On the existence of infinite energy solutions for nonlinear Schrödinger equations

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Abstract

We derive new results about existence and uniqueness of local and global solutions for nonlinear Schrödinger equation, including self-similar global solutions. Our analysis is performed in the framework of Marcinkiewicz spaces.

1 Introduction

We consider the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = \lambda |u|^\rho u, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}, \tag{1.1}$$

$$u(0,x) = \phi(x), \ x \in \mathbb{R}^n, \tag{1.2}$$

where $u = u(t,x)$ is a complex valued function, λ is a fixed complex number, and $0 < \rho < \infty$. The initial value $\phi : \mathbb{R}^n \to \mathbb{C}$ is given. The Cauchy problem (1.1)-(1.2) is formally equivalent to the integral equation

$$u(t) = S(t)\phi - i\lambda \int_0^t S(t-s)(|u(s)|^\rho u(s))ds, \tag{1.3}$$

where $S(t)$ is the unitary group determined by the linear Schrödinger equation

$$\partial_t u - i\Delta u = 0, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}.$$

If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and u is defined by $\widehat{u(t)}(\xi) = e^{-i|\xi|^2 t}\widehat{\phi}(\xi)$, for $\xi \in \mathbb{R}^n$, then

$$\widehat{u}_t + i|\xi|^2 \widehat{u} = 0$$

in $\mathbb{R} \times \mathbb{R}^n$. In this case, the solution of

$$\partial_t u - i\Delta u = 0, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}, \tag{1.4}$$

$$u(0,x) = \phi(x), \ x \in \mathbb{R}^n, \tag{1.5}$$

is given by $u(t) = S(t)\phi = K_t * \phi$, where $K_t(x) = \left(e^{-i|\xi|^2 t}\right)^{\check{}}$.

Existence and uniqueness of local and global solutions of problem (1.1)-(1.2) have been much studied in the framework of the Sobolev spaces H^s, $s \geq 0$, i.e, the solutions and their derivatives have finite energy. See, for instance, Ginibre and Velo [11]-[15], Kato [9]-[10], Cazenave and Weissler[3]-[6], and the references therein.

As far as we know, the first authors to study infinite energy solutions of (1.1)-(1.2) were Cazenave and Weissler in [7]. There, they consider the space

$$X_\rho = \{u \in L^\infty_{loc}((0,\infty), \ L^{\rho+2}(\mathbb{R}^n)); \ \sup_{t>0} t^{\frac{\alpha}{2}} \|u(t)\|_{L^{\rho+2}} < \infty\},$$

where $\frac{\alpha}{2} = \frac{1}{\rho} - \frac{n}{2(\rho+2)}$ and $\|\cdot\|_{L^{\rho+2}}$ denotes the usual $L^{\rho+2}$ norm. Under a suitable smallness condition on the initial data, they prove the existence of global solutions of (1.1)-(1.2) in X_ρ, for ρ in the range

$$\frac{\rho+2}{\rho+1} < \frac{n\rho}{2} < \rho+2. \tag{1.6}$$

If $n = 1$ or $n = 2$, condition (1.6) is equivalent to $\rho_0 < \rho < \infty$, where ρ_0 is the positive value of ρ for which $\frac{\rho+2}{\rho+1} = \frac{n\rho}{2}$. If $n \geq 3$, it is equivalent to $\rho_0 < \rho < \frac{4}{n-2}$. Later on, in [2], the Cauchy problem (1.1)-(1.2) was studied in the framework of weak-L^p spaces. Using a Strichartz-type inequality, the authors obtained existence of solutions in the class $L^{(p,\infty)}(\mathbb{R}^{n+1}) \equiv L_t^{(p,\infty)}\left(L_x^{(p,\infty)}\right)$, where $(t,x) \in \mathbb{R} \times \mathbb{R}^n$ and $p = \frac{\rho(n+2)}{2(\rho+1)}$, for ρ in the range

$$\rho_0 < \frac{4(n+1)}{n(n+2)} < \rho < \frac{4(n+1)}{n^2} < \frac{4}{n-2}. \tag{1.7}$$

In [16], the existence of solutions with initial data in the Besov space $\dot{B}_2^{s_\rho,\infty}(\mathbb{R}^n)$, with positive regularity $s_\rho = \frac{n}{2} - \frac{2}{\rho} > 0$, was proved for ρ in the range $\rho_0 < \frac{4}{n} < \rho < \infty$. Note that if $f \in \dot{B}_2^{s_\rho,\infty}(\mathbb{R}^n)$, then f has at least local finite energy.

We study equation (1.3) in functional spaces of infinite energy. In the first theorem proved here, we consider the initial data ϕ belonging to the Marcinkiewicz space $L^{(\frac{\rho+2}{\rho+1},\infty)}$, and show existence and uniqueness of local in time solutions in the class

$$E_{\alpha,\beta}^T = \left\{u; \ \|u\|_{\alpha,\beta} = \sup_{-T<t<T} |t|^{\frac{\alpha-\beta}{2}} \|u(t)\|_{(\rho+2,\infty)} < \infty\right\},$$

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where $\frac{(\alpha-\beta)}{2} = \frac{n\rho}{2(\rho+2)}$, with $\frac{n\rho}{2} < \frac{\rho+2}{\rho+1}$. Note that $\frac{n\rho}{2} < \frac{\rho+2}{\rho+1}$ is equivalent to $0 < \rho < \rho_0 < \frac{4}{n}$. So, our range for ρ is different from the ones in [7, 2, 16]. The norm $\|\cdot\|_{\alpha,\beta}$ is not invariant by the scaling $u_\mu(t,x) = \mu^{\frac{2}{\rho}} u(\mu^2 t, \mu x)$. This is a key point to obtain local in time solutions in Marcinkiewicz spaces. It is also worth noting that our result allows one to consider singular initial data as, for example, homogeneous functions $|x|^{-\frac{n(\rho+1)}{\rho+2}} \in L^{(\frac{\rho+2}{\rho+1},\infty)}$.

Our second theorem concerns global in time solutions. We show existence of such solutions in norms of type $\sup_{|t|>0} |t|^{\alpha/2} \|u(t)\|_{L^{(\rho+2,\infty)}}$, where $\frac{\alpha}{2} = \frac{1}{\rho} - \frac{n}{2(\rho+2)}$ and

$$\rho_0 < \rho < \frac{4}{n-2}. \tag{1.8}$$

This extends the result of Cazenave and Weissler [7] to the context of Lorentz spaces. Note that range (1.8) is greater than range (1.7).

As a corollary, we show that when the initial data ϕ is a homogeneous function of degree $-\frac{2}{\rho}$, we obtain a self-similar solution, if $\|S(1)\phi\|_{(\rho+2,\infty)}$ is sufficiently small. Moreover, we discuss asymptotic stability of the global solutions, and show that regular perturbations of the linear Schrödinger equations are negligible for large times. We also analyze the behavior of the local solutions as $t \to 0$ in the space $L^{(\rho+2,\infty)}$.

Our approach is different from the methods used in [2, 16], where the authors use a Strichartz-type inequality in weak-L^p and Besov spaces, respectively. Indeed, our existence results are based on bounds for the Schrödinger linear group $S(t)$ in the context of Lorentz spaces. In Lemma 2.1, we state and prove these bounds via real interpolation techniques. They generalize the bounds for usual L^p spaces used in [7].

In section 2, we carefully state our results and discuss their improvement in the light of previous results. We prove them in section 3.

2   Main Results

We first recall some facts about the Lorentz spaces. For more details see, for instance, [1] and [17].

Let $1 < p \le \infty$ and $1 \le q \le \infty$. A measurable function f defined on $\mathbb{R}^n$ belongs to Lorentz space $L^{(p,q)}(\mathbb{R}^n)$ if the quantity

$$\|f\|_{(p,q)} = \begin{cases} \left(\frac{p}{q} \int_0^\infty \left[t^{\frac{1}{p}} f^{**}(t)\right]^q \frac{dt}{t}\right)^{\frac{1}{q}} & , \text{ if } \quad 1 < p < \infty, 1 \le q < \infty, \\[4mm] \sup_{t>0} t^{\frac{1}{p}} f^{**}(t) & , \text{ if } \quad 1 < p \le \infty, q = \infty, \end{cases}$$

is finite, where the f^{**} is defined for $t > 0$ by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds,$$

where

$$f^*(t) = \inf \left\{ s > 0; m\{x \in \mathbb{R}^n : |f(x)| > s\} \le t \right\}, \, t > 0.$$

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Note that $L^p(\mathbb{R}^n) = L^{(p,p)}(\mathbb{R}^n)$. The spaces $L^{(p,\infty)}(\mathbb{R}^n)$ are called weak-L^p spaces or Marcinkiewicz spaces. Lorentz spaces have the same scaling relation as L^p spaces, that is, for all $\lambda > 0$ one has $\|f(\lambda x)\|_{(p,q)} = \lambda^{-\frac{n}{p}} \|f\|_{(p,q)}$, where $1 \le p < \infty$ and $1 \le q \le \infty$. Morover, Lorentz spaces can be constructed via real interpolation [1]. Indeed,

$$L^{(p,q)}(\mathbb{R}^n) = (L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{1-\frac{1}{p},q}, \ 1 < p < \infty.$$

They have the interpolation property

$$(L^{(p_0,q_0)}(\mathbb{R}^n), L^{(p_1,q_1)}(\mathbb{R}^n))_{\theta,q} = L^{(p,q)}(\mathbb{R}^n),$$

provided $0 < p_0 < p_1 < \infty$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $1 \le q_0, q_1, q \le \infty$, where $(\cdot, \cdot)_{\theta,q}$ stands for the real interpolation spaces constructed via the K-method [1].

We begin by bounding the Schrödinger group $S(t)$ in Lorentz spaces.

Lemma 2.1. *Let $1 \le d \le \infty$, and $1 < p < 2$. If p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$, then there exists a constant $C = C(n, \gamma, p) > 0$ such that*

$$\|S(t)\varphi\|_{(p',d)} \le C|t|^{-\frac{n}{2}(\frac{2}{p}-1)} \|\varphi\|_{(p,d)}, \tag{2.1}$$

for all $\varphi \in L^{(p,d)}(\mathbb{R}^n)$ and all $t \ne 0$.

Proof. Fix $t \ne 0$ and let $1 < p_0 < p < p_1 < 2$ such that $\frac{1}{p'} = \frac{\lambda}{p_0} + \frac{1-\lambda}{p_1}$ and $0 < \lambda < 1$. By the well known $L^p = L^{(p,p)}$ estimate of Schrödinger group, we have that $S(t) : L^{p_0} \to L^{p_0'}$ and $S(t) : L^{p_1} \to L^{p_1'}$, where the operator norms are respectively bounded by

$$\|S(t)\|_{p_0 \to p_0'} \le C|t|^{-\frac{n}{2}(\frac{2}{p_0}-1)},$$
$$\|S(t)\|_{p_1 \to p_1'} \le C|t|^{-\frac{n}{2}(\frac{2}{p_1}-1)}.$$

Through real interpolation,

$$\begin{aligned}
\|S(t)\|_{(p,d) \to (p',d)} &\le \|S(t)\|_{p_0 \to p_0'}^\lambda \|S(t)\|_{p_1 \to p_1'}^{1-\lambda} \\
&\le \left(C|t|^{-\frac{n}{2}(\frac{2}{p_0}-1)}\right)^\lambda \left(C|t|^{-\frac{n}{2}(\frac{2}{p_1}-1)}\right)^{1-\lambda} \\
&= C|t|^{-\frac{n}{2}(\frac{2}{p}-1)},
\end{aligned}$$

which is equivalent to (2.1). □

From now on, we denote $\alpha := \dfrac{2}{\rho} - \dfrac{n}{\rho+2}$ and $\beta := \dfrac{2}{\rho} - \dfrac{n(\rho+1)}{(\rho+2)}$.

Definition 2.2. *Let $0 < \rho < \infty$ and $0 < T \le \infty$. We denote by E_α and $E_{\alpha,\beta}^T$ the Banach spaces*

$$E_\alpha = \left\{u; |t|^{\frac{\alpha}{2}} u \in BC((-\infty,\infty); L^{(\rho+2,\infty)})\right\}, \tag{2.2}$$

$$E_{\alpha,\beta}^T = \left\{u; |t|^{\frac{\alpha-\beta}{2}} u \in BC((-T,T); L^{(\rho+2,\infty)})\right\}, \tag{2.3}$$

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with respective norms

$$\|u\|_\alpha = \sup_{-\infty < t < \infty} |t|^{\frac{\alpha}{2}} \|u(t)\|_{(\rho+2,\infty)},$$

and

$$\|u\|_{\alpha,\beta} = \sup_{-T < t < T} |t|^{\frac{\alpha-\beta}{2}} \|u(t)\|_{(\rho+2,\infty)},$$

which are weakly continuous in the sense of distributions at $t = 0$.

Definition 2.3. *Let $0 < T \leq \infty$. A mild solution of the initial value problem (1.1)-(1.2) in the space $E_{\alpha,\beta}^T$ (respectively, in the space E_α) is a complex valued function $u \in E_{\alpha,\beta}^T$ (respectively, $u \in E_\alpha$) satisfying equation (1.3) for all $0 < |t| < T$, such that $u(t) \rightharpoonup \phi$ when $t \to 0$ in the sense of distributions.*

Our main results are

Theorem 2.4. *(Local in time solutions) Let $0 < \rho < \infty$ and $\dfrac{n\rho}{2} < \dfrac{\rho+2}{\rho+1}$.*

1. *If $\phi \in L^{(\frac{\rho+2}{\rho+1},\infty)}$, then there exists $0 < T < \infty$ such that the initial value problem (1.1)-(1.2) has a unique mild solution $u(t,x) \in E_{\alpha,\beta}^T$, with $T = T(\phi) = C \|\phi\|_{(\frac{\rho+2}{\rho+1},\infty)}^{-\frac{\rho}{\delta}}$, where $\delta = 1 - \frac{\alpha-\beta}{2}(\rho+1) > 0$.*

2. *Moreover, if $\phi_n \in L^{(\frac{\rho+2}{\rho+1},\infty)}$ is a sequence of functions satisfying $\phi_n \to \phi$ in $L^{(\frac{\rho+2}{\rho+1},\infty)}$, then there exists $0 < T_0 < \infty$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, the solutions u_n and u with respective initial data ϕ_n and ϕ lie in $E_{\alpha,\beta}^{T_0}$ and $u_n \to u$ in $E_{\alpha,\beta}^{T_0}$. Actually, the solution map $\phi \mapsto u$ is Lipschitz continuous.*

Theorem 2.5. *(Global in time solutions) Let $0 < \rho < \infty$ and $\dfrac{\rho+2}{\rho+1} < \dfrac{n\rho}{2} < \rho+2$.*

1. *If ϕ is a distribution such that $\sup_{-\infty < t < \infty} |t|^{\frac{\alpha}{2}} \|S(t)\phi\|_{(\rho+2,\infty)} < \varepsilon$, for $\varepsilon > 0$ small enough, then the initial value problem (1.1)-(1.2) has a global in time mild solution $u(t,x) \in E_\alpha$. This solution is the only one satisfying $\|u\|_\alpha \leq 2\varepsilon$.*

2. *Futhermore, if (ϕ_n) is a sequence of distributions such that $\|S(t)\phi_n - S(t)\phi\|_{E_\alpha} \to 0$ when $n \to \infty$, and u_n, u are the solutions with respective initial data ϕ_n and ϕ, then $u_n \to u$ in E_α.*

We compare the theorems above with previous results.

- In [7], the existence of solutions in spaces of infinite energy was obtained for $\rho_0 < \rho < \frac{4}{n-2}$, where ρ_0 is the value of ρ for which $\frac{\rho+2}{\rho+1} = \frac{n\rho}{2}$. In [2], using Strichartz-type inequalities, the existence of global solutions in the class $L^{(p,\infty)}(R^{n+1}) \equiv L_t^{(p,\infty)}\left(L_x^{(p,\infty)}\right)$ was established, where $p = \frac{\rho(n+2)}{2(\rho+1)}$ and $\rho_0 < \frac{4(n+1)}{n(n+2)} < \rho < \frac{4(n+1)}{n^2}$. So, Theorem 2.4 extends the set of exponents ρ where such solutions exist by including the interval $0 < \rho < \rho_0$.

- In the range $\rho_0 < \rho < \frac{4}{n-2}$, Theorem 2.5 extends the global solutions results derived in [7] to the framework of Marcinkiewicz spaces. Our range for ρ is also greater than the one in [2] (see 1.7).

- Theorem 2.4 assures the existence of local in time solutions even for singular initial data $\phi(x) = P_k(x) |x|^{-k - \frac{n(\rho+1)}{\rho+2}} \in L^{(\frac{\rho+2}{\rho+1}, \infty)}$, where $P_k(x)$ is a homogeneous polynomial of degree k. As far as we know, there were no previous existence results covering this case. On the other hand, we were not able to obtain self-similar solutions in $E_{\alpha,\beta}$ though, since the norm $\| \cdot \|_{\alpha,\beta}$ is not invariant by the scaling relation $u_\mu(t,x) = \mu^{\frac{2}{\rho}} u(\mu^2 t, \mu x)$.

As a direct consequence of Theorem 2.5, one can show the existence of a self-similar solution.

Corollary 2.6. *(self-similar solutions) In addition to the hypothesis of Theorem 2.5, if the initial data ϕ is a sufficiently small homogeneous function of degree $-\frac{2}{\rho}$, then the solution $u(t,x)$ provided by Theorem 2.5 is self-similar, that is, $u(t,x) = \mu^{\frac{2}{\rho}} u(\mu^2 t, \mu x)$ for all $\mu > 0$, almost everywhere for $x \in \mathbb{R}^n$ and $t > 0$.*

Remark 2.7. *Let $P_k(x)$ be a homogeneous polynomial of degree k. The set of functions ϕ which are finite linear combinations of functions of the form $\frac{P_k(x)}{|x|^{k+\frac{2}{\rho}}}$ is an admissible class for the existence of self-similar solutions for problem (1.1)-(1.2).*

We also analyze the large time behaviour of the solutions given by Theorem 2.5, and study the behaviour of the solutions given in Theorem 2.4 near to time $t = 0$. These are the content of the following theorem.

Theorem 2.8. 1. *(Asymptotic stability) Suppose $0 \le h < 1 - \frac{\alpha}{2}(\rho+1)$, and let $u, v \in E_\alpha$ be two global solutions of problem (1.1)-(1.2) obtained through Theorem 2.5, corresponding to respective initial conditions $\phi, \varphi \in L^{(\frac{\rho+2}{\rho+1}, \infty)}$. If $\lim_{|t| \to \infty} |t|^{\frac{\alpha}{2}+h} \|S(t)(\phi - \varphi)\|_{(\rho+2, \infty)} = 0$, then*

$$\lim_{|t| \to \infty} |t|^{\frac{\alpha}{2}+h} \|u(t) - v(t)\|_{(\rho+2, \infty)} = 0. \tag{2.4}$$

2. *(Decay rate as $t \to 0$) Suppose $\delta = 1 - \frac{\alpha - \beta}{2}(\rho+1) > 0$, and $h > -\delta$. Let $u, v \in E_{\alpha,\beta}$ be two local solutions of (1.1)-(1.2) obtained through Theorem 2.4, corresponding to initial conditions $\phi, \varphi \in L^{(\frac{\rho+2}{\rho+1}, \infty)}$, respectively. If $\lim_{t \to 0} |t|^{\frac{\alpha-\beta}{2}-h} \|S(t)(\phi - \varphi)\|_{(\rho+2, \infty)} = 0$, then*

$$\lim_{t \to 0} |t|^{\frac{\alpha-\beta}{2}-h} \|u(t) - v(t)\|_{(\rho+2, \infty)} = 0. \tag{2.5}$$

Let us comment some improvements produced by Theorem 2.8.

- (Asymptotic stability) Theorem 2.5 already gives

$$\sup_{|t|>0} |t|^{\frac{\alpha}{2}} \|u(t) - v(t)\|_{(\rho+2, \infty)} < \infty.$$

Thus, it is obvious that the estimate (2.4) holds for $h < 0$. On the other hand, the first item in Theorem 2.8 extends this property for the range $0 \le h < 1 - \frac{\alpha}{2}(\rho+1)$. However, more regularity on the initial perturbation $\phi - \varphi$ is required though. For instance, assuming (in addition) that $\phi - \varphi \in L^{\frac{\rho+2}{\rho+1}}$, one obtains

$$\lim_{|t| \to \infty} |t|^{\frac{\alpha}{2}+h} \|S(t)(\phi - \varphi)\|_{(\rho+2, \infty)} = 0,$$

with $0 \le h < -\frac{\beta}{2}$. Observe that $-\frac{\beta}{2} = 1 - \frac{\alpha}{2}(\rho+1) > 0$, when $\rho_0 < \rho < \frac{4}{n-2}$.

- (Decay rate when $t \to 0$) By bound (3.2), one can see that

$$|t|^{\frac{\alpha-\beta}{2}-h} \|u(t) - v(t)\|_{(\rho+2,\infty)} \leq |t|^{\frac{\alpha-\beta}{2}-h} \|S(t)(\phi - \varphi)\|_{(\rho+2,\infty)} + C\,|t|^{\delta-h},$$

which implies the bound (2.5) for $h < \delta$. Assuming further regularity for $\phi - \varphi$, the second item of Theorem 2.8 extends this property for the range $h > -\delta$.

3 Proofs

The following Lemma is important to our ends. For its proof, see [8].

Lemma 3.1. *Let $0 < \rho < \infty$ and X to be a Banach space with norm $\|\cdot\|$. Suppose $B : X \to X$ to be a map satisfying*

$$\|B(x) - B(z)\| \leq K\|x - z\|\left(\|x\|^\rho + \|z\|^\rho\right), \tag{3.1}$$

$B(0) = 0$, and let $R > 0$ be the unique positive root of equation $2^{\rho+1}K(R)^\rho - 1 = 0$. Given $0 < \varepsilon < R$ and $y \in X$, $y \neq 0$, such that $\|y\| \leq \varepsilon$, there exists a solution $x \in X$ for the equation $x = y + B(x)$ such that $\|x\| \leq 2\varepsilon$. The solution x is unique in the ball $B_{2\varepsilon} := \overline{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the following sense: If $\|\tilde{y}\| \leq \varepsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x})$, and $\|\tilde{x}\| \leq 2\varepsilon$, then

$$\|x - \tilde{x}\| \leq \frac{1}{1 - 2^{\rho+1}K\varepsilon^\rho}\|y - \tilde{y}\|.$$

Now, we state and prove the necessary estimates in order to apply Lemma 3.1 in our case.

Lemma 3.2. *Let $0 < \rho < \infty$ and B be defined as*

$$B(u) = -i\lambda \int_0^t S(t-s)(|u(s)|^\rho u(s))ds.$$

If $\dfrac{n\rho}{2} < \dfrac{\rho+2}{\rho+1}$, then there exists a positive constant $K_{\alpha,\beta}$ such that

$$\|B(u) - B(v)\|_{\alpha,\beta} \leq K_{\alpha,\beta}T^{1 - \frac{(\alpha-\beta)(\rho+1)}{2}} \|u - v\|_{\alpha,\beta}\left(\|u\|_{\alpha,\beta}^\rho + \|v\|_{\alpha,\beta}^\rho\right), \tag{3.2}$$

for all $u,\,v \in E_{\alpha,\beta}^T$. On the other hand, if $\dfrac{\rho+2}{\rho+1} < \dfrac{n\rho}{2} < \rho + 2$, then there exists a positive constant K_α such that

$$\|B(u) - B(v)\|_\alpha \leq K_\alpha\|u - v\|_\alpha\left(\|u\|_\alpha^\rho + \|v\|_\alpha^\rho\right), \tag{3.3}$$

for all $u,\,v \in E_\alpha$.

Proof. Without loss of generality, we assume $t > 0$. First note that if $\frac{n\rho}{2} < \frac{\rho+2}{\rho+1} < \rho + 2$, then

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$\frac{\alpha-\beta}{2}(\rho+1) < 1$ and $\frac{n}{2}(\frac{2(\rho+1)}{\rho+2}-1) < 1$. Therefore,

$$\|B(u) - B(v)\|_{(\rho+2,\infty)} \leq \int_0^t \|S(t-s)(|u|^\rho u - |v|^\rho v)\|_{(\rho+2,\infty)} ds$$

$$\leq C \int_0^t (t-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{\rho+2}-1)} \|(|u-v|)(|u|^\rho + |v|^\rho)\|_{(\frac{\rho+2}{\rho+1},\infty)} ds$$

$$\leq C \int_0^t (t-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{\rho+2}-1)} \|u-v\|_{(\rho+2,\infty)} \left(\|u\|_{(\rho+2,\infty)}^\rho + \|v\|_{(\rho+2,\infty)}^\rho \right) ds$$

$$\leq C \left(\sup_{0<t<T} t^{\frac{\alpha-\beta}{2}} \|u-v\|_{(\rho+2,\infty)} \sup_{0<t<T} \left(t^{\frac{(\alpha-\beta)\rho}{2}} \|u\|_{(\rho+2,\infty)}^\rho + t^{\frac{(\alpha-\beta)\rho}{2}} \|v\|_{(\rho+2,\infty)}^\rho \right) \right) \int_0^t (t-s)^{-\frac{\alpha-\beta}{2}} s^{-\frac{\alpha-\beta}{2}(\rho+1)} ds$$

$$= K_{\alpha,\beta} t^{-\frac{\alpha-\beta}{2}} t^{1-\frac{\alpha-\beta}{2}(\rho+1)} \|u-v\|_{\alpha,\beta} \left(\|u\|_{\alpha,\beta}^\rho + \|v\|_{\alpha,\beta}^\rho \right),$$

which proves (3.2). On the other hand, if $\frac{\rho+2}{\rho+1} < \frac{n\rho}{2} < \rho+2$, then $\frac{\alpha}{2}(\rho+1) < 1$ and $\frac{n}{2}(\frac{2(\rho+1)}{\rho+2}-1) < 1$. In this case,

$$\|B(u)-B(v)\|_{(\rho+2,\infty)} \leq C \int_0^t (t-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{\rho+2}-1)} \|u-v\|_{(\rho+2,\infty)} \left(\|u\|_{(\rho+2,\infty)}^\rho + \|v\|_{(\rho+2,\infty)}^\rho \right) ds$$

$$\leq C \left(\sup_{t>0} t^{\frac{\alpha}{2}} \|u-v\|_{(\rho+2,\infty)} \sup_{t>0} \left(t^{\frac{\alpha\rho}{2}} \|u\|_{(\rho+2,\infty)}^\rho + t^{\frac{\alpha\rho}{2}} \|v\|_{(\rho+2,\infty)}^\rho \right) \right) \int_0^t (t-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{\rho+2}-1)} s^{-\frac{\alpha}{2}(\rho+1)} ds$$

$$= K_\alpha t^{-\frac{\alpha}{2}} \|u-v\|_\alpha \left(\|u\|_\alpha^\rho + \|v\|_\alpha^\rho \right),$$

which proves (3.3). $\qquad \square$

3.1 Proof of Theorem 2.4

Let $y = S(t)\phi$. Due to Lemma 2.1, one has

$$\|y\|_{\alpha,\beta} = \sup_{-T<t<T} |t|^{\frac{\alpha-\beta}{2}} \|S(t)\phi\|_{(\rho+2,\infty)} \leq C \|\phi\|_{(\frac{\rho+2}{\rho+1},\infty)} < \infty.$$

Using Lemma 3.2, one gets

$$\|B(u) - B(v)\|_{\alpha,\beta} \leq K_{\alpha,\beta} T^\delta \|u-v\|_{\alpha,\beta} \left(\|u\|_{\alpha,\beta}^\rho + \|v\|_{\alpha,\beta}^\rho \right), \tag{3.4}$$

where $\delta = 1 - \frac{\alpha-\beta}{2}(\rho+1) > 0$. Now, choose $0 < T < \infty$ sufficiently small, and $\varepsilon > 0$ such that $\|y\|_{\alpha,\beta} \leq C \|\phi\|_{(\frac{\rho+2}{\rho+1},\infty)} = \varepsilon < R := \left(\frac{1}{2^{(\rho+1)}K_{\alpha,\beta}T^\delta} \right)^{\frac{1}{\rho}}$. Using Lemma 3.1 with $X = E_{\alpha,\beta}^T$, one assures the existence of a local mild solution $u \in E_{\alpha,\beta}^T$. Moreover, this solution is unique in the ball $B_{2\varepsilon} := \overline{B}(0, 2\varepsilon) \subset E_{\alpha,\beta}^T$. Furthermore, through standard arguments on can prove that $u(t) \to \phi$ in the sense of distributions when $t \to 0$. So, solutions of the integral equation are indeed mild solutions in the sense of Definition 2.3.

Finally, let u_n and u be the solutions with respective initial data ϕ_n and ϕ. By Lemma 3.1, one has

$$\|u_n - u\|_{E_{\alpha,\beta}} \leq \frac{1}{1 - 2^{\rho+1}K_{\alpha,\beta}T^\delta \varepsilon^\rho} \|S(t)\phi_n - S(t)\phi\|_{E_{\alpha,\beta}} \leq \frac{C}{1 - 2^{\rho+1}K_{\alpha,\beta}T^\delta \varepsilon^\rho} \|\phi_n - \phi\|_{(\frac{\rho+2}{\rho+1},\infty)}.$$

This finishes the proof.

3.2 Proof of Theorem 2.5

Apply Lemma 3.1 to the integral equation (1.3) with $X = E_\alpha$ and $y = S(t)\phi$. In this case, the bound (3.3) gives

$$\|B(u) - B(v)\|_\alpha \le K_\alpha \|u - v\|_\alpha \left(\|u\|_\alpha^\rho + \|v\|_\alpha^\rho\right).$$

Now, one considers $\varepsilon > 0$ small enough so that $\|S(t)\phi\|_\alpha = \sup_{|t|>0} |t|^{\frac{\alpha}{2}} \|S(t)\phi\|_{(\rho+2,\infty)} < \varepsilon$ allows one to apply Lemma 3.1 repeatedly, in order to obtain the existence of a global mild solution $u \in E_\alpha$. This solution is unique in the ball $B_{2\varepsilon} := \overline{B}(0, 2\varepsilon) \subset E_\alpha$.

The continuity of the solutions with respect to the initial conditions, as well as the continuity of the solutions in the sense of distributions, follow as in the proof of Theorem 2.4.

3.3 Proof of Corollary 2.6

Let $t > 0$. If the initial data $\phi(x)$ is a homogeneous function of degree $-\frac{2}{\rho}$, then $S(t)\phi$ satisfies the self-similar property $u(t,x) = \mu^{\frac{2}{\rho}} u(\mu^2 t, \mu x)$. Thus,

$$t^{\alpha/2} \|S(t)\phi\|_{(\rho+2,\infty)} = t^{\frac{\alpha}{2}} t^{\frac{n}{2(\rho+2)} - \frac{1}{\rho}} \|S(1)\phi\|_{(\rho+2,\infty)} = \|S(1)\phi\|_{(\rho+2,\infty)}.$$

Moreover, $\|S(1)\phi\|_{L^{\rho+2}}$ is finite(see [7]). Since the inclusion $L^{\rho+2} \hookrightarrow L^{(\rho+2,\infty)}$ continuously, one has

$$\|S(1)\phi\|_{(\rho+2,\infty)} \le \|S(1)\phi\|_{L^{\rho+2}} < \infty.$$

Therefore, if $\|S(1)\phi\|_{(\rho+2,\infty)}$ is small enough , it is straightforward to show that the solution $u(t,x)$ obtained in Theorem 2.4 is self-similar.

3.4 Proof of Theorem 2.8

Without loss of generality, assume $t > 0$. Subtracting the integral equations satisfied by u and v, one gets

$$t^{\frac{\alpha}{2}+h} \|u(t) - v(t)\|_{(\rho+2,\infty)} \le t^{\frac{\alpha}{2}+h} \|S(t)(\phi - \varphi)\|_{(\rho+2,\infty)}$$
$$+ t^{\frac{\alpha}{2}+h} \left\| \int_0^t S(t-s)(u\,|u|^\rho - v\,|v|^\rho)ds \right\|_{(\rho+2,\infty)}.$$

Since $\|u\|_\alpha, \|v\|_\alpha \le 2\varepsilon$, one uses the change of variable $s \longmapsto ts$, and bound

$$t^{\frac{\alpha}{2}+h} \left\| \int_0^t S(t-s)(u|u|^\rho - v|v|^\rho)ds \right\|_{(\rho+2,\infty)}$$

$$\le C t^{\frac{\alpha}{2}+h} \int_0^t (t-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{(\rho+2)}-1)} s^{-\frac{\alpha(\rho+1)}{2}-h} (s^{\frac{\alpha\rho}{2}} \|u(s)\|_{(\rho+2,\infty)}^\rho + s^{\frac{\alpha\rho}{2}} \|v(s)\|_{(\rho+2,\infty)}^\rho) s^{\frac{\alpha}{2}+h} \|u(s) - v(s)\|_{(\rho+2,\infty)} ds$$

$$\le C 2^{\rho+1} \varepsilon^\rho \int_0^1 (1-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{(\rho+2)}-1)} s^{-\frac{\alpha(\rho+1)}{2}-h} (ts)^{\frac{\alpha}{2}+h} \|u(ts) - v(ts)\|_{(\rho+2,\infty)} ds.$$

Therefore,

$$t^{\frac{\alpha}{2}+h} \|u(t) - v(t)\|_{(\rho+2,\infty)} \le t^{\frac{\alpha}{2}+h} \|S(t)(\phi - \varphi)\|_{(\rho+2,\infty)} \tag{3.5}$$

$$+ C 2^{\rho+1} \varepsilon^\rho \int_0^1 (1-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{(\rho+2)}-1)} s^{-\frac{\alpha}{2}(\rho+1)-h} (ts)^{\frac{\alpha}{2}+h} \|u(ts) - v(ts)\|_{(\rho+2,\infty)} ds,$$

for all $t > 0$. Now, define
$$A := \limsup_{t \to \infty} t^{\frac{\alpha}{2} + h} \|u(t) - v(t)\|_{(\rho+2,\infty)}.$$

Using the assumption on the initial perturbation $\phi - \varphi$, it is not difficult to show that $A < \infty$. Now, note that

$$\limsup_{t \to \infty} \int_0^1 (1-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{(\rho+2)}-1)} s^{\frac{-\alpha(\rho+1)}{2}-h}(ts)^{\frac{\alpha}{2}+h}\|u(ts) - v(ts)\|_{(\rho+2,\infty)}ds$$

$$\leq A \int_0^1 (1-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{(\rho+2)}-1)} s^{-\frac{\alpha}{2}(\rho+1)-h}ds.$$

So, taking $\limsup_{t \to \infty}$ in (3.5), one obtains

$$A \leq \left(C2^{\rho+1}\varepsilon^\rho \int_0^1 (1-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{(\rho+2)}-1)} s^{-\frac{\alpha(\rho+1)}{2}-h}ds \right) A.$$

Now, let $\Gamma := C2^{\rho+1} \int_0^1 (1-s)^{-\frac{n}{2}(\frac{2(\rho+1)}{(\rho+2)}-1)} s^{-\frac{\alpha(\rho+1)}{2}-h}ds$. Choosing $\varepsilon > 0$ sufficiently small such that $\varepsilon^\rho \Gamma < 1$, one concludes that $A = 0$. This proves part 1 of the theorem.

In order to prove part 2, let $\delta = 1 - \frac{\alpha-\beta}{2}(\rho+1)$ and $0 < t < T$ as in Theorem 2.4. One can write $\delta = \frac{\alpha-\beta}{2} - h - \frac{\alpha-\beta}{2} - \frac{\alpha-\beta}{2}(\rho+1) + h + 1$. Again, one subtracts the equations for u and v, and bound

$$t^{\frac{\alpha-\beta}{2}-h} \left\| \int_0^t S(t-s)(u\,|u|^\rho - v\,|v|^\rho)ds \right\|_{(\rho+2,\infty)}$$

$$\leq Ct^{\frac{\alpha-\beta}{2}-h} \int_0^t (t-s)^{-\frac{\alpha-\beta}{2}} s^{-\frac{\alpha-\beta}{2}(\rho+1)+h}(s^{\frac{\alpha-\beta}{2}\rho}(\|u(s)\|_{(\rho+2,\infty)}^\rho + \|v(s)\|_{(\rho+2,\infty)}^\rho))\ s^{\frac{\alpha-\beta}{2}-h}\|u(s) - v(s)\|_{(\rho+2,\infty)}ds$$

$$\leq C2^{\rho+1}\varepsilon^\rho t^{\frac{\alpha-\beta}{2}-h-\frac{\alpha-\beta}{2}-\frac{\alpha-\beta}{2}(\rho+1)+h+1} \int_0^1 (1-s)^{-\frac{\alpha-\beta}{2}} s^{-\frac{\alpha-\beta}{2}(\rho+1)+h}(ts)^{\frac{\alpha-\beta}{2}-h}\|u(ts) - v(ts)\|_{(\rho+2,\infty)}ds.$$

Hence,

$$t^{\frac{\alpha-\beta}{2}-h} \|u(t) - v(t)\|_{(\rho+2,\infty)} \leq t^{\frac{\alpha-\beta}{2}-h} \|S(t)(\phi - \varphi)\|_{(\rho+2,\infty)}$$

$$+ C2^{\rho+1}\varepsilon^\rho t^\delta \int_0^1 (1-s)^{-\frac{\alpha-\beta}{2}} s^{-\frac{\alpha-\beta}{2}(\rho+1)+h}(ts)^{\frac{\alpha-\beta}{2}-h}\|u(ts) - v(ts)\|_{(\rho+2,\infty)}ds$$

Writing $A := \limsup_{t \to 0} t^{\frac{\alpha-\beta}{2}-h}\|u(t) - v(t)\|_{(\rho+2,\infty)} < \infty$, one takes $\limsup_{t \to 0}$ in the last inequality to obtain

$$0 \leq A \leq C2^{\rho+1}\varepsilon^\rho A \int_0^1 (1-s)^{-\frac{\alpha-\beta}{2}} s^{-\frac{\alpha-\beta}{2}(\rho+1)+h}ds \lim_{t \to 0} t^\delta = 0.$$

This concludes the proof.

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