

In attempting to resolve the decreasing monotonicity of Brown-York's quasi-local mass [1, 2], the author [18] propose a new quasi-local mass and prove its positivity essentially for 2-spheres with positive Gauss curvature. It is still open when the 2-spheres have nonnegative Gauss curvature because the isometric embedding into $\mathbb{R}^3$ in this case is only proved to be $C^{1,1}$ by Guan-Li and Hong-Zuily [5, 6]. However, we expect the $C^{1,1}$ regularity is sufficient for our propose, and we address it elsewhere.

In this note, we use the idea of Wang and Yau to extend the quasi-local mass in [18] to the case of 2-spheres with negative Gauss curvature. We embed such 2-spheres into the (spacelike) hyperbola in the Minkowski spacetime which has the nontrivial second fundamental form. By using the constant spinors in the Minkowski spacetime, we can solve a boundary problem for the Dirac-Witten equation. Then,

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the method in [18] gives rise to the quasi-local mass as well as its positivity. We would like to point out that our quasi-local mass is only one quantity, while the one defined by Wang and Yau is a 4-vectors. This difference is due to the hyperbola in our approach goes to null infinity in the Minkowski spacetime, and the one in Wang-Yau's approach goes to spatial infinity in the Anti-de Sitter spacetime, which has trivial second fundamental form. The positive mass theorem near null infinity in asymptotically Minkowski spacetimes was established in [16, 17].

## 2. Dirac-Witten equations

In this section, we will review the existences of the Dirac-Witten equations proved in [18]. Let $(N, \widetilde{g})$ be a 4-dimensional spacetime which satisfies the Einstein fields equations. Let $(M, g, p)$ be a smooth *initial data set*. Fix a point $p \in M$ and an orthonormal basis $\{e_\alpha\}$ of $T_pN$ with $e_0$ future-time-directed normal to $M$ and $e_i$ tangent to $M$ $(1 \leq i \leq 3)$.

Denote by $\mathbb{S}$ the (local) spinor bundle of $N$. It exists globally over $M$ and is called the hypersurface spinor bundle of $M$. Let $\widetilde{\nabla}$ and $\overline{\nabla}$ be the Levi-Civita connections of $\widetilde{g}$ and $g$ respectively, the same symbols are used to denote their lifts to the hypersurface spinor bundle. There exists a Hermitian inner product $( \, , \, )$ on $\mathbb{S}$ along $M$ which is compatible with the spin connection $\widetilde{\nabla}$. The Clifford multiplication of any vector $\widetilde{X}$ of $N$ is symmetric with respect to this inner product. However, this inner product is not positive definite and there exists a positive definite Hermitian inner product defined by $\langle \, , \, \rangle = (e_0 \cdot \, , \, )$ on $\mathbb{S}$ along $M$.

Define the second fundamental form of the initial data set $p_{ij} = \widetilde{g}(\widetilde{\nabla}_i e_0, e_j)$. Suppose that $M$ has boundary $\Sigma$ which has finitely many connected components $\Sigma^1, \cdots, \Sigma^l$, each of which is a topological 2-sphere, endowed with its induced Riemannian and spin structures. Fix a point $p \in \Sigma$ and an orthonormal basis $\{e_i\}$ of $T_pM$ with $e_r = e_1$ outward normal to $\Sigma$ and $e_a$ tangent to $\Sigma$ for $2 \leq a \leq 3$. Let $h_{ab} = \langle \overline{\nabla}_a e_r, e_b \rangle$ be the second fundamental form of $\Sigma$. Let $H = tr(h)$ be its mean curvature. $\Sigma$ is a *future/past apparent horizon* if

$$H \mp tr(p|_\Sigma) \geq 0 \tag{2.1}$$

holds on $\Sigma$. When $\Sigma$ has multi-components, we require that (2.1) holds (with the same sign) on each $\Sigma_i$. The spin connection has the following relation

$$\widetilde{\nabla}_a = \nabla_a + \frac{1}{2} h_{ab} e_r \cdot e_b \cdot - \frac{1}{2} p_{aj} e_0 \cdot e_j \cdot . \tag{2.2}$$

The Dirac-Witten operator along $M$ is defined by $\widetilde{D} = e_i \cdot \widetilde{\nabla}_i$. The Dirac operator of $M$ but acting on $\mathbb{S}$ is defined by $\overline{D} = e_i \cdot \overline{\nabla}_i$. Denote by $\nabla$ the lift of the Levi-Civita connection of $\Sigma$ to the spinor bundle $\mathbb{S}|_\Sigma$. Let $D = e_a \cdot \nabla_a$ be the Dirac operator of $\Sigma$ but acting on $\mathbb{S}|_\Sigma$. The Weitzenböck type formula gives rise to

$$\begin{aligned}&\int_M |\widetilde{\nabla}\phi|^2 + \langle \phi, \mathcal{T}\phi\rangle - |\widetilde{D}\phi|^2 \\ = \;&\int_\Sigma \langle \phi, (e_r \cdot D - \frac{H}{2} + \frac{tr(p|_\Sigma)}{2} e_0 \cdot e_r \cdot - \frac{p_{ar}}{2} e_0 \cdot e_a \cdot)\phi\rangle. \quad (2.3)\end{aligned}$$

where $\mathcal{T} = \frac{1}{2}(T_{00} + T_{0i} e_0 \cdot e_i \cdot)$. If the spacetime satisfies the *dominant energy condition*, then $\mathcal{T}$ is a nonnegative operator. Let

$$P_\pm = \frac{1}{2}(Id \pm e_0 \cdot e_r \cdot)$$

be the projective operators on $\mathbb{S}|_\Sigma$. In [18], we prove the following existences:

(i) If $tr_g(p) \geq 0$ and $\Sigma$ is a past apparent horizon, then the following Dirac-Witten equation has a unique smooth solution $\phi \in \Gamma(\mathbb{S})$

$$\begin{cases} \widetilde{D}\phi = 0 & in \quad M \\ P_+\phi = P_+\phi_0 & on \quad \Sigma_{i_0} \\ P_+\phi = 0 & on \quad \Sigma_i \; (i \neq i_0) \end{cases} \quad (2.4)$$

for any given $\phi_0 \in \Gamma(\mathbb{S}\big|_\Sigma)$ and for fixed $i_0$;

(ii) If $tr_g(p) \leq 0$ and $\Sigma$ is a future apparent horizon, then the following Dirac-Witten equation has a unique smooth solution $\phi \in \Gamma(\mathbb{S})$

$$\begin{cases} \widetilde{D}\phi = 0 & in \quad M \\ P_-\phi = P_-\phi_0 & on \quad \Sigma_{i_0} \\ P_-\phi = 0 & on \quad \Sigma_i \; (i \neq i_0) \end{cases} \quad (2.5)$$

for any given $\phi_0 \in \Gamma(\mathbb{S}\big|_\Sigma)$ and for fixed $i_0$.

## 3. Embedding 2-spheres

Let $(M, g, p)$ be a smooth *initial data set* where $M$ has boundary $\Sigma$ which has finitely many connected components $\Sigma_1, \cdots, \Sigma_l$, each of which is a topological 2-sphere. Suppose that some $\Sigma_{i_0}$ can be smoothly isometrically embedded into a smooth spacelike hypersurface $\breve{M}^3$ in the Minkowski spacetime $\mathbb{R}^{3,1}$ and denote by $\aleph$ the isometric embedding. Let $\breve{\Sigma}_{i_0}$ be the image of $\Sigma_{i_0}$ under the map $\aleph$. Let $\breve{e}_r$ the unit vector outward normal to $\breve{\Sigma}_{i_0}$ and $\breve{h}_{ij}$, $\breve{H}$ are the second fundamental form, the mean curvature of $\breve{\Sigma}_{i_0}$ respectively. Denote by $p_0 = \breve{p} \circ \aleph$, $H_0 = \breve{H} \circ \aleph$ the pullbacks to $\Sigma$.

The isometric embedding $\aleph$ also induces an isometry between the (intrinsic) spinor bundles of $\Sigma_{i_0}$ and $\breve{\Sigma}_{i_0}$ together with their Dirac operators which are isomorphic to $e_r \cdot D$ and $\breve{e}_r \cdot \breve{D}$ respectively. This isometry can be extended to an isometry over the complex 2-dimensional sub-bundles of their hypersurface spinor bundles. Denote by $\breve{\mathbb{S}}^{\breve{\Sigma}_{i_0}}$ this sub-bundle of $\breve{\mathbb{S}}|_{\breve{\Sigma}_{i_0}}$. Let $\breve{\phi}$ be a constant section of $\breve{\mathbb{S}}^{\breve{\Sigma}_{i_0}}$ and denote $\phi_0 = \breve{\phi} \circ \aleph$. Denote by $\breve{\Xi}$ the set of all these constant spinors $\breve{\phi}$ with the unit norm. This set is isometric to $S^3$.

Let $\breve{D}$ be the (induced) Dirac operator on $\breve{\Sigma}_{i_0}$ which acts on the hypersurface spinor bundle $\breve{\mathbb{S}}$ of $\breve{M}$. Let $\breve{\phi}$ be the covariant constant spinor of the trivial spinor bundle on $\mathbb{R}^{3,1}$ with unit norm taking by the positive Hermitian metric on $\breve{\mathbb{S}}$. Then (2.2) implies

$$\breve{\nabla}_a \breve{\phi} + \frac{1}{2}\, \breve{h}_{ab} \breve{e}_r \cdot \breve{e}_b \cdot \breve{\phi} - \frac{1}{2}\, \breve{p}_{aj} \breve{e}_0 \cdot \breve{e}_j \cdot \breve{\phi} = 0$$

over $\breve{\Sigma}_{i_0}$. Pullback to $\Sigma_{i_0}$, we obtain

$$e_r \cdot D\phi_0 = \frac{H_0}{2}\phi_0 - \frac{1}{2} p_{0aa} e_0 \cdot e_r \cdot \phi_0 + \frac{1}{2} p_{0ar} e_0 \cdot e_a \cdot \phi_0 \tag{3.1}$$

over $\Sigma_{i_0}$. Denote $\phi_0^{\pm} = P_{\pm}\phi_0$. Since $e_r \cdot D \circ P_{\pm} = P_{\mp} \circ e_r \cdot D$, (3.1) gives rise to

$$\begin{aligned} e_r \cdot D\phi_0^+ &= \frac{H_0}{2}\phi_0^- + \frac{1}{2} p_{0aa}\phi_0^- + \frac{1}{2} p_{0ar} e_0 \cdot e_a \cdot \phi_0^+, \\ e_r \cdot D\phi_0^- &= \frac{H_0}{2}\phi_0^+ - \frac{1}{2} p_{0aa}\phi_0^+ + \frac{1}{2} p_{0ar} e_0 \cdot e_a \cdot \phi_0^-. \end{aligned}$$

Therefore, using

$$\int_{\Sigma_{i_0}} \langle \phi_0^-, e_r \cdot D\phi_0^+ \rangle = \int_{\Sigma_{i_0}} \langle e_r \cdot D\phi_0^-, \phi_0^+ \rangle,$$

we obtain

$$\int_{\Sigma_{i_0}} (H_0 - p_{0aa})|\phi_0^+|^2 = \int_{\Sigma_{i_0}} (H_0 + p_{0aa})|\phi_0^-|^2. \tag{3.2}$$

In this paper, we introduce the following conditions on $M$:

(i) $tr_g(p) \geq 0$, $H|_{\Sigma_i} + tr(p|_{\Sigma_i}) \geq 0$ for all $i$;
(ii) $tr_g(p) \leq 0$, $H|_{\Sigma_i} - tr(p|_{\Sigma_i}) \geq 0$ for all $i$.

**Lemma 1.** *Let $(N^{3,1}, \widetilde{g})$ be a spacetime which satisfies the dominant energy condition. Let $(M, g, p)$ be a smooth spacelike (orientable) hypersurface which has boundary $\Sigma$ with finitely many multi-components $\Sigma_i$, each of which is a topological sphere. Suppose that $\Sigma_{i_0}$ can be smoothly isometrically embedded into some spacelike hypersurface $(\breve{M}, \breve{g}, \breve{p})$ in the Minkowski spacetime $\mathbb{R}^{3,1}$. Let $\aleph$ be the isometric embedding and let $\breve{\Sigma}_{i_0}$ be the image of $\Sigma_{i_0}$. Suppose either condition $(i)$ holds and $\breve{\Sigma}_{i_0}$ are past apparent horizons, i.e.,*

$$\breve{H} + tr(\breve{p}|_{\breve{\Sigma}_{i_0}}) \geq 0,$$

*or condition $(ii)$ holds and $\breve{\Sigma}_{i_0}$ are future apparent horizons, i.e.,*

$$\breve{H} - tr(\breve{p}|_{\breve{\Sigma}_{i_0}}) \geq 0.$$

*Let $\phi$ be the unique solution of (2.4) or (2.5) for some $\breve{\phi} \in \breve{\Xi}$. Then*

$$\int_{\Sigma_{i_0}} \langle \phi, e_r \cdot D\phi \rangle \leq \frac{1}{2} \int_{\Sigma_{i_0}} \langle \phi, (H_0 - p_{0aa} e_0 \cdot e_r \cdot + p_{0ar} e_0 \cdot e_a \cdot) \phi \rangle.$$

*Proof :* Assume condition $(i)$ holds and $\breve{\Sigma}_{i_0}$ are past apparent horizons. Let $\phi$ be the smooth solution of (2.4) with the prescribed $\phi_0$ on $\Sigma_{i_0}$. Denote $\phi^{\pm} = P_{\pm}\phi$. Denote $\phi_0^{\pm} = P_{\pm}\phi_0$. By the boundary condition, we have $\phi^+ = \phi_0^+$. Thus

$$\begin{aligned}
\int_{\Sigma_{i_0}} \langle \phi, e_r \cdot D\phi \rangle &= 2\Re \int_{\Sigma_{i_0}} \langle \phi^-, e_r \cdot D\phi_0^+ \rangle \\
&= \Re \int_{\Sigma_{i_0}} \langle \phi^-, H_0 \phi_0^- + p_{0aa} \phi_0^- + p_{0ar} e_0 \cdot e_r \cdot \phi_0^+ \rangle \\
&\leq \frac{1}{2} \int_{\Sigma_{i_0}} (H_0 + p_{0aa})(|\phi^-|^2 + |\phi_0^-|^2) \\
&\quad + \Re \int_{\Sigma_{i_0}} \langle \phi^-, p_{0ar} e_0 \cdot e_a \cdot \phi_0^+ \rangle \\
&= \frac{1}{2} \int_{\Sigma_{i_0}} (H_0 + p_{0aa})|\phi^-|^2 + (H_0 - p_{0aa})|\phi_0^+|^2 \\
&\quad + \Re \int_{\Sigma_{i_0}} \langle \phi^-, p_{0ar} e_0 \cdot e_a \cdot \phi^+ \rangle \\
&= \frac{1}{2} \int_{\Sigma_{i_0}} H_0 |\phi|^2 + p_{0aa}(|\phi^-|^2 - |\phi^+|^2) \\
&\quad + \Re \int_{\Sigma_{i_0}} \langle \phi^-, p_{0ar} e_0 \cdot e_a \cdot \phi^+ \rangle.
\end{aligned}$$

Note that

$$\langle \phi, p_{0aa} e_0 \cdot e_r \cdot \phi \rangle = p_{0aa}(|\phi^+|^2 - |\phi^-|^2).$$

Moreover, that $e_0 \cdot e_a \cdot P_\pm = P_\mp \cdot e_0 \cdot e_a \cdot$ gives rise to

$$\langle \phi, p_{0ar} e_0 \cdot e_a \cdot \phi \rangle = 2\Re \langle \phi^-, p_{0ar} e_0 \cdot e_a \cdot \phi^+ \rangle.$$

Same argument is applied under condition $(ii)$. We finally prove the lemma. Q.E.D.

## 4. Quasi-local mass

Now we use the idea of Wang and Yau [14] (see also [11]) to extend the definition of quasi-local mass in [18] to the case of 2-spheres with negative Gauss curvature.

We first review the definition for 2-spheres with nonnegative Gauss curvature in [18]: Suppose some $\Sigma_{i_0}$ can be smoothly isometrically embedded into $\mathbb{R}^3$ in the Minkowski spacetime $\mathbb{R}^{3,1}$ and denote $\breve{\Sigma}_{i_0}$ its image. (It exists if $\Sigma_{i_0}$ has positive Gauss curvature.) In this case, $\breve{p} = 0$.

Let $\phi$ be the unique solution of (2.4) or (2.5) for some $\breve{\phi} \in \breve{\Xi}$. Denote

$$\begin{aligned} m(\Sigma_{i_0}, \breve{\phi}) \ = \ & \frac{1}{8\pi} \Re \int_{\Sigma_{i_0}} \Big[ (H_0 - H)|\phi|^2 \\ & + tr(p|_{\Sigma_{i_0}}) \langle \phi, e_0 \cdot e_r \cdot \phi \rangle \\ & - p_{ar} \langle \phi, e_0 \cdot e_a \cdot \phi \rangle \Big]. \end{aligned} \qquad (4.1)$$

The *quasi local mass of* $\Sigma_{i_0}$ is defined as

$$m(\Sigma_{i_0}) = \min_{\breve{\Xi}} m(\Sigma_{i_0}, \breve{\phi}). \qquad (4.2)$$

If all $\Sigma_i$ can be isometrically embedded into $\mathbb{R}^3$ in the Minkowski spacetime $\mathbb{R}^{3,1}$, we define the *quasi local mass of* $\Sigma$ as

$$m(\Sigma) = \sum_i m(\Sigma_i). \qquad (4.3)$$

If the mean curvature of $\breve{\Sigma}_{i_0}$ is further nonnegative (it is true if $\Sigma_{i_0}$ has positive Gauss curvature), we can prove the positivity of the quasi-local mass (4.2) (Theorem 1 in [18]).

Now suppose some $\Sigma_{i_0}$ has negative Gauss curvature and let

$$K_{\Sigma_{i_0}} \geq -\kappa^2$$

($\kappa > 0$) where $-\kappa^2$ is the minimum of the Gauss curvature. (Here we must choose the minimum of the Gauss curvature instead of arbitrary lower bound, otherwise the quasi-local mass defined in the following way might depend on this arbitrary lower bound.) By [9, 3], $\Sigma_{i_0}$ can be smoothly isometrically embedded into the hyperbolic space $\mathbb{H}^3_{-\kappa^2}$ with constant curvature $-\kappa^2$ as a convex surface which bounds a convex domain in $\mathbb{H}^3_{-\kappa^2}$. Let $(t, x_1, x_2, x_3)$ be the spacetime coordinates of $\mathbb{R}^{3,1}$. Then $\mathbb{H}^3_{-\kappa^2}$ is one-fold of the spacelike hypersurfaces

$$\big\{(t, x_1, x_2, x_3)\big|t^2 - x_1^2 - x_2^2 - x_3^2 = \frac{1}{\kappa^2}\big\}.$$

The induced metric of $\mathbb{H}^3_{-\kappa^2}$ is

$$\breve{g}_{\mathbb{H}^3_{-\kappa^2}} = \frac{1}{1+\kappa^2 r^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\psi^2)$$

It has the second fundamental form $\breve{p}^+_{\mathbb{H}^3_{-\kappa^2}} = \kappa\breve{g}_{\mathbb{H}^3_{-\kappa^2}}$ for the upper-fold $\{t > 0\}$ and $\breve{p}^-_{\mathbb{H}^3_{-\kappa^2}} = -\kappa\breve{g}_{\mathbb{H}^3_{-\kappa^2}}$ for the lower-fold $\{t < 0\}$ with respect to the future-time-directed normal. Denote also $\breve{\Sigma}_{i_0}$ its image.

Let $\phi$ be the unique solution of (2.4) or (2.5) for some $\breve{\phi} \in \breve{\Xi}$. Denote

$$\begin{aligned}\hat{m}_\pm(\Sigma_{i_0}, \breve{\phi}) &= \frac{1}{8\pi}\Re\int_{\Sigma_{i_0}}\Big[(H_0 - H)|\phi|^2 \\ &\quad -\big(tr(p_0|_{\Sigma_{i_0}}) - tr(p|_{\Sigma_{i_0}})\big)\langle\phi, e_0\cdot e_r\cdot\phi\rangle \\ &\quad +(p_{0ar} - p_{ar})\langle\phi, e_0\cdot e_a\cdot\phi\rangle\Big] \qquad (4.4)\end{aligned}$$

where

$$p_0 = \begin{cases} \text{pullback of } \breve{p}^+_{\mathbb{H}^3_{-\kappa^2}} : & \text{if } \Sigma_{i_0} \text{ is isometrically embedded into} \\ & \text{the upper-fold } \{t>0\}, \\ \text{pullback of } \breve{p}^-_{\mathbb{H}^3_{-\kappa^2}} : & \text{if } \Sigma_{i_0} \text{ is isometrically embedded into} \\ & \text{the lower-fold } \{t<0\}. \end{cases}$$

It is easy to see that $tr(p_0|_{\Sigma_{i_0}}) = \pm 2$, thus

$$\begin{aligned}\hat{m}_\pm(\Sigma_{i_0}, \breve{\phi}) &= \frac{1}{8\pi}\Re\int_{\Sigma_{i_0}}\Big[(H_0 - H)|\phi|^2 \\ &\quad +tr(p|_{\Sigma_{i_0}})\langle\phi, e_0\cdot e_r\cdot\phi\rangle \\ &\quad -p_{ar}\langle\phi, e_0\cdot e_a\cdot\phi\rangle\Big] \\ &\quad \mp\frac{\kappa}{4\pi}\int_{\Sigma_{i_0}}\langle\phi, e_0\cdot e_r\cdot\phi\rangle.\end{aligned}$$

Now we define the quasi local mass of $\Sigma_{i_0}$ under conditions $(i)$, $(ii)$ which are introduced in the previous section.

If condition $(i)$ holds, we embed $\Sigma_{i_0}$ into upper-fold $\{t > 0\}$. Since $\breve{\Sigma}_{i_0}$ is convex, we have

$$\breve{H} + tr(\breve{p}|_{\breve{\Sigma}_{i_0}}) > 0.$$

If condition $(ii)$ holds, we embed $\Sigma_{i_0}$ into lower-fold $\{t < 0\}$. We have

$$\breve{H} - tr(\breve{p}|_{\breve{\Sigma}_{i_0}}) > 0$$

in this case.

The *quasi local mass* of $\Sigma_{i_0}$ is defined as

$$\hat{m}(\Sigma_{i_0}) = \begin{cases} \min_{\breve{\Xi}} \hat{m}_+(\Sigma_{i_0}, \breve{\phi}): & \text{if condition } (i) \text{ holds,} \\ \min_{\breve{\Xi}} \hat{m}_-(\Sigma_{i_0}, \breve{\phi}): & \text{if condition } (ii) \text{ holds.} \end{cases} \tag{4.5}$$

Note that it might have two different values via embedding to the upper-fold and to the lower-fold respectively when $tr(p) = 0$. However, since $\widetilde{D}\phi = 0$, $\widetilde{D}(e_0 \cdot \phi) = -tr_g(p)\phi = 0$, we have

$$\int_\Sigma \langle e_r \cdot \phi, e_0 \cdot \phi \rangle = \int_M \langle \widetilde{D}\phi, e_0 \cdot \phi \rangle - \langle \phi, \widetilde{D}(e_0 \cdot \phi) \rangle = 0.$$

This implies $\hat{m}_+(\Sigma_{i_0}, \breve{\phi}) = \hat{m}_-(\Sigma_{i_0}, \breve{\phi})$. Hence $\hat{m}(\Sigma_{i_0})$ is unique in this case. Furthurmore, (4.5) approaches (4.2) when $\kappa \to 0$.

If $\Sigma_1, \cdots, \Sigma_{l_0}$ can be isometrically embedded into $\mathbb{R}^3$ in the Minkowski spacetime $\mathbb{R}^{3,1}$, and $\Sigma_{l_0+1}, \cdots, \Sigma_l$ can be isometrically embedded into $\mathbb{H}^3_{-\kappa^2_{l_0+1}}, \cdots, \mathbb{H}^3_{-\kappa^2_l}$ in the Minkowski spacetime $\mathbb{R}^{3,1}$ respectively, we define the *quasi local mass of* $\Sigma$ as

$$\hat{m}(\Sigma) = \sum_{1 \le i \le l_0} m(\Sigma_i) + \sum_{l_0+1 \le i \le l} \hat{m}(\Sigma_i). \tag{4.6}$$

**Theorem 1.** *Let $(N, \widetilde{g})$ be a spacetime which satisfies the dominant energy condition. Let $(M, g, p)$ be a smooth initial data set with the boundary $\Sigma$ which has finitely many multi-components $\Sigma_i$, each of which is topological 2-sphere. Suppose that some $\Sigma_{i_0}$ has negative Gauss curvature and let $K_{\Sigma_{i_0}} \ge -\kappa^2$ $(\kappa > 0)$ where $-\kappa^2$ is the minimum of the Gauss curvature. If either condition $(i)$ or condition $(ii)$ holds, then*

(1) $\hat{m}(\Sigma_{i_0}) \ge 0$*;*
(2) *that $\hat{m}(\Sigma_{i_0}) = 0$ implies the energy-momentum of spacetime satisfies*

$$T_{00} = |f||\phi|^2, \quad T_{0i} = f\langle \phi, e_0 \cdot e_i \cdot \phi \rangle$$

*along $M$, where $f$ is a real function, $\phi$ is the unique solution of (2.4) or (2.5) for some $\breve{\phi} \in \breve{\Xi}$.*

(3) *Furthermore, if $p_{ij} = 0$, then $\hat{m}(\Sigma_{i_0}) = 0$ implies that $M$ is flat with connected boundary; if $p_{ij} = \pm\kappa g_{ij}$, then $\hat{m}(\Sigma_{i_0}) = 0$ implies that $M$ has constant curvature $-\kappa^2$.*

*Proof :* By Lemma 1, statements (1), (2) and the first part of statement (3) can be proved by the same argument as the proof of Theorem 1 in [18]. For the proof of the second part of the statement (3), the vanishing quasi local mass implies

$$\overline{\nabla}_i \phi \pm \frac{\kappa}{2} e_0 \cdot e_i \cdot \phi = 0.$$

Since $\overline{\nabla}_i(e_0 \cdot \phi) = e_0 \cdot \overline{\nabla}_i \phi$, we find the $M$ has constant Ricci curvature with the scalar curvature $-6\kappa^2$. Therefore $M$ has constant curvature $-\kappa^2$ because the dimension is 3. Q.E.D.

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## References

[1] J.D. Brown and J.W. York, *Quasilocal energy in general relativity*, Mathematical aspects of classical field theory (Seattle, WA, 1991), 129-142, Contemp. Math., 132, Amer. Math. Soc., Providence, RI, 1992.

[2] J.D. Brown and J.W. York, *Quasilocal energy and conserved charges derived from the gravitational action*, Phys. Rev. D(3) 47, 1407-1419 (1993).

[3] M.P. do Carmo and F.W. Warner, *Rigidity and convexity of hypersurfaces in spheres*, J. Diff. Geom. 4, 133-144 (1970).

[4] P. Chruściel, M. Herzlich, *The mass of asymptotically hyperbolic Riemannian manifolds*, Pacific J. Math., 212, 231-264 (2003).

[5] F. Guan and Y.Y. Li, *The Weyl problem with nonnegative Gauss curvature*, J. Diff. Geom., 39, 331-342 (1994).

[6] J.X. Hong and C. Zuily, *Isometric embedding of the 2-sphere with nonnegative curvature in* $\mathbb{R}^3$, Math. Z., 219, 323-334 (1995).

[7] C-C.M. Liu and S.T. Yau, *Positivity of quasilocal mass*, Phys. Rev. Lett. 90, 231102 (2003).

[8] C-C.M. Liu and S.T. Yau, *Positivity of quasilocal mass II*, J. Amer. Math. Soc. 19, 181 (2006).

[9] A.V. Pogorelov, *Some results on surface theory in the large*, Adv. Math. 1, 191-264 (1964).

[10] Y. Shi and L-F. Tam, *Positive mass theorem and the boundary behavior of compact manifolds with nonnegative scalar curvature*, J.Diff.Geom. 62, 79 (2002).

[11] Y. Shi and L-F. Tam, *Boundary behaviors and scalar curvature of compact manifolds*, math/0611253.

[12] R. Schoen, S.T. Yau, *On the proof of the positive mass conjecture in general relativity*, Commun. Math. Phys. 65, 45-76 (1979).

[13] R. Schoen, S.T. Yau, *Proof of the positive mass theorem. II*, Commun. Math. Phys. 79, 231-260 (1981).
[14] M.T. Wang and S.T. Yau, *A generalization of Liu-Yau's quasi-local mass*, math.DG/0602321.
[15] X. Wang, *The mass of asymptotically hyperbolic manifolds*, J. Diff. Geom., 57, 273-299 (2001).
[16] X.Zhang, *A definition of total energy-momenta and the positive mass theorem on asymptotically hyperbolic 3-manifolds I*, Commun. Math. Phys., 249, 529-548 (2004).
[17] X. Zhang, *The positive mass theorem near null infinity*, Proceedings of ICCM 2004, December 17-22, Hong Kong (eds. S.T. Yau, etc.), AMS/International Press, Boston, to appear, math/0604154.
[18] X. Zhang, *A new quasi-local mass and positivity*, Acta Mathematica Sinica, English Series, to appear.

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