On the generation of the coefficient field of a newform by a single Hecke eigenvalue

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Abstract

Let f be a non-CM newform of weight $k \geq 2$. Let L be a subfield of the coefficient field of f. We completely settle the question of the density of the set of primes p such that the p-th coefficient of f generates the field L. This density is determined by the inner twists of f. As a particular case, we obtain that in the absence of non-trivial inner twists, the density is 1 for L equal to the whole coefficient field. We also present some new data on reducibility of Hecke polynomials, which suggest questions for further investigation.

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1 Statement of the results

The principal result of this paper is the following theorem. Its corollaries below completely resolve the question of the density of the set of primes p such that the p-th coefficient of f generates a given field.

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Theorem 1. *Let f be a newform (i.e., a new normalized cuspidal Hecke eigenform) of weight $k \geq 2$, level N and Dirichlet character χ which does not have complex multiplication (CM, see [R80, p. 48]). Let $E_f = \mathbf{Q}(a_n(f) : (n, N) = 1)$ be the field of coefficients of f and $F_f = \mathbf{Q}\left(\frac{a_n(f)^2}{\chi(n)} : (n, N) = 1\right)$.*

The set

$$\left\{ p \text{ prime} : \mathbf{Q}\left(\frac{a_p(f)^2}{\chi(p)}\right) = F_f \right\}$$

has density 1.

A twist of f by a Dirichlet character ϵ is said to be *inner* if there exists a (necessarily unique) field automorphism $\sigma_\epsilon : E_f \to E_f$ such that

$$a_p(f \otimes \epsilon) = a_p(f)\epsilon(p) = \sigma_\epsilon(a_p(f)) \tag{1}$$

for almost all primes p. For a discussion of inner twists we refer the reader to [R80, §3] and [R85, §3]. Here we give several statements that will be needed for the sequel. The σ_ϵ belonging to the inner twists of f form an abelian subgroup Γ of the automorphism group of E_f. The field F_f is the subfield of E_f fixed by Γ. It is well-known that the coefficient field E_f is either a CM field or totally real. In the former case, the formula

$$\overline{a_p(f)} = \chi(p)^{-1} a_p(f), \tag{2}$$

which is easily derived from the behaviour of the Hecke operators under the Petersson scalar product, shows that f has a non-trivial inner twist by χ^{-1} with $\sigma_{\chi^{-1}}$ being complex conjugation. If N is square free, $k = 2$ and the Dirichlet character χ of f is the trivial character, then there are no nontrivial inner twists of f.

Lemma 1. *The field F_f is totally real and $\mathbf{Q}(a_p(f))$ contains $\frac{a_p(f)^2}{\chi(p)}$.*

Proof. Equation 2 gives $\frac{a_p(f)^2}{\chi(p)} = a_p(f)\overline{a_p(f)}$, whence F_f is totally real. Since every subfield of a CM field is preserved by complex conjugation, $\mathbf{Q}(a_p(f))$ contains $\overline{a_p(f)}$, thus it also contains $\frac{a_p(f)^2}{\chi(p)}$. □

We immediately obtain the following two results.

Corollary 1. *Let f and E_f be as in Theorem 1. If f does not have any nontrivial inner twists (e.g. if $k = 2$, N is square free and χ is trivial), then the set*

$$\{p \text{ prime} : \mathbf{Q}(a_p(f)) = E_f\}$$

has density 1.

Corollary 2. *Let f and F_f be as in Theorem 1. The set*

$$\{p \text{ prime} : F_f \subseteq \mathbf{Q}(a_p(f))\}$$

has density 1.

To any subgroup H of Γ, we associate a number field K_H as follows. Consider the inner twists as characters of the absolute Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and let $\epsilon_1, \ldots, \epsilon_r$ be the inner twists such that $H = \{\sigma_{\epsilon_1}, \ldots, \sigma_{\epsilon_r}\}$. Let K_H be the minimal number field on which all ϵ_i for $1 \leq i \leq r$ are trivial, i.e. the field such that its absolute Galois group is the kernel of the map

$$\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \xrightarrow{\epsilon_1,\ldots,\epsilon_r} \mathbf{C}^\times \times \cdots \times \mathbf{C}^\times.$$

We use this field to express the density of the set of primes p such that $a_p(f)$ is *contained* in a given subfield of the coefficient field.

Corollary 3. *Let f, E_f and F_f be as in Theorem 1. Let L be any subfield of E_f. Let M_L be the set*

$$\{p \text{ prime} : a_p(f) \in L\}\,.$$

(a) If L does not contain F_f, then M_L has density 0.

(b) If L contains F_f, then $L = E_f^H$ for some subgroup $H \subseteq \Gamma$ and M_L has density $1/[K_H : \mathbf{Q}]$.

Proof. Suppose first that L does not contain F_f. Then $a_p(f) \in L$ implies that F_f is not a subfield of $\mathbf{Q}(a_p(f))$. Thus by Corollary 2, M_L is a subset of a set of density 0 and is consequently itself of density 0. We now assume that $L = E_f^H$. Then we have

$$\begin{aligned} M_L &= \{p \text{ prime} : \sigma(a_p(f)) = a_p(f)\, \forall \sigma \in H\} \\ &= \{p \text{ prime} : a_p(f)\epsilon_i(p) = a_p(f)\, \forall i \in \{1,\ldots,r\}\}\,. \end{aligned}$$

Since the set of p with $a_p(f) = 0$ has density 0 (see for instance [S81], p. 174), the density of M_L is equal to the density of

$$\{p \text{ prime} : \epsilon_i(p) = 1\, \forall i \in \{1,\ldots,r\}\} = \{p \text{ prime} : p \text{ splits completely in } K_H\}\,,$$

yielding the claimed formula. ∎

A complete answer as to the density of the set of p such that $a_p(f)$ *generates* a given field $L \subseteq E_f$ is given by the following immediate result.

Corollary 4. *Let f, E_f and F_f be as in Theorem 1. Let L be E_f^H with H some subgroup of Γ. The density of the set*

$$\{p \text{ prime} : \mathbf{Q}(a_p(f)) = L\}\,.$$

is equal to the density of the set

$$\{p \text{ prime} : \epsilon_i(p) = 1\, \forall i \in \{1, \ldots, r\} \text{ and } \epsilon_j(p) \neq 1\, \forall j \in \{r+1, \ldots, s\}\}\,,$$

where the ϵ_j for $j \in \{r+1, \ldots, s\}$ are the inner twists of f that belong to elements of $\Gamma - H$.

This corollary means that the above density is completely determined by the inner twists of f. We illustrate this by giving two examples. In weight 2 there is a newform on $\Gamma_0(63)$ with coefficient field $\mathbf{Q}(\sqrt{3})$. It has an inner twist by the Legendre symbol $p \mapsto \left(\frac{p}{3}\right)$. Consequently, the field F_f is $\mathbf{Q}$ and the set of p such that $a_p(f) \in \mathbf{Q}$ has density $\frac{1}{2}$.

For the next example we consider the newform of weight 2 on $\Gamma_0(512)$ whose coefficient field has degree 4 over $\mathbf{Q}$. More precisely, the coefficient field E_f is $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ and $F_f = \mathbf{Q}$. Hence, $\Gamma = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z} = \{1, \sigma_1, \sigma_2, \sigma_3\}$. There are thus nontrivial inner twists ϵ_1, ϵ_2 and ϵ_3, all of which are quadratic, as their values must be contained in the totally real field E_f. As $\sigma_1\sigma_2 = \sigma_3$, it follows that $\epsilon_1(p)\epsilon_2(p) = \epsilon_3(p)$. This equation already excludes the possibility that all $\epsilon_i(p) \neq 1$, whence there is not a single p such that $a_p(f)$ generates E_f. Furthermore, the set of p such that a_p generates the quadratic field $E_f^{\langle\sigma_1\rangle}$ is equal to the density of $\{p \text{ prime} : \epsilon_1(p) = 1 \text{ and } \epsilon_2(p) \neq 1\}$, which is $\frac{1}{4}$. Similar arguments apply to the other two quadratic fields. The set of p such that $a_p \in \mathbf{Q}$ also has density $\frac{1}{4}$.

In the literature there are related but weaker results concerning Corollary 1, which are situated in the context of Maeda's conjecture, i.e., they concern the case of level 1 and assume that the space $S_k(1)$ of cusp forms of weight k and level 1 consists of a single Galois orbit of newforms (see, e.g., [JO98] and [BM03]). We now show how Corollary 1 extends the principal results of these two papers.

Let f be a newform of level N, weight $k \geq 2$ and trivial Dirichlet character $\chi = 1$ which neither has CM nor nontrivial inner twists. This is for instance true when $N = 1$. Let $\mathbb{T}$ be the $\mathbf{Q}$-algebra generated by all T_n with $n \geq 1$ inside $\mathrm{End}(S_k(N, 1))$ and let $\mathfrak{P}$ be the kernel of the $\mathbf{Q}$-algebra homomorphism

$\mathbb{T} \xrightarrow{T_n \mapsto a_n(f)} E_f$. As $\mathbb{T}$ is reduced, the map $\mathbb{T}_{\mathfrak{P}} \xrightarrow{T_n \mapsto a_n(f)} E_f$ is a ring isomorphism with $\mathbb{T}_{\mathfrak{P}}$ the localization of $\mathbb{T}$ at $\mathfrak{P}$. Non canonically $\mathbb{T}_{\mathfrak{P}}$ is also isomorphic as a $\mathbb{T}_{\mathfrak{P}}$-module (equivalently as an E_f-vector space) to its $\mathbf{Q}$-linear dual, which can be identified with the localization at $\mathfrak{P}$ of the $\mathbf{Q}$-vector space $S_k(N, 1; \mathbf{Q})$ of cusp forms in $S_k(N, 1)$ with q-expansion in $\mathbf{Q}[[q]]$. Hence, $\mathbf{Q}(a_p(f)) = E_f$ precisely means that the characteristic polynomial $P_p \in \mathbf{Q}[X]$ of T_p acting on the localization at $\mathfrak{P}$ of $S_k(N, 1; \mathbf{Q})$ is irreducible. Corollary 1 hence shows that the set of primes p such that P_p is irreducible has density 1.

This extends Theorem 1 of [JO98] and Theorem 1.1 of [BM03]. Both theorems restrict to the case $N = 1$ and assume that there is a unique Galois orbit of newforms, i.e., a unique $\mathfrak{P}$, so that no localization is needed. Theorem 1 of [JO98] says that

$$\#\{p < X \text{ prime } : P_p \text{ is irreducible in } \mathbf{Q}[X]\} \gg \frac{X}{\log X}$$

and Theorem 1.1 of [BM03] states that there is $\delta > 0$ such that

$$\#\{p < X \text{ prime } : P_p \text{ is reducible in } \mathbf{Q}[X]\} \ll \frac{X}{(\log X)^{1+\delta}}.$$

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2 Group theoretic input

Lemma 2. *Let q be a prime power and ϵ a generator of the cyclic group $\mathbb{F}_q^\times$.*

(a) The conjugacy classes c in $\mathrm{GL}_2(\mathbb{F}_q)$ have the following four kinds of representatives:

$$S_a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad T_a = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \quad U_{a,b} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad V_{x,y} = \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}$$

where $a \neq b$, and $y \neq 0$.

(b) The number of elements in each of these conjugacy classes are: $1, q^2-1, q^2+q$, *and* q^2-q, *respectively.*

Proof. See Fulton-Harris [FH91], page 68. ∎

We use the notation $[g]_G$ for the conjugacy class of g in G.

Proposition 1. *Let q be a prime power and r a positive integer. Let further $R \subseteq \widetilde{R} \subseteq \mathbb{F}_{q^r}^\times$ be subgroups. Put $\sqrt{\widetilde{R}} = \{s \in \mathbb{F}_{q^r}^\times \;:\; s^2 \in \widetilde{R}\}$. Set*

$$H = \{g \in \mathrm{GL}_2(\mathbb{F}_q) \;:\; \det(g) \in R\}$$

and let

$$G \subseteq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^r}) \;:\; \det(g) \in \widetilde{R}\}$$

be any subgroup such that H is a normal subgroup of G. Then the following statements hold.

(a) The group $G/(G \cap \mathbb{F}_{q^r}^\times)$ (with $\mathbb{F}_{q^r}^\times$ identified with scalar matrices) is either equal to $\mathrm{PSL}_2(\mathbb{F}_q)$ or to $\mathrm{PGL}_2(\mathbb{F}_q)$. More precisely, if we let $\{s_1, \dots, s_n\}$ be a system of representatives for $\sqrt{\widetilde{R}}/R$, then for all $g \in G$ there is i such that $g \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i^{-1} \end{pmatrix} \in G \cap \mathrm{GL}_2(\mathbb{F}_q)$ and $\begin{pmatrix} s_i & 0 \\ 0 & s_i \end{pmatrix} \in G$.

(b) Let $g \in G$ such that $g \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i^{-1} \end{pmatrix} \in G \cap \mathrm{GL}_2(\mathbb{F}_q)$ and $\begin{pmatrix} s_i & 0 \\ 0 & s_i \end{pmatrix} \in G$. Then

$$[g]_G = [g \begin{pmatrix} s_i^{-1} & 0 \\ 0 & s_i^{-1} \end{pmatrix}]_{G \cap \mathrm{GL}_2(\mathbb{F}_q)} \begin{pmatrix} s_i & 0 \\ 0 & s_i \end{pmatrix}.$$

(c) Let $P(X) = X^2 - aX + b \in \mathbb{F}_{q^r}[X]$ be a polynomial. Then the inequality

$$\sum_C |C| \;\le\; 2|\widetilde{R}/R|(q^2+q)$$

holds, where the sum runs over the conjugacy classes C of G with characteristic polynomial equal to $P(X)$.

Proof. (a) The classification of the finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_q)$ yields that the group $G/(G \cap \mathbb{F}_{q^r}^\times)$ is either $\mathrm{PGL}_2(\mathbb{F}_{q^u})$ or $\mathrm{PSL}_2(\mathbb{F}_{q^u})$ for some $u \mid r$. This, however, can only occur with $u = 1$, as $\mathrm{PSL}_2(\mathbb{F}_{q^u})$ is simple. The rest is only a reformulation.

(b) This follows from (a), since scalar matrices are central.

(c) From (b) we get the inclusion

$$\bigsqcup_C C \subseteq \bigsqcup_{i=1}^{n} \bigsqcup_D D \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right),$$

where C runs over the conjugacy classes of G with characteristic polynomial equal to $P(X)$ and D runs over the conjugacy classes of $G \cap \mathrm{GL}_2(\mathbb{F}_q)$ with characteristic polynomial equal to $X^2 - as_i^{-1}X + bs_i^{-2}$ (such a conjugacy class is empty if the polynomial is not in $\mathbb{F}_q[X]$). The group $G \cap \mathrm{GL}_2(\mathbb{F}_q)$ is normal in $\mathrm{GL}_2(\mathbb{F}_q)$, as it contains $\mathrm{SL}_2(\mathbb{F}_q)$. Hence, any conjugacy class of $\mathrm{GL}_2(\mathbb{F}_q)$ either has an empty intersection with $G \cap \mathrm{GL}_2(\mathbb{F}_q)$ or is a disjoint union of conjugacy classes of $G \cap \mathrm{GL}_2(\mathbb{F}_q)$. Consequently, by Lemma 2, the disjoint union $\bigsqcup_D D \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right)$ is equal to one of

(i) $[U_{a,b}]_{\mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right)$,

(ii) $[V_{x,y}]_{\mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right)$ or

(iii) $[S_a]_{\mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right) \sqcup [T_a]_{\mathrm{GL}_2(\mathbb{F}_q)} \left(\begin{smallmatrix} s_i & 0 \\ 0 & s_i \end{smallmatrix}\right)$.

Still by Lemma 2, the first set contains q^2+q, the second set q^2-q and the third one q^2 elements. Hence, the set $\bigsqcup_C C$ contains at most $2|\widetilde{R}/R|(q^2+q)$ elements. ☐

3 Proof

The proof of Theorem 1 relies on the following important theorem by Ribet, which, roughly speaking, says that the image of the mod ℓ Galois representation attached to a fixed newform is as big as it can be for almost all primes ℓ.

Theorem 2 (Ribet)**.** *Let f be a Hecke eigenform of weight $k \geq 2$, level N and Dirichlet character $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$. Suppose that f does not have CM. Let E_f and F_f be as in Theorem 1 and denote by $\mathcal{O}_{E_f}$ and $\mathcal{O}_{F_f}$ the corresponding rings of integers. For almost all prime numbers ℓ the following statement holds:*

> *Let $\widetilde{\mathcal{L}}$ be a prime ideal of $\mathcal{O}_{E_f}$ dividing ℓ. Put $\mathcal{L} = \widetilde{\mathcal{L}} \cap \mathcal{O}_{F_f}$ and $\mathcal{O}_{F_f}/\mathcal{L} \cong \mathbb{F}$. Consider the residual Galois representation*

$$\overline{\rho}_{f,\widetilde{\mathcal{L}}} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathcal{O}_{E_f}/\widetilde{\mathcal{L}})$$

attached to f. Then the image $\overline{\rho}_{f,\widetilde{\mathcal{L}}}(\mathrm{Gal}(\overline{\mathbf{Q}}/K_\Gamma))$ is equal to

$$\{g \in \mathrm{GL}_2(\mathbb{F}) \,:\, \det(g) \in \mathbb{F}_\ell^{\times(k-1)}\},$$

where K_Γ is the field defined in Section 1.

Proof. It suffices to take Ribet [R85, Thm. 3.1] mod $\widetilde{\mathcal{L}}$. ☐

Theorem 3. *Let f be a non-CM newform of weight $k \geq 2$, level N and Dirichlet character χ. Let F_f be as in Theorem 1 and let $L \subset F_f$ be any proper subfield. Then the set*

$$\left\{ p \text{ prime} : \frac{a_p(f)^2}{\chi(p)} \in L \right\}$$

has density zero.

Proof. Let $L \subsetneq F_f$ be a proper subfield and $\mathcal{O}_L$ its integer ring. We define the set

$$S := \{\mathcal{L} \subset \mathcal{O}_{F_f} \text{ prime ideal} : [\mathcal{O}_{F_f}/\mathcal{L} : \mathcal{O}_L/(L \cap \mathcal{L})] \geq 2\}.$$

Notice that this set is infinite. For, if it were finite, then all but finitely many primes would split completely in the extension F_f/L, which is not the case by Chebotarev's density theorem.

Let $\mathcal{L} \in S$ be any prime, ℓ its residue characteristic and $\widetilde{\mathcal{L}}$ a prime of $\mathcal{O}_{E_f}$ lying over $\mathcal{L}$. Put $\mathbb{F}_q = \mathcal{O}_L/(L \cap \mathcal{L})$, $\mathbb{F}_{q^r} = \mathcal{O}_{F_f}/\mathcal{L}$ and $\mathbb{F}_{q^{rs}} = \mathcal{O}_{E_f}/\widetilde{\mathcal{L}}$. We have $r \geq 2$. Let W be the subgroup of $\mathbb{F}_{q^{rs}}^\times$ consisting of the values of χ modulo $\widetilde{\mathcal{L}}$; its size $|W|$ is less than or equal to $|(\mathbf{Z}/N\mathbf{Z})^\times|$. Let $R = \mathbb{F}_\ell^{\times(k-1)}$ be the subgroup of $(k-1)$st powers of elements in the multiplicative group $\mathbb{F}_\ell^\times$ and let $\widetilde{R} = \langle R, W \rangle \subset \mathbb{F}_{q^{rs}}^\times$. The size of $\widetilde{R}$ is less than or equal to $|R| \cdot |W|$. Let $H = \{g \in \mathrm{GL}_2(\mathbb{F}_{q^r}) \,:\, \det(g) \in R\}$ and $G = \mathrm{Gal}(\overline{\mathbf{Q}}^{\ker \overline{\rho}_{f,\tilde{\mathcal{L}}}}/\mathbf{Q})$. By Galois theory, G can be identified with the image of the residual representation $\overline{\rho}_{f,\widetilde{\mathcal{L}}}$, and we shall make this identification from now on. By Theorem 2 we have the inclusion of groups

$$H \subseteq G \subseteq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^{rs}}) : \det(g) \in \widetilde{R}\}$$

with H being normal in G.

If C is a conjugacy class of G, by Chebotarev's density theorem the density of

$$\{p \,\text{prime} :\ [\overline{\rho}_{f,\widetilde{\mathcal{L}}}(\mathrm{Frob}_p)]_G = C\}$$

equals $|C|/|G|$. We consider the set

$$M_{\mathcal{L}} := \bigsqcup_C \{p \text{ prime} : [\overline{\rho}_{f,\widetilde{\mathcal{L}}}(\mathrm{Frob}_p)]_G = C\} \supseteq \left\{p \text{ prime} : \overline{\left(\frac{a_p(f)^2}{\chi(p)}\right)} \in \mathbb{F}_q\right\},$$

where the reduction modulo $\mathcal{L}$ of an element $x \in \mathcal{O}_{F_f}$ is denoted by $\overline{x}$ and C runs over the conjugacy classes of G with characteristic polynomials equal to some $X^2 - aX + b \in \mathbb{F}_{q^{rs}}[X]$ such that

$$a^2 \in \{t \in \mathbb{F}_{q^{rs}} : \exists u \in \mathbb{F}_q \, \exists w \in W : t = uw\}$$

and automatically $b \in \widetilde{R}$. The set $M_{\mathcal{L}}$ has the density $\delta(M_{\mathcal{L}}) = \sum_C \frac{|C|}{|G|}$ with C as before. There are at most $2q|W|^2 \cdot |R|$ such polynomials. We are now precisely in the situation to apply Prop. 1, Part (c), which yields the inequality

$$\delta(M_{\mathcal{L}}) \leq \frac{4|W|^3 q(q^{2r} + q^r)}{(q^{3r} - q^r)} = O\left(\frac{1}{q^{r-1}}\right) \leq O\left(\frac{1}{q}\right),$$

where for the denominator we used $|G| \geq |H| = |R| \cdot |\mathrm{SL}_2(\mathbb{F}_{q^r})|$.

Since q is unbounded for $\mathcal{L} \in S$, the intersection $M := \bigcap_{\mathcal{L} \in S} M_{\mathcal{L}}$ is a set having a density and this density is 0. The inclusion

$$\left\{p \text{ prime} : \frac{a_p(f)^2}{\chi(p)} \in L\right\} \subseteq M$$

finishes the proof. □

Proof of Theorem 1. It suffices to apply Theorem 3 to each of the finitely many subextension of F_f. □

4 Reducibility of Hecke polynomials: questions

Motivated by a conjecture of Maeda, there has been some speculation that for every integer k and prime number p, the characteristic polynomial of T_p acting on $S_k(1)$ is irreducible. See, for example, [FJ02], which verifies this for all $k < 2000$ and $p < 2000$. The most general such speculation might be the following question: *if f is a non-CM newform of level $N \geq 1$ and weight $k \geq 2$ such that some $a_p(f)$ generates the field $E_f = \mathbf{Q}(a_n(f) : n \geq 1)$, do all but finitely many prime-indexed Fourier coefficients $a_p(f)$ generate E_f?* The answer in general is no. An example is given by the newform in level 63 and weight 2 that has an inner twist by $\left(\frac{\cdot}{3}\right)$. Also for non-CM newforms of weight 2 without nontrivial inner twists such that $[E_f : \mathbf{Q}] = 2$, we think that the answer is likely no.

Let $f \in S_k(\Gamma_0(N))$ be a newform of weight k and level N. The *degree* of f is the degree of the field E_f, and we say that f is a *reducible newform* if $a_p(f)$ does not generate E_f for infinitely many primes p.

For each even weight $k \leq 12$ and degree $d = 2, 3, 4$, we used [SAGE] to find newforms f of weight k and degree d. For each of these forms, we computed the *reducible primes* $p < 1000$, i.e., the primes such $a_p(f)$ does not generate E_f. The result of this computation is given in Table 1. Table 2 contains the number of reducible primes $p < 10000$ for the first 20 newforms of degree 2 and weight 2. This data inspires the following question.

Question 1. *If $f \in S_2(\Gamma_0(N))$ is a newform of degree 2, is f necessarily reducible? That is, are there infinitely many primes p such that $a_p(f) \in \mathbf{Z}$?*

Tables 4–6 contain additional data about the first few newforms of given degree and weight, which may suggest other similar questions. In particular, Table 4 contains data for all primes up to 10^6 for the first degree 2 form f with $L(f, 1) \neq 0$, and for the first degree 2 form g with $L(g, 1) = 0$. We find that there are 386 primes $< 10^6$ with $a_p(f) \in \mathbf{Z}$ and 309 with $a_p(g) \in \mathbf{Z}$.

Question 2. *If $f \in S_2(\Gamma_0(N))$ is a newform of degree 2, can the asymptotic behaviour of the function*

$$N(x) := \#\{p \, \text{prime} : \, p < x, a_p(f) \in \mathbf{Z}\}$$

be described as a function of x?

The authors intend to investigate these questions in a subsequent paper.

Table 1: Counting Reducible Characteristic Polynomials

k	d	N	reducible $p < 1000$
2	2	23	13, 19, 23, 29, 43, 109, 223, 229, 271, 463, 673, 677, 883, 991
2	3	41	17, 41
2	4	47	47
4	2	11	11
4	3	17	17
4	4	23	23
6	2	7	7
6	3	11	11
6	4	17	17
8	2	5	5
8	3	17	17
8	4	11	11
10	2	5	5
10	3	7	7
10	4	13	13
12	2	5	5
12	3	7	7
12	4	21	3, 7

Table 2: First 20 Newforms of Degree 2 and Weight 2

k	d	N	$\#\{$reducible $p < 10000\}$
2	2	23	47
2	2	29	42
2	2	31	78
2	2	35	48
2	2	39	71
2	2	43	43
2	2	51	64
2	2	55	95
2	2	62	77
2	2	63	622 (inner twist by $\left(\frac{\cdot}{3}\right)$)

k	d	N	$\#\{$reducible $p < 10000\}$
2	2	65	43
2	2	65	90
2	2	67	51
2	2	67	19
2	2	68	53
2	2	69	47
2	2	73	43
2	2	73	55
2	2	74	52
2	2	74	21

Table 3: Newforms 23a and 67b: values of $\psi(x) = \#\{\text{reducible } p < x \cdot 10^5\}$

k	d	N	r_{an}	1	2	3	4	5	6	7	8	9	10
2	2	23	0	127	180	210	243	277	308	331	345	360	386
2	2	67	1	111	159	195	218	240	257	276	288	301	309

Table 4: First 5 Newforms of Degrees 3, 4 and Weight 2

k	d	N	reducible $p < 10000$
2	3	41	17, 41
2	3	53	13, 53
2	3	61	61, 2087
2	3	71	23, 31, 71, 479, 647, 1013, 3181
2	3	71	13, 71, 509, 3613

k	d	N	reducible $p < 10000$
2	4	47	47
2	4	95	5, 19
2	4	97	97
2	4	109	109, 4513
2	4	111	3, 37

Table 5: First 5 Newforms of Degrees 2, 3 and Weight 4

k	d	N	reducible $p < 1000$
4	2	11	11
4	2	13	13
4	2	21	3, 7
4	2	27	3, 7, 13, 19, 31, 37, 43, 61, 67, 73, 79, 97, 103, 109, 127, 139, 151, 157, 163, 181, 193, 199, 211, 223, 229,241, 271, 277, 283, 307, 313, 331, 337, 349, 367, 373, 379, 397, 409, 421, 433, 439, 457, 463, 487, 499, 523, 541, 547, 571, 577, 601, 607, 613, 619, 631, 643, 661, 673, 691, 709, 727, 733, 739, 751, 757, 769, 787, 811, 823, 829, 853, 859, 877, 883, 907, 919, 937, 967, 991, 997 (has inner twists)
4	2	29	29

k	d	N	reducible $p < 1000$
4	3	17	17
4	3	19	19
4	3	35	5, 7
4	3	39	3, 13
4	3	41	41

Table 6: Newforms on $\Gamma_0(389)$ of Weight 2

k	d	N	reducible $p < 10000$
2	1	389	none (degree 1 polynomials are all irreducible)
2	2	389	5, 11, 59, 97, 157, 173, 223, 389, 653, 739, 859, 947, 1033, 1283, 1549, 1667, 2207, 2417, 2909, 3121, 4337, 5431, 5647, 5689, 5879, 6151, 6323, 6373, 6607, 6763, 7583, 7589, 8363, 9013, 9371, 9767
2	3	389	7, 13, 389, 503, 1303, 1429, 1877, 5443
2	6	389	19, 389
2	20	389	389

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