Generation of polycyclic groups

Martin Kassabov* & Nikolay Nikolov

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To Dan Segal on occasion of his 60-th birthday

Abstract

We give a new and self-contained proof of a theorem of Linnell and Warhurst that $d(G) - d(\widehat{G}) \leq 1$ for virtually polycyclic groups G. We also give a simple sufficient condition for equality $d(G) = d(\widehat{G})$ when G is virtually abelian.

Introduction

Let G be a finitely generated residually finite group. By $d(G)$ we denote the minimal size of a generating set for G, and by $d(\widehat{G})$ the minimal size of a generating set for the profinite completion $\widehat{G}$ of G. In other words

$$d(\widehat{G}) = \max\left\{d(G/N) \mid N \lhd G,\ G/N < \infty\right\}.$$

Polycyclic groups are one of the best understood class of groups. For example most of the decision problems are decidable in this class, see [7].

It seems surprising therefore that it is still an open problem whether there exists an algorithm which finds $d(G)$ for any polycyclic group G (given by say a set of generators and relations). This is unknown even in the case when G is virtually abelian.

It is obvious that $d(G) \geq d(\widehat{G})$ and when there is equality both the value of $d(G)$ and a minimal generating set for G can indeed be found algorithmically. (Say by enumerating both the finite images and all possibilities for generating sets for G).

In general $d(G) - d(\widehat{G})$ can be arbitrarily large even for metabelian groups G, see [4]. In fact Wise [9] has proved that there exist groups G with arbitrarily large $d(G)$ while $d(\widehat{G}) = 3$.

Fortunately for polycyclic groups the situation is not that bad. In [3] Linnell and Warhurst proved the following theorem using methods from commutative algebra and lattices over orders.

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Theorem 1. *Let G be a virtually polycyclic group. Then $d(G) \leq d(\widehat{G}) + 1$.*

Note that the inequality is sharp even for virtually abelian groups: many examples with $d(G) = d(\widehat{G}) + 1$ are constructed in [5].

In this note we give an alternative proof of Theorem 1. While not claiming anything new we believe that our argument is much simpler that the original one in [3]. Moreover our result gives some sufficient condition when $d(G) = d(\widehat{G})$ which can be verified quite easily in the case when G is virtually abelian.

Theorem 2. *Let G be a group with normal finitely generated abelian subgroup U such that G/U is finite. Let $d_p(G) = d(G/U^p)$ for any prime p. Then*

$$\alpha = d(\widehat{G}) \leq d(G) \leq k := \max\{\alpha, \beta + 1\} \leq d(\widehat{G}) + 1, \tag{1}$$

where $\alpha = \max_p d_p(G)$ and $\beta = \min_p d_p(G)$.

Moreover for any integer $N \in \mathbb{N}$ there exists a generating set S for G of size k, such that the first $d(\widehat{G})$ elements generate a subgroup of index co-prime to N. The same result holds for finitely generated virtually nilpotent groups.

In particular if there are two primes p and q such that $d_p(G) \neq d_q(G)$ then $d(G) = d(\widehat{G})$.

Note that Theorem 2 easily implies a weaker version of Theorem 1, namely that $d(G) \leq d(\widehat{G}) + 2$, however obtaining the right bound $d(\widehat{G}) + 1$ is harder. For that we need a general and somewhat technical result (Theorem 7 on lifting generators) proved in Section 1. The proofs of Theorems 2 and 1 are then immediate and are given in Section 2.

Notation

For elements $a, b \in G$ in a group G the commutator $[a, b]$ of a and b is $aba^{-1}b^{-1}$.

1 Lifting generators

In this section we shall prove a general result which under certain condition produces a generating set of a group G starting from a generating set of some quotient G/V of G. The reason for stating it in such generality is because in the next section we shall apply it in two settings: when G a virtually abelian group and then when G is virtually metabelian.

First recall the following result by Gaschütz ([1]).

Theorem 3. *Let G be a finite group with a normal subgroup N. Let $d \geq d(G)$ and let $a_1, \ldots a_d$ be any d elements which generate G mod N, i.e., $G = N\langle a_1, \ldots, a_d\rangle$. Then we can find elements $g_i \in a_i N$ $(i = 1, 2, \ldots, d)$ such that $G = \langle g_1, \ldots, g_d\rangle$.*

Definition 4. Let G be a group and p be a prime. A normal subgroup L of finite index in G is *p-good* if for any subgroup $H \leq G$ with $HL = G$ we have that $[G : H]$ is finite and coprime to p.

It is not difficult to see that p-good subgroups exists for any virtually polycyclic group G, see Lemma 11 below.

Now let G be a finitely generated group with an abelian normal subgroup V of finite rank. Suppose that for every prime number p we have chosen a p-good subgroup G_p of G such that $G_p \geq V^p$.

Definition 5. We say that $h_1, \ldots, h_k$ generate G mod p if $\langle h_1, \ldots, h_k \rangle G_p = G$. Let $d_p(G) = d(G/G_p)$ denote the minimal size of a set of generators for G mod p.

Definition 6. Let $w = w(x_1, \ldots, x_n)$ be a group word (element in the free group F). The Fox derivatives $\frac{\partial w}{\partial x_i}$ are elements in the group ring $\mathbb{Z}[F]$, which are defined by $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$ and

$$\frac{\partial uv}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i}.$$

Let G be a group and V be a G-module. For any n-tuple $\underline{g} = (g_1, \ldots, g_n) \in G^n$ the Fox derivative $\frac{\partial w}{\partial x_i}$ naturally defines a map $\frac{\partial w}{\partial x_i}(\underline{g}) : V \to V$.

An equivalent way to define this map is the following: Let Γ be an extension of G by the abelian group V then

$$\frac{\partial w}{\partial x_i}(\underline{g})(a) = w(\gamma_1, \ldots, \gamma_{i-1},\ a\gamma_i, \gamma_{i+1}, \ldots, \gamma_n) \cdot w(\gamma_1, \ldots, \gamma_n)^{-1}$$

for any lifts $\gamma_i \in \Gamma$ of $g_i \in G$.

Theorem 7. *Let $\underline{\gamma} = (\gamma_1, \ldots \gamma_k)$ be a set of elements in G which generate G/V. Suppose that $w(x_1, \ldots, x_k)$ is a word such that $w(\underline{\gamma}) \in V$.*

Assume that

1. *The image of the map $\pi : V \to V$ defined by $\pi(v) = \frac{\partial w}{\partial x_k}(\underline{\gamma}) \circ v$ has finite index M in V.*
2. *For any choice of elements $g_1 \in \gamma_1 V, \ldots, g_{k-1} \in \gamma_{k-1} V$ the group $\langle g_1, \ldots, g_{k-1} \rangle$ generated by them has finite index in G.*
3. *We have that $d_p(G) \leq k$ for any prime p.*

Then there exist lifts $g_1, \ldots, g_k$ of $\gamma_1, \ldots, \gamma_k$ (i.e. such that $g_i \in \gamma_i V$) which generate G.

Moreover there is an algorithm for finding g_i from the γ_i (provided all the objects from conditions 1,2,3 above are computable).

Proof. We say that the element g_i is a lift of γ_i whenever $g_i \in \gamma_i V$. Note that the Fox derivative $\frac{\partial w}{\partial x_i}(\underline{g}) = \frac{\partial w}{\partial x_i}(\underline{\gamma})$ does not depend on the choice of lifts $\underline{g} = (g_1, \dots, g_k)$ of $\underline{\gamma} = (\gamma_1, \dots, \gamma_k)$.

For each $i = 0, 1, \dots, k$ let $Q(i)$ be the following statement.

$\mathbf{Q(i)}$: *There exist lifts $S_i = \{g_1, \dots, g_i\}$ of $\gamma_1, \dots, \gamma_i$ and a finite set of prime numbers P_i with the following property.*

- *For each prime $p \in P_i$ there exist lifts $g_j^{(p)} \in \gamma_j V$, $j = i+1, \dots, k$ such that $S_i \cup \{g_j^{(p)}\}_{j>i}$ are k generators for G mod p.*
- *For each prime $p \notin P_i$, there exist lifts $g_j^{(p)}$, $j = i+1, \dots, k-1$ such that for any lift $g_k^{(p)} \in \gamma_k V$ we have that $S_i \cup \{g_j^{(p)}\}_{j>i}$ are k generators for G mod p.*

The proof of $Q(i)$ is by induction on i.

The base case $i = 0$ is proved as follows: Take $S_0 = \emptyset$ and choose any lifts $\gamma_1, \dots, \gamma_{k-1}$ of the γ_j. They generate a subgroup of finite index L in G, therefore for any $p \not| L$ and any lift $g_k \in \gamma_k V$ the elements $\{g_1, \dots, g_k\}$ generate G mod p.

Define P_0 to be the set consisting of all primes which divide L or M. We only have to show that for all $p \in P_0$ there exist lifts $g_1^{(p)}, \dots, g_k^{(p)}$, which generate G mod p. Consider the k images $\bar{\gamma}_j$ of $\gamma_1, \dots, \gamma_k$ in G/VG_p. They generate G/VG_p (since γ_j generate G/V) and also we know that G/G_p is k-generated. By Gaschutz Theorem we can find elements $g_j^{(p)} \in \gamma_j V G_p$ which generate G/G_p. These can be further adjusted by elements from G_p so that $g_j^{(p)}\gamma_j^{-1} \in V$. This proves the base case $i = 0$ of the induction.

Suppose that we have already found S_i and P_i. By the Chinese Remainder Theorem there exists a lift g_{i+1} such that $g_{i+1} = g_{i+1}^{(p)} (\text{mod } G_p)$ for all $p \in P_i$.

Choose any lifts $g_{i+2}, \dots g_{k-1}$. By one of the assumptions the group H generated by $\{g_i\}_{i=1}^{k-1}$ is of finite index N_i in G. Denote $P_{i+1} = \{p \mid \ p \text{ divides } N_i\} \cup P_i$.

We want to show that $S_{i+1} = S_i \cup \{g_{i+1}\}$ and the set P_{i+1} satisfy the induction hypothesis.

It is very easy to check that the second part of the induction hypothesis is satisfied for this definition of the set P_{i+1} (just choose $g_j^{(p)} = g_j$ for $j = i+2, \dots, k-1$). It remains to show that for all primes $p \in P_{i+1}$ the first condition is satisfied. This is clearly the case if $p \in P_i$. Let $p \notin P_i$ then by the induction assumption there exist lifts $\{g_j^{(p)}\}_{j=i}^{k-1}$ which together with S_i and any lift $g_k^{(p)} \in \gamma_k V$ generate G mod p. We will show that we can chose a lift g_k of $\gamma_k$such that the group L generated by S_{i+1}, $\{g_j^{(p)}\}_{j=i+1}^{k-1}$ and g_k contains an element $u \equiv g_{i+1}^{-1} g_{i+1}^{(p)} (\text{mod } V^p)$, which implies that these elements generate G mod p.

The key observation here is that for $x \in V$ the element $w\big(g_1^{(p)}, \dots, g_{k-1}^{(p)}, xg_k^{(p)}\big)$ is equal to

$$\left(\frac{\partial w}{\partial x_k}(\underline{\gamma}) \cdot x\right) w\big(g_1^{(p)}, \dots, g_{k-1}^{(p)}, g_k^{(p)}\big) = \pi'(x) \in V$$

which by one of the assumptions takes any value in V/V^p as x ranges over V/V^p (since p does not divide the index M of the image of $\frac{\partial w}{\partial x_k}(\underline{\gamma})$ in V). So we can indeed find $x \in V$ such that the element $\pi'(x) \in V$ satisfies $\pi'(x) \equiv g_{i+1}^{-1} g_{i+1}^{(p)}$ mod V^p.

This shows that there exist lifts g_k which generate G mod p which completes the induction step.

The statement $Q(k)$ gives a set S_k which generates G mod p for any prime p and therefore $\langle S_k \rangle = G$.

It is clear that this argument in fact produces an algorithm for finding the set S_k in a very efficient way, of course provided the various subgroup indices, words and maps $G \to G/G_p$ involved in the induction are computable. Theorem 7 is proved. □

Remark 8. A slight modification of the proof gives that for any finite set P of primes such that $d_p(G) < k$, we can find lifts $g_1, \dots, g_{k-1}$ which together with any lift of γ_k generate a subgroup of index not divisible by any prime in P.

Remark 9. If $\gamma_k = e$ then we can take the word $w = x_k$. Its Fox derivative is $\frac{\partial w}{\partial x_k} = e$ and defines the identity map from V to V, which is clearly surjective.

Remark 10. If we replace the assumption that V is abelian with V nilpotent, then all results remain valid, since a set generates a nilpotent group if and only if it generates the abelianization of the group.

2 Applications of Theorem 7

2.1 Proof of Theorem 2

Proof. Clearly $d(\widehat{G}) = \alpha$. Let $k := \max\{\beta + 1, \alpha\}$.

Take $V := U^q$ where q is a prime such that $\beta = d_q(G)$. Then V is q-good, i.e., any collection of elements which generates G/V generates a subgroup of finite index (coprime to q) in G.

Take elements $\gamma_1, \dots, \gamma_\beta$ which generate G/V. Set $\gamma_i = 1$ for $j = \beta+1, \dots, k$. It is easy to see that the group G, subgroups V, $G_p = V^p$ (for any prime p), the elements γ_i above, and the word $w = x_k$ satisfy the conditions of Theorem 7. We conclude that G can be generated by some lifts of $\gamma_1, \dots, \gamma_k$ and so $d(G) \le k$ as claimed.

For the second part of the theorem start with $V = U^N$ instead and with any generating set $\gamma_1, \dots, \gamma_s$ for G/V and again take $\gamma_j = 1,\ s < j \le k$. The rest of the argument is similar. □

2.2 Proof of Theorem 1

We begin with the following straightforward

Lemma 11. *If G is a virtually polycyclic group and p is a prime then G has a p-good subgroup L.*

Proof. We use induction on the Hirsch length $h(G)$ of G. When $h(G) = 0$ then G is finite and we can simply take $L = 1$. Suppose that the Lemma has been proved for all groups of Hirsch length less than $h > 0$. Consider a virtually polycyclic group G with $h(G) = h$. Since G is infinite it has an infinite normal abelian subgroup N. Let $\overline{G} = G/N^p$. Clearly $h(\overline{G}) < h$ and so $\overline{G}$ has a p-good subgroup, say $\overline{L} = L/N^p$ for some normal subgroup L containing N^p.

Then L is p-good for G: Suppose $H \leq G$ with $HL = G$. Then $\overline{HL} = \overline{G}$ where $\overline{H} = HN^p/N^p$ and so the index $[G : HN^p]$ is finite and coprime to p. Therefore $HN = HN^p$ (since N/N^p is a power of p while $[HN : HN^p]$ must be coprime to p) and so $H_1N^p = N$ where $H_1 = H \cap N$. The last equality implies that $[N : H_1]$ is finite and coprime to p and hence so is $[G : H] = [G : NH][N : H_1]$.

Notice that the p-good subgroup L we found contains N^p. □

Lemma 11 raises the following natural

Problem 1. Which groups G possess a subgroup N of finite index such that $NH = G$ for a subgroup $H < G$ implies that $[G : H] < \infty$?

The purpose of the following three Lemmas is to ensure the existence of suitable word w, subgroup V and elements $g_1, \dots g_k$ in a virtually metabelian and polycyclic group G which meet the conditions of Theorem 7.

Lemma 12. *Let G be a virtually metabelian and polycyclic group. Then there exist normal subgroups $G_0 \rhd V$ of G such that*

1. *G/G_0 is finite, V is a torsion free abelian group,*

2. *if H is a subgroup of G such that $HG_0 = G$ then H is of finite index in G,*

3. *G_0/V is a nilpotent group which acts commutatively on V, i.e., $G_0/C_{G_0}(V)$ is abelian, and*

4. *$\mathbb{Q} \otimes V$ is a perfect $\mathbb{Q}[G_0/V]$ module, i.e., $(G_0 - 1) \cdot V$ has finite index in V.*

Proof. Let $A < B$ be normal subgroups of G such that A and B/A are torsion free abelian and $[G : B]$ is finite. Let L be a p-good subgroup of G for some prime p and take $G_0 = B \cap L$. Then G_0 also a p-good subgroup of G and item 2 follows.

Let $W = A \cap G_0$. We have that W and G_0/W are torsion free abelian groups and W is a module for $\bar{G}_0 = G_0/W$. Consider the chain of submodules $W \geq (G_0 - 1)W \geq (G_0 - 1)^2W \geq \cdots$. This is a chain of subgroups of the finitely generated abelian group W , so let V be the first module in that series such that $(G_0 - 1)V$ has finite index in V. Clearly G_0/V is a nilpotent group (since G_0 acts nilpotently on W/V). Item 4 is clear since $[V : (G_0 - 1)V]$ is finite while item 3 follows since $C_{G_0}(V) \geq W$ and G_0/W is abelian. □

Lemma 13. *Let Γ_0 be a finitely generated torsion free abelian group and let V be a finitely generated torsion free $\mathbb{Z}[\Gamma_0]$ module such that $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$ is a perfect $\mathbb{Q}[\Gamma_0]$-module. Then there exists an integer N such that for any subgroup $\Gamma < \Gamma_0$ of index co-prime to N we have that $V_{\mathbb{Q}}$ is a perfect $\mathbb{Q}[\Gamma]$-module.*

Further when this happens then we can find an integer $M \in \mathbb{N}$ (depending on Γ), integers s_i and group elements $h_i \in \Gamma$ such that

$$\sum s_i(1 - h_i) = M \cdot id$$

as operators on V.

Proof. Let χ be a irreducible character (over $\mathbb{C}$) of Γ_0. We will call χ a character of finite order if all values of χ are roots of 1, in this case the order of χ is the least integer n such that all its values are n-th roots of 1.

Since V is a perfect $\mathbb{Q}[\Gamma_0]$ module it does not contain a trivial submodule. Let N be the gcd of all orders of irreducible characters which appears in $\mathbb{C}V$. If $\Gamma \leq \Gamma_0$ is a subgroup of index co-prime to N then the restriction of any irreducible characters in V to Γ is non trivial. Therefore $\mathbb{C}V$ is a perfect $\mathbb{C}[\Gamma]$ module, which implies that $V_{\mathbb{Q}} = \mathbb{Q} \otimes V$ is perfect $\mathbb{Q}[\Gamma]$ module.

For the second part, let I be the augmentation ideal of $\mathbb{Q}\Gamma$. Since $V_{\mathbb{Q}} = IV_{\mathbb{Q}}$ and $V_{\mathbb{Q}}$ is a finite dimensional vector space over $\mathbb{Q}$ we have by Nakayama's lemma that $id + T$ annihilates $V_{\mathbb{Q}}$ for some $T \in I$.

Expressing T in the basis of I and clearing the common denominator M of the rational coefficients gives the integers s_i and the elements $h_i \in G$. □

Lemma 14. *Let Γ be a group with an abelian normal subgroup V such that $[\Gamma', V] = 1$ and Γ/V is nilpotent of class d. Let $g_1, \ldots, g_{k-1} \in \Gamma$.*

For any integers $t \in \mathbb{N}, s_i$ and group elements $h_i \in \langle g_1, \ldots, g_{k-1} \rangle$ $(i = 1, \ldots, t)$ there exists a word w on $x_1, \ldots, x_k$ such that

- *$w(g_1, \ldots, g_{k-1}, g) \in V$ for any $g \in \Gamma$*
- *The action of Fox derivative $\frac{\partial w}{\partial x_k}(g_1, \ldots, g_{k-1}, g)$ on $x \in V$ is given by*

$$x \mapsto \left(\sum_{i=1}^{t} s_i(1 - h_i) \right)^d \cdot x.$$

Proof. Consider the word $w'(g_1, \ldots, g_{k-1}, g) = \prod [g, h_i]^{s_i}$ where h_i are expressed as words on $g_1, \ldots, g_{k-1}$. A direct computation gives that the Fox derivative $\frac{\partial w}{\partial x_k}$ at $(g_1, \ldots, g_{k-1}, g)$ with respect to the last variable acts on V as multiplication by $\sum s_i(1 - h_i) \in \mathbb{Z}[\Gamma/V]$ (use that $[ag, h_i] = a[g, h_i]({}^{h_i}a)^{-1} = {}^{(1-h_i)}a \cdot [g, h_i]$ and each $[g, h_i]$ acts trivially on V).

Iterating the map $g \to w'(g_1, \dots, g_{k-1}, g)$ d times gives a word w:

$$w(x_1, \dots, x_k) = w'\Big(x_1, \dots, x_{k-1}, w'\big(x_1, \dots, x_{k-1}, w(\cdots w'(x_1, \dots, x_{k-1}, x_k) \dots)\big)\Big).$$

The Fox derivative of w with respect to x_k is $\big(\sum s_i(1-h_i)\big)^d$, because substitution of words corresponds to multiplication of Fox derivatives. The word w always evaluates to one on Γ/V, because the group Γ/V is nilpotent of class d. □

We now have all the ingredients to prove Theorem 1. It will follow from the corresponding result for metabelian groups:

Theorem 15. *Let G be a virtually metabelian polycyclic group. Then*

$$d(\widehat{G}) \leq d(G) \leq d(\widehat{G}) + 1.$$

Proof. Let $k = d(\widehat{G}) + 1$. Let G_0 and V are the subgroups provided by Lemma 12. Now Lemma 13 applied to the group $G_0/[G_0, G_0]V$ acting on V, gives us an integer N such than any subgroup of index co-prime to N in G_0/V acts perfectly on V (as a rational module).

By Lemma 2 there exists a generating set $S = \{\gamma_1, \dots, \gamma_k\}$ of G/V such that $\gamma_k \in G_0$ and the subgroup $\Gamma = \langle \gamma_1, \dots, \gamma_{k-1}\rangle V \cap G_0$ has index $[G_0 : \Gamma]$ co-prime to N. Therefore V is a perfect rational $\mathbb{Q}[\Gamma/V]$-module and for some integer M and element T in the augmentation ideal of $\mathbb{Z}[\Gamma/V]$ we have that T acts on V as multiplication by M.

For each prime p pick a p-good subgroup G_p of G containing V^p. (In fact by replacing G_p with a normal subgroup of G of finite index we may even assume $G_p \cap V = V^p$.)

Now apply Lemma 14 to Γ and V with $\gamma_i = g_i$, $(i = 1, \dots, k-1)$, $g = \gamma_k$ and $s_i \in \mathbb{N}$, $h_i \in \langle \gamma_1, \dots, \gamma_{k-1}\rangle$ chosen so that $T \equiv \sum_i s_i(1-h_i)$ in $\mathbb{Z}[\Gamma/V]$. We conclude that there is a word $w(x_1, \dots, x_k)$ such that $w(\underline{\gamma}) \in V$ and $\frac{\partial w}{\partial x_k}(\underline{\gamma})$ acts on V as multiplication by M^d where d is the nilpotency class of Γ/V.

We can now apply Theorem 7 to G with these choices of w, V, G_p and γ_j. The conditions 1,2 and 3 are satisfied by the construction of the subgroups G_0 and V, the word w and the definition of the number k. So by Theorem 7 we can find lifts $a_i \in \gamma_i V$ such that $G = \langle a_1, \dots, a_k\rangle$. Theorem 15 is proved. □

Remark 16. If we have that $d(G/V) < d(\widehat{G})$ then the argument above gives that $d(G) = d(\widehat{G})$.

Proof of Theorem 1. In general a virtually polycyclic group G is virtually nilpotent by abelian, i.e., it has normal subgroups $G_1 > G_2$ such that G/G_1 is finite, G_1/G_2 is abelian while G_2 is nilpotent. (See Theorem 2, Chapter 2 in [6]).

Now every group which generates $H = G/G_2'$ generates G and so we have $d(G) = d(H)$, $d(\widehat{G}) = d(\widehat{H})$. Thus Theorem 1 becomes a corollary of Theorem

15. Moreover its proof gives an efficient algorithm for generating a polycyclic group G with $d(\widehat{G}) + 1$ elements, even with $d(\widehat{G})$ elements if the condition of Remark 16 holds. $\square$

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