A NEGATIVE MASS THEOREM FOR THE 2-TORUS

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Abstract. Let M be a closed surface. For a metric g on M, denote the area element by dA and the Laplace-Beltrami operator by $\Delta = \Delta_g$. We define the Robin mass $m(p)$ at the point $p \in M$ to be the value of the Green function $G(p,q)$ at $q = p$ after the logarithmic singularity has been subtracted off, and we define $\operatorname{trace}\Delta^{-1} = \int_M m(p)\,dA$. This regularized trace can also be obtained by regularization of the spectral zeta function and is hence a spectral invariant. Furthermore, $(\operatorname{trace}\Delta^{-1})/A$ is a non-trivial analog for closed surfaces of the ADM mass for higher dimensional asymptotically flat manifolds. We define the Δ-mass of (M,g) to equal $(\operatorname{trace}\Delta_g^{-1} - \operatorname{trace}\Delta_{S^2,A}^{-1})/A$, where $\Delta_{S^2,A}$ is the Laplacian on the round sphere of area A. In this paper we show that in each conformal class $\mathcal{C}$ for the 2-torus, there exists a metric with negative Δ-mass. From this it follows that the minimum of the Δ-mass on $\mathcal{C}$ is negative and attained by some metric $g \in \mathcal{C}$. For this minimizing metric g, one gets a sharp logarithmic Hardy-Littlewood-Sobolev inequality and an Onofri-type inequality. We remark that if the flat metric in $\mathcal{C}$ is sufficiently long and thin then the minimizing metric g is non-flat. The proof of our result depends on analyzing the ordinary differential equation $\phi'' = 1 - e^{\phi}$ which is equivalent to $h'' = 1 - 1/h$. The solutions are periodic and we need to establish quite delicate, asymptotically sharp inequalities relating the period to the maximum value.

Section 1. Introduction, Main Results and Summary of the Proof.

Let M be a smooth, closed, compact surface with a (Riemannian) metric g. Denote the area element of g by dA and the area by A. Let $\Delta = \Delta_g$ denote the Laplace-Beltrami operator for g, given in local coordinates $(x_1, \dots, x_n)$ by

$$\Delta \;=\; -\sum_{i,j} \frac{1}{\sqrt{\det g}}\,\frac{\partial}{\partial x_i}\,\sqrt{\det g}\;g^{ij}\frac{\partial}{\partial x_j}. \tag{1.1}$$

The kernel of Δ is the constants. Let Δ^{-1} denote the inverse operator

$$\Delta^{-1}\Delta f \;=\; f \;-\; \frac{1}{A}\int_M f\,dA.$$

The Green function $G(p,q)$ for Δ is the smooth function on $M \times M \setminus \{(p,p) : p \in M\}$ which satisfies

$$\Delta^{-1} f(p) \;=\; \int_M G(p,q) f(q)\,dA(q).$$

Denoting the distance from p to q in the metric g by $d(p,q)$, the function $G(p,q)$ is smooth away from the diagonal and has an expansion at the diagonal of the form

$$G(p,q) \;=\; -\frac{1}{2\pi}\log d(p,q) \;+\; m(p) \;+\; o(d(p,q)). \tag{1.2}$$

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We call the value $m(p) = m_g(p)$ the *Robin mass at the point p*. For a smooth function ϕ on M, write A_ϕ for the area of M in the metric $e^\phi g$, so

$$A_\phi \;=\; \int_M e^\phi \, dA.$$

Conformal change of the Robin mass. *If ϕ is a smooth function on M then*

$$m_{e^\phi g}(p) \;=\; m_g(p) \;+\; \frac{\phi}{4\pi} \;-\; \frac{2}{A_\phi}(\Delta_g^{-1} e^\phi)(p) \;+\; \frac{1}{A_\phi^2}\int_M e^\phi \Delta_g^{-1} e^\phi \, dA. \tag{1.3}$$

For the proof, see for example [S1], [S2], [M2] or [O2]. We define

$$\operatorname{trace}\Delta_g^{-1} \;=\; \int_M m_g(p)\, dA(p).$$

This is a spectral invariant for Δ, since it can be obtained from the spectral zeta function associated to Δ, see [S1], [S2], [M3], or [O2].

Remark. Writing $K(p)$ for the Gaussian curvature of g at p, it is shown in [S1], [S2], that for any metric g on the 2-sphere, we have

$$m_g(p) \;-\; \frac{1}{2\pi}\Delta^{-1}K(p) \;=\; \frac{1}{A}\operatorname{trace}\Delta_g^{-1}. \tag{1.4}$$

The left hand side (and hence the right hand side) is a 2-sphere analog of the ADM mass from general relativity. Indeed, the (Riemannian) ADM mass is defined for asymptotically flat manifolds. However, if M is a compact Riemannian manifold of dimension greater than 2, with positive conformal Laplacian, then given a point $p \in M$ we can define a mass at p by blowing up the metric around p using the Green function for the conformal Laplacian, and taking the ADM mass of the resulting asymptotically flat metric. This amounts to taking the constant term in the asymptotic expansion of the Green function for the conformal Laplacian around the point p. The left hand side of (1.4) is the natural non-trivial analog of this for the 2-sphere. Formula (1.4) does not hold for surfaces of higher genus. The left hand side is no longer pointwise constant and its fluctuation does not have obvious geometric significance. Therefore we consider the right hand side of (1.4) as a natural non-trivial analog of the ADM mass for compact surfaces.

Now (1.3) immediately gives the following formula, see also [M1].

Conformal change of $\operatorname{trace}\Delta^{-1}$ (Morpurgo's Formula). *If ϕ is a smooth function on M, then*

$$\operatorname{trace}\Delta_{e^\phi g}^{-1} \;=\; \int_M m_g e^\phi \, dA \;+\; \frac{1}{4\pi}\int_M \phi\, e^\phi \, dA \;-\; \frac{1}{A_\phi}\int_M e^\phi \Delta_g^{-1} e^\phi \, dA. \tag{1.5}$$

On the round sphere, the right hand side of (1.5) occurs in the logarithmic Hardy-Littlewood-Sobolev inequality.

Sharp logarithmic Hardy-Littlewood-Sobolev inequality on the S^2. *If g is a round metric on S^2 of area A,*

$$\frac{1}{4\pi}\int_{S^2} \phi\, e^\phi \, dA \;-\; \frac{1}{A}\int_{S^2} e^\phi \Delta^{-1} e^\phi \, dA \;\geq\; 0$$

holds for all functions $\phi : S^2 \to \mathbb{R}$ with $\int_{S^2} e^\phi \, dA = A$ such that $\int_{S^2} \phi\, e^\phi \, dA$ is finite. Moreover equality is attained exactly when e^ϕ is the Jacobian of a conformal transformation of S^2.

For the proof, see [On], [CL], [B]. Combining this with (2), Morpurgo obtained the following.

Spectral interpretation of the logarithmic HLS inequality. *Among all metrics on the* 2*-sphere of area* A*, the round metric attains the minimum value of* $\operatorname{trace}\Delta^{-1}$.

The behavior of $\operatorname{trace}\Delta^{-1}$ for non-flat metrics on the torus was first considered in [M1]. Suppose g_0 is any flat metric of unit area on the 2-torus, and let $\lambda_1(g_0)$ denote the lowest eigenvalue of the Laplace-Beltrami operator for g_0. Let $\mathcal{C}_1$ denote the class of metrics conformal to g_0 having unit area. It was shown in [M1] that if $\lambda_1(g_0) > 8\pi$, then g_0 is a *local* minimum for $\operatorname{trace}\Delta^{-1}$ on $\mathcal{C}_1$. In [LL1], [LL2], this was improved to a global result in most cases. Indeed, it was shown that g_0 minimizes $\operatorname{trace}\Delta^{-1}$ on $\mathcal{C}_1$ provided $\lambda_1(g_0) \geq \pi^3$, or g_0 is rectangular and $\lambda_1 \geq 8\pi$. It is well understood that g_0 cannot minimize $\operatorname{trace}\Delta^{-1}$ on $\mathcal{C}_1$ when $\lambda_1(g_0)$ is small. Indeed, it can be observed from the Kronecker limit formula that when $\lambda_1(g_0)$ is small, the value of $\operatorname{trace}\Delta^{-1}$ for g_0 is greater than the value for the round sphere of unit area, as was pointed out in [DS2]. However, by blowing a spherical bubble, one can construct a family of metrics in $\mathcal{C}_1$ for which $\operatorname{trace}\Delta^{-1}$ approaches the value for the round sphere (see [O2], [DS2] for different approaches to this). In this paper, we show that if T is a flat torus of unit area with $\lambda_1(T) < 8\pi$, then the minimum value of $\operatorname{trace}\Delta^{-1}$ among conformal metrics of unit area is attained by a non-flat metric. Although we do not identify this minimizing metric explicitly, we do construct a candidate, which is approximately spherical except for a short wormhole joining the poles.

Theorem 1. *Let* T *be a* 2*-dimensional torus with metric* g_0*. Then there exists a metric* g *in the same conformal class as* g_0 *and having the same area* A*, such that the Robin mass* $m(x)$ *for* g *is constant, and strictly less than the Robin mass for the round sphere of area* A*.*

This leads to the following result.

Theorem 2. *Let* T *be a* 2*-dimensional torus with metric* g_0*. Then among metrics in the same conformal class as* g_0 *and having the same area* A*, there exists a metric* g *which attains the minimum value of* $\operatorname{trace}\Delta^{-1}$*. Moreover* g *has constant Robin mass* $m(x)$*, and this is less than the Robin mass of the round sphere of area* A*.*

We remark that if g_0 is flat with $\lambda_1(g_0) < 8\pi$, then the metric g is not flat and the Robin mass for g is less than that for g_0.

Corollary 3. *(Analogs of Logarithmic HLS inequality and Onofri's Inequality for the torus.) For the minimizing metric* g *of Theorem 2, we have*

$$\frac{1}{4\pi}\int_T \phi\, e^{\phi}\, dA \;-\; \frac{1}{A}\int_T e^{\phi}\Delta^{-1}e^{\phi}\, dA \;\geq\; 0 \tag{1.6}$$

for all functions $\phi : T \to \mathbb{R}$ *with* $\int_T e^{\phi}\, dA = A$ *such that* $\int_T \phi\, e^{\phi}\, dA$ *is finite. Here,* dA *and* Δ *are associated to* g*. Moreover, for* $\phi \in C^\infty(T)$,

$$\frac{1}{16\pi}\int_T \phi\Delta\phi\, dA \;-\; \log\left(\frac{1}{A}\int_T e^{\phi}\, dA\right) \;+\; \frac{1}{A}\int_T \phi\, dA \;\geq\; 0.$$

To deduce Theorem 2 from Theorem 1, we appeal to Theorem 1 of [O2], which states that the minimum value of $\operatorname{trace}\Delta^{-1}$ among metrics conformal to g_0 having the same area is attained, provided there exists a metric conformal to g_0 for which the value of $\operatorname{trace}\Delta^{-1}$ is lower than the value for the round sphere of the same area. The proof of that result is a variational argument very similar in spirit to the proof of the Yamabe theorem in the non-positive case. One is trying to find ϕ to minimize (1.5).

First one modifies the equation to break the lack of compactness by replacing Δ^{-1} in the integral on the right by $\Delta^{-1-\varepsilon}$. One can construct a minimizer for the resulting functional, and one wants this minimizer to converge to a limit as $\varepsilon \to 0$. It is here that one uses the fact that the value of trace Δ^{-1} is lower than that for the round sphere, which is what prevents bubbles from forming and ensures the existence of a convergent subsequence as $\varepsilon \to 0$. To deduce Corollary 3 from Theorem 2, we appeal to Theorem 3 in [O2], which is just an explicit formulation of the duality between the logarithmic Sobolev inequality and the Onofri inequality. For some related results, see [Ch], [M2], [M3], [O1], [OsPS1], [S2]. For a probabilistic interpretation of trace Δ^{-1}, see [DS1].

Proof of Theorem 1. We will quickly show that our result is related to the problem of establishing somewhat delicate inequalities between the period and the maximum value of solutions to the ordinary differential equation $\phi'' = 1 - e^{\phi}$. These inequalities are established by making just the right Taylor expansion of the integral formula for the period.

We first remark that under scaling by a constant e^{λ}, the Robin mass scales as

$$m_{e^{\lambda}g}(p) \;=\; m_g(p) \;+\; \frac{\lambda}{4\pi}.$$

Hence if we can prove the Theorem for area $A = 1$, it follows for arbitrary values of A. Furthermore, by the classical Uniformization Theorem we can assume that g_0 is a flat metric on T with area 1, and we seek the metric $g = e^{\phi} g_0$ of area 1. From (1.3), the condition that the mass $m_{e^{\phi}g_0}(p)$ is constant is

$$\phi \;-\; 8\pi(\Delta_0^{-1} e^{\phi}) \;\; \text{is constant,}$$

where Δ_0 is the Laplacian for g_0. Applying Δ_0 we find that this is equivalent to

$$\Delta_0 \phi \;=\; 8\pi(e^{\phi} - 1). \tag{1.7}$$

We remark that if ϕ satisfies this condition then the metric $e^{\phi} g_0$ automatically has area 1, since

$$0 \;=\; \int_T \Delta_0 \phi \, dA \;=\; 8\pi \int_T (e^{\phi} - 1)\, dA_0.$$

where dA_0 is the area element for g_0. We assume that ϕ satisfies (1.7). Then (1.5) gives

$$\operatorname{trace} \Delta_{e^{\phi}g_0}^{-1} \;=\; \operatorname{trace} \Delta_0^{-1} \;+\; \frac{1}{8\pi} \int_T \phi(1 + e^{\phi})\, dA_0, \tag{1.8}$$

Now we work on a torus with flat metric g of area 1, given by $\mathbb{C}/\Lambda$ where Λ is the lattice generated by $1/b$ and $a + ib$. A fundamental domain for the torus is given by

$$\left\{ x + iy : 0 \le y \le b, \qquad \frac{ay}{b} \le x \le \frac{ay+1}{b} \right\}. \tag{1.9}$$

It is a fact that every metric on the torus is conformal to such a flat metric, with

$$b \;\ge\; \left(\frac{3}{4}\right)^{1/4} \;=\; 0.9306... \tag{1.10}$$

For the flat metric g on this torus, we compute in the appendix using the first Kronecker limit formula that setting

$$\beta \;=\; \sqrt{\pi}\, b, \tag{1.11}$$

we have

$$\operatorname{trace}\Delta_0^{-1} \;-\; \operatorname{trace}\Delta_{S^2,1}^{-1} \;=\; \frac{1}{4\pi}\left(\frac{\beta^2}{3} \;-\; \log(4\beta^2) \;+\; 1 \;-\; 4\sum_{n=1}^{\infty}\log\left|1-e^{-2n(\beta^2-i\sqrt{\pi}\,\beta a)}\right|\right), \tag{1.12}$$

where $\Delta_{S^2,1}^{-1}$ is the Laplacian on the round 2-sphere of area 1, see also [Chiu], [S1], [S2]. From this we see that

$$\operatorname{trace}\Delta_0^{-1} \;-\; \operatorname{trace}\Delta_{S^2,1}^{-1} \;\le\; \frac{1}{4\pi}\left(\frac{\beta^2}{3} \;-\; \log(4\beta^2) \;+\; 1 \;-\; 4\sum_{n=1}^{\infty}\log\left|1-e^{-2n\beta^2}\right|\right). \tag{1.13}$$

From this point, the proof involves some simple numerical evaluations as well as exact formulas and asymptotic estimates. It is a fact first pointed out in [DS2] that that the left hand side of (1.13) is negative when β is small. To see this, note that

$$-4\sum_{n=1}^{\infty}\log\left|1-e^{-2n\beta^2}\right|$$

is decreasing in β and is thus bounded by the value at the endpoint $\beta = \pi^{1/2}(3/4)^{1/4}$, which is

$$-4\sum_{n=1}^{\infty}\log\left|1-e^{-3^{1/2}\pi n}\right| \;<\; 0.02.$$

On the other hand,

$$\frac{\beta^2}{3} \;-\; \log(4\beta^2) \;+\; 1$$

is convex on the interval $[\pi^{1/2}(3/4)^{1/4}, 2.6]$, and hence is bounded above there by -0.04. Adding these terms, we find that the right hand side of (1.13) is negative when $\beta \le 2.6$. We see then that in this case the flat metric $g = g_0$ satisfies the conclusion of Theorem 1. We only need prove Theorem 1 when $\beta > 2.6$. Noting that $2.6 > \pi/\sqrt{2}$, we now complete the proof of Theorem 1, by explaining how to find g in the case $\beta > \pi/\sqrt{2}$.

Remark. If $b > 1$, then the length of the shortest geodesic is $1/b$ and the lowest eigenvalue of the Laplace-Beltrami operator is $\lambda_1 = 4\pi^2/b^2 = 4\pi^3/\beta^2$, so the value $\beta = \pi/\sqrt{2}$ corresponds to $\lambda_1 = 8\pi$. The value $\beta = 2$ corresponds to $\lambda_1 = \pi^3$. We remark that when $\beta \le 2$, it is shown in [LL1] that the flat metric minimizes $\operatorname{trace}\Delta^{-1}$. Since the minimum must beat the round sphere, this again confirms for the case $\beta \le 2$, that (1.13) is negative.

Assuming ϕ satisfies (1.7), combining (1.8) and (1.13) gives

$$\begin{aligned} &\operatorname{trace}\Delta_{e^{\phi}g_0}^{-1} \;-\; \operatorname{trace}\Delta_{S^2,1}^{-1} \\ &\quad\le\; \frac{1}{4\pi}\left(\frac{1}{2}\int_T \phi(1+e^{\phi})\,dA_0 \;+\; \frac{\beta^2}{3} \;-\; \log(4\beta^2) \;+\; 1 \;-\; 4\sum_{n=1}^{\infty}\log\left|1-e^{-2n\beta^2}\right|\right). \end{aligned} \tag{1.14}$$

We will find $\phi \in C^{\infty}(T)$ satisfying (1.7) such that $\phi(x+iy)$ is a function of y alone, and the right hand side of (1.14) is negative. We can recast (1.7) and (1.14) in terms of the single variable y so that Theorem 1 follows from the following:

Theorem 1′. *For each $b > (\pi/2)^{1/2}$, there exists a smooth function $\phi \in C^\infty(\mathbb{R})$ satisfying*

$$\frac{d^2\phi}{dy^2} = 8\pi(1 - e^\phi), \tag{1.15}$$

$$\phi(y + b) = \phi(y) \quad \textit{for every } y \in \mathbb{R}, \tag{1.16}$$

$$\phi \textit{ attains its maximum value } \phi_0 \textit{ at } y = 0, \tag{1.17}$$

and such that writing $\beta = \pi^{1/2} b$, we have

$$\frac{1}{2b}\int_0^b \phi(1 + e^\phi)\, dy \; + \; \frac{\beta^2}{3} \; - \; \log(4\beta^2) \; + \; 1 \; - \; 4\sum_{n=1}^\infty \log\left|1 - e^{-2n\beta^2}\right| \; < \; 0. \tag{1.18}$$

Remarks. 1. The condition (1.17) is just thrown in to eliminate the degree of freedom given by translation invariance. In fact we choose ϕ to have smallest period b, which together with (1.15) and (1.17) determines ϕ uniquely.

2. In proving Theorem 1′, we will establish a relationship between the maximum value ϕ_0 of ϕ and the period b. A simplified version is that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$e^{\phi_0} + \log 4 + \varepsilon_1 e^{-\phi_0} \; \le \; \pi b^2 \; \le \; e^{\phi_0} + \log 4 + \varepsilon_2 e^{-\phi_0}, \qquad \text{for } b \ge \left(\frac{\pi}{2}\right)^{1/2}.$$

The precise version is that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$e^{\phi_0} - \phi_0 + \log 4 + \varepsilon_1 e^{-\phi_0} \; \le \; \pi b^2 - \log(\pi b^2) \; \le \; e^{\phi_0} - \phi_0 + \log 4 + \varepsilon_2 e^{-\phi_0}, \qquad \text{for } b \ge \left(\frac{\pi}{2}\right)^{1/2}. \tag{1.19}$$

3. In [DS2], conformal factors were chosen for long skinny flat tori of area 1, so that as the length of the flat torus tends to infinity, the Robin mass of the new metric converges to that of the round sphere. From [O2], one sees this can easily be accomplished by conformal factors which concentrate at a point, but the conformal factors in [DS2] depend only on the length variable y. In this paper we choose conformal factors which minimize the Robin mass among one-variable candidates, yielding optimal metrics which beat the mass of the sphere on every torus. It is unknown whether our conformal factors give the true minimizer in any case.

The rest of the paper is dedicated to proving Theorem 1′. We begin by giving a summary of the proof, and then supply the details,

Outline of the proof of Theorem 1′.

In Proposition 2.1, we will show that for $b > \sqrt{\pi/2}$, there exists a unique function ϕ satisfying (1.15)–(1.17) and having smallest period b. Moreover, the initial condition ϕ_0 increases with b. Next write

$$\beta \; = \; \sqrt{\pi}\, b, \qquad f_0 \; = \; e^{\phi_0} \; - \; \phi_0, \qquad M \; = \; \frac{1}{2b}\int_0^b \phi(1 + e^\phi)\, dy. \tag{1.20}$$

Let us emphasize that although we are now using 4 variables, b, β, ϕ_0, f_0, each one is an increasing function of any of the others. The non-trivial relationship between them is the differential equation which relates b to ϕ_0. We are trying to prove inequality 1.18, which we write as

$$M \; + \; \frac{\beta^2}{3} \; - \; \log(4\beta^2) \; + \; 1 \; - \; 4\sum_{n=1}^\infty \log\left|1 - e^{-2n\beta^2}\right| \; < \; 0. \tag{1.21}$$

In Proposition 2.4 we show that

$$\frac{d(\beta M)}{d\beta} \;=\; 1 \;-\; f_0. \tag{1.22}$$

We then investigate how f_0 behaves as a function of β, so that we can estimate the left hand side of (1.21). Set

$$\begin{aligned} \varepsilon(\beta) \;&=\; \beta^2 \;-\; \log(4\beta^2) \;-\; f_0 \\ &=\; \frac{d(\beta M)}{d\beta} \;+\; \beta^2 \;-\; \log(4\beta^2) \;-\; 1. \end{aligned} \tag{1.23}$$

We will prove the three key estimates, (1.24)–(1.26). Set β_1 to be the value of β corresponding to the initial value $\phi_0 = \log 5$.

$$\varepsilon(\beta) \;>\; 0, \qquad \text{for} \quad \frac{\pi}{2^{1/2}} < \beta \leq \beta_1, \tag{1.24}$$

$$\varepsilon(\beta) \;>\; \frac{0.03}{\beta^2} \qquad \text{for} \quad \beta_1 \leq \beta. \tag{1.25}$$

For some $\gamma > 0$, we have

$$\varepsilon(\beta) \;<\; \frac{\gamma}{\beta^2}, \qquad \text{for } \frac{\pi}{2^{1/2}} < \beta. \tag{1.26}$$

Thus $\varepsilon(\beta)$ is integrable. For the proof of (1.24), see Proposition 2.6–Corollary 2.8. For the other two inequalities, see Lemma 2.9 and Proposition 2.10.

In Corollary 2.5, we obtain a simple upper bound on β in terms of ϕ_0 which yields

$$\beta_1 \;\leq\; 3.8, \qquad \text{for } \phi_0 \leq \log 5.$$

Hence integrating (1.24), (1.25) from β to infinity yields

$$\frac{1}{\beta}\int_\beta^\infty \varepsilon(\tilde\beta)\, d\tilde\beta \;>\; \frac{0.01}{\beta^2}, \qquad \text{for} \quad \beta > \frac{\pi}{2^{1/2}}. \tag{1.27}$$

Now integrating (1.23) gives

$$M \;+\; \frac{\beta^2}{3} \;-\; \log(4\beta^2) \;+\; 1 \;=\; \frac{C}{\beta} \;-\; \frac{1}{\beta}\int_\beta^\infty \varepsilon(\tilde\beta)\, d\tilde\beta, \tag{1.28}$$

where C is the constant of integration. In Proposition 2.11 we rework some of the asymptotic formulas required in the proof of (1.25)-(1.26) to show that $C = 0$. Hence combining this with (1.27) gives

$$M \;+\; \frac{\beta^2}{3} \;-\; \log(4\beta^2) \;+\; 1 \;\leq\; -\frac{0.01}{\beta^2}, \qquad \text{for} \quad \beta > \frac{\pi}{2^{1/2}}. \tag{1.29}$$

Finally, one can check with a simple numerical calculation that

$$-4\sum_{n=1}^{\infty} \log\left|1 - e^{-2n\beta^2}\right| \;<\; \frac{0.002}{\beta^2}, \tag{1.30}$$

holds at the value $\beta = \pi/2^{1/2}$. But then in Lemma 2.12 we see that (1.30) must hold at all values $\beta > \pi/2^{1/2}$. Adding (1.29) and (1.30) gives (1.21), thus completing the proof of Theorem 1′.

Now we fill in the results stated in the outline to complete the proof.

Section 2. Auxiliary Results and Proofs.

Proposition 2.1. *There exists a smooth function* $\psi : \left(\sqrt{\pi/2}\ ,\ \infty\right) \times \mathbb{R} \to \mathbb{R}$ *such that for each fixed* $b \in (\sqrt{\pi/2}, \infty)$ *the function*

$$\phi(y) = \psi(b, y)$$

satisfies (1.15)–(1.17), has smallest period b*, and attains its minimum value at* $y = b/2$*. Moreover, writing*

$$f(\phi) \ = \ e^{\phi} \ - \ \phi,$$

ϕ *is also characterized by having period* b *and satisfying the following two conditions:*

$$\phi(-y) = \phi(y), \tag{2.1}$$

$$y \ = \ \frac{1}{4\sqrt{\pi}} \int_{\phi(y)}^{\phi_0} \frac{d\phi}{\sqrt{f_0 - f(\phi)}}, \qquad\qquad y \in (0, b/2). \tag{2.2}$$

Furthermore, the map

$$b \ \mapsto \ \phi_0 \ = \ \psi(b, 0)$$

is smooth from the interval $(\sqrt{\pi/2}, \infty)$ *onto the interval* $(0, \infty)$*, and*

$$\frac{db}{d\phi_0} \ > \ 0.$$

Remarks. 1. Every solution of (1.15)–(1.17) has the form

$$\phi(y) \ = \ \psi(b/n, y),$$

for some $n \in \mathbb{N}$.

2. By making the change of variables $h = e^{\phi}$, and $d\alpha = e^{\phi}\, dy$, we can transform equation (2.4) to

$$\frac{d^2 h}{d\alpha^2} \ = \ 8\pi \left(1 - \frac{1}{h}\right). \tag{2.3}$$

Now $d\alpha$ is a measure of the change in area, and in some respects it turns out to be more natural to analyze (2.3) than (1.15). However, we will require a delicate estimate on the relationship between b and ϕ_0, and although we work with the variable h at some points, there are places where it is better to work with (1.15). (For example Proposition 2.4.)

Proof of Proposition 2.1. This result is standard and is part of the standard theory of ordinary differential equations, see for example [A] and [Chi]. We give the proof here to set up notation for later. For $\phi \in \mathbb{R}$, set

$$f(\phi) = e^{\phi} - \phi.$$

We start by constructing the inverse of f. Indeed, f maps $\mathbb{R}$ onto $[1, \infty)$, and for each $f_1 \in [1, \infty)$ there exist at most two solutions of the equation $f(\phi) = f_1$, given by $\phi = \phi_*(f_1)$ and $\phi = \phi^*(f_1)$, where

$$\phi_*(f_1) \leq 0, \qquad\qquad \phi^*(f_1) \geq 0. \tag{2.4}$$

For $\phi_0 > 0$, we consider the initial value problem

$$\frac{d^2\phi}{dy^2} = 8\pi\left(1 - e^{\phi}\right), \tag{2.5}$$

$$\phi(0) = \phi_0. \tag{2.6}$$

$$\frac{d\phi}{dy}(0) = 0, \tag{2.7}$$

Set

$$f_0 = f(\phi_0). \tag{2.8}$$

Multiplying (2.5) by $d\phi/dy$ and integrating from $y = 0$ gives

$$\left(\frac{d\phi}{dy}\right)^2 = 16\pi\left(e^{\phi_0} - \phi_0 - (e^{\phi} - \phi)\right) = 16\pi(f_0 - f(\phi)). \tag{2.9}$$

Hence

$$\frac{dy}{d\phi} = \frac{\pm 1}{4\sqrt{\pi}} \frac{1}{\sqrt{f_0 - f(\phi)}}. \tag{2.10}$$

Set

$$\ell = \ell(\phi_0) := \frac{1}{4\sqrt{\pi}} \int_{\phi_*(f_0)}^{\phi_0} \frac{d\phi}{\sqrt{f_0 - f(\phi)}}. \tag{2.11}$$

Then the function $\phi(y)$, assuming it exists, satisfies

$$y = I(\phi(y)) \qquad \text{for } 0 \le y \le \ell, \qquad \text{where} \qquad I(z) = \frac{1}{4\sqrt{\pi}} \int_{z}^{\phi_0} \frac{d\phi}{\sqrt{f_0 - f(\phi)}}. \tag{2.12}$$

Defining ϕ to be the inverse of the function I, we find that ϕ is decreasing and smooth on $(0, \ell)$ and it extends to be continuously differentiable on $[0, \ell]$, and satisfies

$$\phi(0) = \phi_0, \qquad \phi(\ell) = \phi_*(f(\phi_0)), \qquad \frac{d\phi}{dy}(0) = \frac{d\phi}{dy}(\ell) = 0.$$

We now extend ϕ to $[-\ell, \ell]$ by requiring that it is even, that is $\phi(-y) = \phi(y)$, and then we extend it to $\mathbb{R}$ by requiring that it is periodic with period 2ℓ. The result is an even, continuously differentiable, periodic function on $\mathbb{R}$ whose smallest period is 2ℓ, and which is smooth on $\mathbb{R} \setminus 2\ell\mathbb{Z}$ and satisfies (2.5) there, and which attains its maximum value at $y = 0$ and its minimum value at $y = \ell$. Now by the general theorem on the uniqueness and smoothness of solutions to ordinary differential equations, this solution ϕ is smooth and satisfies (2.5) everywhere on $\mathbb{R}$. Moreover by the smooth dependence of solutions to ordinary differential equations on the initial conditions, we see that defining

$$\eta(\phi_0, y) = \phi(y), \qquad \text{where } \phi \text{ satisfies (2.5)–(2.7)},$$

then $\eta \in C^\infty((0,\infty)\times\mathbb{R})$. The final step is to show that the function

$$\phi_0 \;\to\; b = 2\ell(\phi_0)$$

is smooth and bijective from $(0,\infty)$ to $(\sqrt{\pi/2},\infty)$, with

$$\frac{db}{d\phi_0} > 0,$$

so the inverse function

$$b \;\to\; \phi_0(b)$$

is smooth and bijective from $(\sqrt{\pi/2},\infty)$ to $(0,\infty)$. We then define the function ψ by

$$\psi(b,y) \;=\; \eta(\phi_0(b),y).$$

Proposition 2.1 is thus reduced to the following.

Proposition 2.2. *The function $\beta : [1,\infty) \to [0,\infty)$ defined by*

$$\beta(f_0) \;:=\; \frac{1}{2}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{d\phi}{\sqrt{f_0 - f(\phi)}} \tag{2.13}$$

is a smooth function mapping $(1,\infty)$ bijectively onto $\left(\pi/\sqrt{2}\;,\;\infty\right)$, with

$$\frac{d\beta}{df_0} \;>\; 0, \qquad \textit{on} \qquad (1,\infty).$$

Proof. See [Chi] for a general proof of this result. See also [ChiJ]. We include the proof here to develop properties of the variable $J = j^* + j_*$ which will be useful later on. To reduce the need for notation, it is convenient to work with physical variables rather than functions. (To be more precise, we suppose that there is a fixed underlying "physical" space which we don't need to specify. A variable is then a continuous function defined on this space.) We suppose then that ϕ is a variable taking values in $\mathbb{R}$, and f and h are variables related to ϕ by

$$f = e^\phi - \phi, \qquad h = e^\phi, \qquad \phi = \log h, \qquad f = h - \log h. \tag{2.14}$$

The variables f and h take values in $[1,\infty)$ and $(0,\infty)$ respectively. Given a value for f, we write $\phi^* \geq 0$ and $\phi_* \leq 0$ for the two corresponding values for ϕ and set

$$h^* = e^{\phi^*}, \qquad h_* = e^{\phi_*}. \tag{2.15}$$

When $f = 1$ we have $\phi^* = \phi_* = 0$ and $h^* = h_* = 1$. For other values of f the values of ϕ^* and ϕ_* are distinct. Then making a change of variables,

$$\begin{aligned}
\frac{1}{2}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{d\phi}{(f_0 - f)^{1/2}} \;&=\; \frac{1}{2}\int_1^{f_0} \frac{1}{(f_0-f)^{1/2}}\left(\frac{1}{e^{\phi^*(f)}-1} \;+\; \frac{1}{1-e^{\phi_*(f)}}\right) df \\
&=\; \frac{1}{2}\int_1^{f_0} \frac{1}{(f_0-f)^{1/2}}\left(\frac{1}{h^*(f)-1} \;+\; \frac{1}{1-h_*(f)}\right) df.
\end{aligned} \tag{2.16}$$

We will now analyze the Jacobian factor in (2.16) and modify it to obtain a positive monotonically increasing function of f.

Lemma 2.3. *Define variables* j^* *and* j_* *by*

$$j^* \;=\; \frac{1}{h^*-1} \;-\; \frac{1}{(2(f-1))^{1/2}}, \qquad\qquad j_* \;=\; \frac{1}{1-h_*} \;-\; \frac{1}{(2(f-1))^{1/2}}. \tag{2.17}$$

Then

(a) As $f \to 1$,

$$j^* \;\to\; -\frac{1}{3}, \qquad\qquad j_* \;\to\; \frac{1}{3}.$$

(b) The variables j^* *and* j_* *are increasing with* f, *indeed*

$$\frac{dj^*}{df} > 0, \qquad \frac{dj_*}{df} > 0, \qquad\qquad \textit{for } f > 1,$$

and

$$\frac{dj^*}{df} \;=\; O((f-1)^{-1/2}), \qquad \frac{dj_*}{df} \;=\; O((f-1)^{-1/2}), \qquad\qquad \textit{as } f \to 1.$$

(c) As functions of the variable f, *the variables* j^* *and* j_* *are concave. More precisely,*

$$\frac{d^2 j^*}{df^2} < 0, \qquad \frac{d^2 j_*}{df^2} < 0, \qquad\qquad \textit{for } f > 1.$$

(d) The variable

$$j^* + j_* \;=\; \frac{1}{h^*-1} \;+\; \frac{1}{1-h_*} \;-\; \left(\frac{2}{f-1}\right)^{1/2} \tag{2.18}$$

satisfies

$$\frac{d(j^*+j_*)}{df} \;>\; 0, \qquad\qquad \frac{d^2(j^*+j_*)}{df^2} \;<\; 0, \qquad\qquad \textit{for } f > 1,$$

and

$$(j^*+j_*) \to 0, \qquad\qquad \frac{d(j^*+j_*)}{df} \;=\; O((f-1)^{-1/2}), \qquad\qquad \textit{as } f \to 1.$$

(e)

$$0 \;<\; j^* + j_* \;<\; 1, \qquad\qquad \textit{when} \quad f > 1.$$

Proof of Lemma 2.3. Clearly (d) follows from (a), (b) and (c). Moreover, see from (2.18) that $j^* + j_* \to 1$, as $f \to \infty$, so (e) follows from (d).

(a) Dealing with the variables j^* and j_* simultaneously, note that as $h \to 1$, we have

$$\begin{aligned} \frac{1}{|h-1|} \;-\; \frac{1}{(2(f-1))^{1/2}} \;&=\; \frac{1}{|h-1|} \;-\; \frac{1}{(2(h-1-\log(1-(1-h))))^{1/2}} \\ &=\; \frac{1}{|h-1|} \;-\; \frac{1}{((1-h)^2 \;+\; 2(1-h)^3/3 \;+\; 2(1-h)^4/4 \;+\; \dots)^{1/2}} \end{aligned} \tag{2.19}$$

$$\to \begin{cases} -1/3 & \text{as } h \downarrow 1, \\ 1/3 & \text{as } h \uparrow 1. \end{cases}$$

(b) We need to show that

$$\frac{d}{df}\left(\frac{1}{|h-1|} - \frac{1}{(2(f-1))^{1/2}}\right) > 0, \qquad \text{when} \quad f > 1. \tag{2.20}$$

Note that

$$\frac{dh}{df} = \frac{h}{h-1}. \tag{2.21}$$

We thus compute the sign of the derivative

$$\frac{d}{df}\left(\frac{1}{|h-1|} - \frac{1}{(2(f-1))^{1/2}}\right) = \frac{-\operatorname{sign}(h-1)}{(h-1)^2}\frac{dh}{df} + \frac{1}{(2(f-1))^{3/2}}$$
$$= \frac{-h}{|h-1|^3} + \frac{1}{(2(f-1))^{3/2}}.$$

Hence (2.20) will follow if we can show that

$$\frac{|h-1|^3}{h} > (2(f-1))^{3/2}, \qquad \text{for } h \neq 1,$$

equivalently

$$h^{-2/3}(h-1)^2 > 2(f-1), \qquad \text{for } h \neq 1. \tag{2.22}$$

But this indeed holds, since

$$h^{-2/3}(h-1)^2 - 2(f-1) \tag{2.23}$$

equals zero at $h = 1$, and

$$\frac{d}{df}\left(h^{-2/3}(h-1)^2 - 2(f-1)\right) = \frac{2h^{-2/3}(2h+1)}{3} - 2 > 0 \qquad \text{for } f > 1.$$

Indeed,

$$h^{-2/3}(2h+1) > 3 \qquad \text{for } f > 1,$$

as one can easily check by cubing both sides or differentiating once more with respect to f. The behavior of the derivative as $f \to 1$ is obtained with a Taylor expansion as in (2.19).

(c) We compute

$$\frac{d^2}{df^2}\left(\frac{1}{|h-1|} - \frac{1}{(2(f-1))^{1/2}}\right) = \frac{d}{df}\left(\frac{-h}{|h-1|^3} + \frac{1}{(2(f-1))^{3/2}}\right)$$
$$= \frac{h(2h+1)}{|h-1|^5} - \frac{3}{(2(f-1))^{5/2}}.$$

In order to show that this is negative, we need to show

$$\frac{(2(f-1))^{5/2}}{3} \;<\; \frac{|h-1|^5}{h(2h+1)} \qquad\qquad \text{for } f>1,$$

or equivalently we need to show

$$2(f-1) \;<\; 3^{2/5}(h(2h+1))^{-2/5}(h-1)^2 \qquad\qquad \text{for } f>1. \tag{2.24}$$

Now defining

$$\tau \;=\; 3^{2/5}(h(2h+1))^{-2/5}(h-1)^2 \;-\; 2(f-1), \tag{2.25}$$

we see that τ vanishes at $f=1$. Differentiating with respect to f we get

$$\frac{d\tau}{df} \;=\; \frac{2\cdot 3^{2/5}}{5}(2h^2+h)^{-7/5}h\,(6h^2+8h+1) \;-\; 2, \tag{2.26}$$

which also vanishes at $f=1$. To show that this is positive, we compute

$$\frac{5}{2\cdot 3^{2/5}}\frac{d^2\tau}{df^2} \;=\; \frac{2h^2(6h^2-2h+1)}{5(2h^2+h)^{12/5}} \;>\; 0 \qquad\qquad \text{for } h>0. \qquad \square$$

Now we can complete the proof of Proposition 2.2. Introduce the function $J:[1,\infty)\to\mathbb{R}$ such that

$$j^* + j_* = J(f).$$

We see that β is smooth by fixing c with $1<c<f_0$ and writing

$$\begin{aligned}\beta(f_0) \;&=\; \frac{1}{2}\int_1^{f_0}\frac{1}{(f_0-f)^{1/2}}\left(\frac{2}{(2(f-1))^{1/2}} \;+\; J(f)\right)df \;=\; \frac{\pi}{2^{1/2}} \;+\; \frac{1}{2}\int_1^{f_0}\frac{J(f)}{(f_0-f)^{1/2}}\,df \\ &=\; \frac{\pi}{2^{1/2}} \;+\; \frac{1}{2}\int_1^{c}\frac{J(f)}{(f_0-f)^{1/2}}\,df \;+\; \frac{1}{2}\int_0^{f_0-c}\frac{J(f_0-f)}{f^{1/2}}\,df.\end{aligned} \tag{2.27}$$

Since J is smooth away from 1, both integrals on the right can be differentiated repeatedly in f_0, and we see β is smooth in f_0. Differentiating and letting $c\to 0$ gives

$$\frac{d\beta(f_0)}{df_0} \;=\; \frac{1}{2}\int_0^{f_0-1}\frac{J'(f_0-f)}{f^{1/2}}\,df \;>\; 0. \qquad \square$$

Our mission is to compute the quantity M in terms of β, and we will prove (1.22) relating M to f_0. We rescale the function ϕ to have period 2, by taking the solution ψ from Proposition 2.1, and setting

$$\rho(b,s) \;=\; \psi(b, bs/2),$$

so that for b fixed, the function $s\mapsto\rho(b,s)$ is even, and attains its maximum value at $s=0$, and

$$\frac{\partial^2\rho}{\partial s^2} \;=\; 2\beta^2(1-e^{\rho}), \tag{2.28}$$

The solution ρ is a smooth function of (β,s), and we are interested in the quantity M, defined in (1.20). Setting $f_0 = f(\phi_0) = e^{\phi_0}-\phi_0$, we have from the definition (1.20), the symmetry of ϕ, and (2.10),

$$M \;=\; \frac{1}{b}\int_0^{b/2}\phi(1+e^{\phi})\,dy \;=\; \frac{1}{2}\int_0^{1}\rho(1+e^{\rho})\,ds \;=\; \frac{1}{4\beta}\int_{\phi_*(f_0))}^{\phi_0}\frac{\phi(1+e^{\phi})}{(f_0-f(\phi))^{1/2}}\,d\phi. \tag{2.29}$$

Proposition 2.4. *(a)*

$$\frac{dM}{d\beta} \;=\; \frac{1}{\beta}\int_0^1 \rho(1-e^\rho)\,ds.$$

(b)

$$\frac{d(\beta M)}{d\beta} \;=\; \frac{1}{2}\int_0^1 \rho(3-e^\rho)\,ds \;=\; 1-f_0.$$

Proof. (a) We differentiate (2.28) to obtain

$$\frac{\partial^2}{\partial s^2}\frac{\partial\rho}{\partial\beta} \;=\; 4\beta(1-e^\rho) \;-\; 2\beta^2\frac{\partial\rho}{\partial\beta}e^\rho. \tag{2.30}$$

Integrating (2.30) we get

$$\int_0^1 \frac{\partial\rho}{\partial\beta}e^\rho\,ds \;=\; 0. \tag{2.31}$$

Hence

$$\frac{dM}{d\beta} \;=\; \frac{1}{2}\int_0^1 \frac{d\rho}{d\beta}(1+e^\rho+\rho e^\rho)\,ds \;=\; \frac{1}{2}\int_0^1 \frac{d\rho}{d\beta}(1-e^\rho+\rho e^\rho)\,ds. \tag{2.32}$$

However, integrating (2.30) against ρ, we get

$$\int_0^1 \frac{\partial\rho}{\partial\beta}\frac{\partial^2\rho}{ds^2}\,ds \;=\; 4\beta\int_0^1 \rho(1-e^\rho)\,ds \;-\; 2\beta^2\int_0^1 \frac{\partial\rho}{\partial\beta}\rho e^\rho\,ds.$$

Hence using the equation (2.28), we get

$$2\beta^2\int_0^1 \frac{\partial\rho}{\partial\beta}(1-e^\rho)\,ds \;=\; -2\beta^2\int_0^1 \frac{\partial\rho}{\partial\beta}\rho e^\rho\,ds \;+\; 4\beta\int_0^1 \rho(1-e^\rho)\,ds.$$

Hence

$$\frac{1}{2}\int_0^1 \frac{\partial\rho}{\partial\beta}(1-e^\rho+\rho e^\rho)\,ds \;=\; \frac{1}{\beta}\int_0^1 \rho(1-e^\rho)\,ds.$$

Combining this with (2.32) gives (a).

(b) The first equality follows directly from (a). For the second, we multiply (2.28) by $d\rho/ds$ and integrating as in (2.9), to get

$$\left(\frac{\partial\rho}{\partial s}\right)^2 \;=\; 4\beta^2(f_0+\rho-e^\rho).$$

But then

$$\frac{1}{2}\int_0^1 \rho(1-e^\rho)\,ds \;=\; \frac{1}{4\beta^2}\int_0^1 \rho\frac{\partial^2\rho}{\partial s^2}\,ds \;=\; -\frac{1}{4\beta^2}\int_0^1 \left(\frac{\partial\rho}{\partial s}\right)^2 ds$$

$$=\; -\int_0^1 (f_0+\rho-e^\rho)\,ds \;=\; 1-f_0-\int_0^1 \rho\,ds. \qquad \square$$

Corollary 2.5.

$$\beta \;\le\; \frac{\pi}{2^{1/2}} \;+\; (f_0-1)^{1/2}.$$

Proof. From (2.27) and Lemma 2.3 (e), we have

$$\beta(f_0) \;=\; \frac{\pi}{2^{1/2}} \;+\; \frac{1}{2}\int_1^{f_0} \frac{J(f)}{(f_0-f)^{1/2}}\,df \;\le\; \frac{\pi}{2^{1/2}} \;+\; \frac{1}{2}\int_1^{f_0} \frac{1}{(f_0-f)^{1/2}}\,df \;=\; \frac{\pi}{2^{1/2}} \;+\; (f_0-1)^{1/2}.$$

Our task now is to work towards the estimate in (1.24). This inequality can be checked quite carefully using mathematica, but we give a concise analytic proof with minimal computation.

Proposition 2.7. *Given a constant $\lambda > 0$, define functions $V, W : [1,\infty) \to \mathbb{R}$ by*

$$\begin{aligned} V(f) \;&=\; \frac{\pi}{2^{1/2}} \;+\; \lambda(f-1)^{3/2}, \\ W(f) \;&=\; V(f)^2 \;-\; \log(4V(f)^2) \;-\; f. \end{aligned}$$

Suppose that for $f_1 > 1$ fixed, there exists λ such that

(a) $$0 < \lambda < \frac{2J(f_1)}{3(f_1-1)},$$

(b) $$W(f_1) > 0,$$

(c) $$W'(f_1) < 0.$$

Then writing $\beta = \beta(f_0)$ for the function defined in (2.13), we have

$$\beta^2 - \log(4\beta^2) - f_0 > 0, \qquad\qquad 1 < f_0 < f_1.$$

Proof. First we show that $W''(f) > 0$ for $f \ge 1$. Indeed, note that $V(f) > 1$ and $V''(f) > 0$, and

$$W'(f) \;=\; 2\left(V(f) - \frac{1}{V(f)}\right)V'(f) \;-\; 1,$$

$$W''(f) \;=\; 2\left(1 + \frac{1}{V(f)^2}\right)(V'(f))^2 \;+\; 2\left(V(f) - \frac{1}{V(f)}\right)V''(f) \;>\; 0.$$

Next note that $W'' > 0$ combined with (c) shows that W is decreasing on $[1, f_1]$, and this combined with (b) shows that $W(f_0) > 0$ for $1 < f_0 < f_1$.

Now we show that for $1 < f_0 < f_1$ we have $\beta(f_0) > V(f_0)$. Indeed, comparing the concave function $J(f)$ with the linear function, we get

$$J(f) \;>\; \frac{J(f_1)}{f_1-1}(f-1), \qquad\qquad 1 < f < f_1.$$

Substituting $t = (f-1)/(f_0-1)$ we get

$$\beta(f_0) \;=\; \frac{\pi}{2^{1/2}} \;+\; \frac{1}{2}\int_1^{f_0} \frac{J(f)}{(f_0-f)^{1/2}}\,df$$

$$\geq\; \frac{\pi}{2^{1/2}} \;+\; \frac{J(f_1)}{2(f_1-1)}\int_1^{f_0} \frac{f-1}{(f_0-f)^{1/2}}\,df \;=\; \frac{\pi}{2^{1/2}} \;+\; \frac{J(f_1)(f_0-1)^{3/2}}{2(f_1-1)}\int_0^1 \frac{t}{(1-t)^{1/2}}\,dt$$

$$=\; \frac{\pi}{2^{1/2}} \;+\; \frac{2J(f_1)(f_0-1)^{3/2}}{3(f_1-1)} \;>\; \frac{\pi}{2^{1/2}} \;+\; \lambda(f_0-1)^{3/2} \;=\; V(f_0).$$

Hence we have

$$\beta^2 \;-\; \log(4\beta^2) \;-\; f_0 \;>\; V(f_0)^2 \;-\; \log(4V(f_0)) \;-\; f_0 \;=\; W(f_0) \;>\; 0.$$

Remark. It will be useful to know the formula

$$h_* \;=\; \sum_{j=0}^{\infty} \frac{(j+1)^{j-1}}{j!}\; e^{-(j+1)f},$$

although we will not prove it or depend on it.

Lemma 2.8. *For $\lambda = 0.098$, the conditions of Proposition 2.7 are satisfied for $f_1 = 5 - \log 5$.*

Proof of Lemma 2.8. Step 1: For $h = 5$, write $5_* = h_*$. Then numerical calculation shows that

$$5 - \log(5) = f(5) \;<\; f(0.034) = 0.034 - \log(0.034).$$

Hence

$$5_* > 0.034,$$

and

$$(j+j_*)(f(5)) \;=\; \frac{1}{5-1} \;+\; \frac{1}{1-5_*} \;-\; \left(\frac{2}{5-\log 5-1}\right)^{1/2} \;\geq\; \frac{1}{4} \;+\; \frac{1}{1-0.034} \;-\; \left(\frac{2}{4-\log 5}\right)^{1/2} \;=\; 0.3705...,$$

and the right hand term in Proposition 2.6 (a) is

$$\frac{2(j+j_*)(f(5))}{3(4-\log 5)} \;=\; 0.1033.... \;>\; 0.098.$$

Step 2:

$$V(f(5)) \;=\; \frac{\pi}{2^{1/2}} \;+\; \lambda(4-\log 5)^{3/2} \;=\; 2.583664...$$

so

$$2.58366 \;<\; V(f(5)) \;<\; 2.584.$$

Hence at the value $f_1 = 5 - \log 5$ we have

$$W \;=\; V^2 \;-\; \log(4V^2) \;-\; (5-\log 5) \;\geq\; 2.58366^2 - \log(4\times 2.58366^2) - (5-\log 5) \;=\; 0.00002... \;>\; 0,$$

while

$$W' \;=\; 3\lambda\left(V - \frac{1}{V}\right)(4{-}\log 5)^{1/2} - 1 \;<\; 0.294\left(2.584 - \frac{1}{2.584}\right)(4{-}\log 5)^{1/2} - 1 \;=\; -0.001... \;<\; 0. \quad \square$$

Corollary 2.8. *Set* $\beta_1 = \beta(5-\log 5)$. *For* $\beta = \beta(f_0)$, *we set*

$$\varepsilon(\beta) \;=\; \beta^2 \;-\; \log(4\beta^2) \;-\; f_0.$$

Then

$$\varepsilon(\beta) \;>\; 0, \qquad\qquad \textit{if} \quad \frac{\pi}{2^{1/2}} < \beta < \beta_1.$$

Proof. By Lemma 2.7, if $1 < f_0 < f_1 = 5 - \log 5$, then $\varepsilon(\beta) > 0$.

Now we will investigate more precisely how $b(f_0)$ depends on f_0 as $f_0 \to \infty$. We will use the fact that we only have positive Taylor coefficients in the expansion

$$(1-x)^{-1/2} \;=\; \sum_0^\infty \gamma_k x^k, \qquad\qquad \gamma_k = \frac{(2k)!}{2^{2k}(k!)^2} \sim \frac{1}{\sqrt{\pi k}}.$$

From (2.5) and the fact that ϕ is even and periodic with period b, we have

$$\int_0^{b/2} (1-e^\phi)\,dy \;=\; 0.$$

Hence using (2.10) and setting $h = e^\phi$, we get

$$\beta \;=\; \sqrt{\pi}b \;=\; 2\sqrt{\pi}\int_0^{b/2} e^\phi\,dy \;=\; \frac{1}{2}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{e^\phi\,d\phi}{\sqrt{f_0 - f(\phi)}} \;=\; \frac{1}{2}\int_{h_*(f_0)}^{h^*(f_0)} \frac{dh}{\sqrt{f_0-(h-\log h)}}. \tag{2.33}$$

Using the notation of (2.14), (2.15) and writing $h_0 = h^*(f_0)$ and $h_{0*} = h_*(f_0)$, so $f_0 = h_0 - \log h_0 = h_{0*} - \log h_{0*}$, and setting $t = 1 - h/h_0$, we get

$$\begin{aligned}
\beta \;&=\; \frac{1}{2}\int_{h_{0*}}^{h_0} \left(h_0 - h + \log\frac{h}{h_0}\right)^{-1/2} dh \\
&=\; \frac{h_0}{2}\int_0^{1-h_{0*}/h_0} (h_0 t \;+\; \log(1-t))^{-1/2}\;dt \\
&=\; \frac{h_0}{2(h_0-1)^{1/2}}\int_0^{1-h_{0*}/h_0} t^{-1/2}\left(1 - \frac{-\log(1-t)-t}{(h_0-1)t}\right)^{-1/2} dt \qquad (2.34)\\
&=\; \frac{h_0}{2(h_0-1)^{1/2}}\sum_{k=0}^\infty \frac{\gamma_k}{(h_0-1)^k}\int_0^{1-h_{0*}/h_0} t^{-1/2}\left(\frac{-\log(1-t)-t}{t}\right)^k dt \\
&=\; \frac{h_0}{(h_0-1)^{1/2}}\sum_{k=0}^\infty \frac{\gamma_k\,\mu_k(1-h_{0*}/h_0)}{(h_0-1)^k}, \qquad (2.35)
\end{aligned}$$

where the series converges by monotone convergence, and

$$\mu_k(\tau) \;=\; \frac{1}{2}\int_0^\tau t^{-1/2}\left(\frac{-\log(1-t)-t}{t}\right)^k dt. \tag{2.36}$$

Clearly $\mu_k(\tau)$ is an increasing function of τ which is strictly positive for $\tau \in (0,1]$. Now

$$\beta^2 \;=\; \frac{h_0^2}{h_0-1}\sum_{k=0}^{\infty}\frac{\nu_k(1-h_{0*}/h_0)}{(h_0-1)^k}, \qquad\qquad \nu_k(\tau) \;=\; \sum_{j=0}^{k}\gamma_j\,\gamma_{k-j}\,\mu_j(\tau)\,\mu_{k-j}(\tau). \tag{2.37}$$

Clearly $\nu_j(\tau)$ is also positive. It is easy to compute

$$\mu_0(\tau) \;=\; \tau^{1/2}, \qquad\qquad \nu_0(\tau) \;=\; \tau.$$

Hence just taking the first term in (2.37) gives

$$\beta^2 \;>\; \frac{h_0^2\nu_0(1-h_{0*}/h_0)}{h_0-1} \;=\; \frac{h_0(h_0-h_{0*})}{h_0-1} \;>\; h_0. \tag{2.38}$$

Now applying the Mean Value Theorem to the function $w \mapsto w - \log w$, we have

$$\begin{aligned}\beta^2-\log(\beta^2) \;-\; (h_0-\log h_0) \;\geq\; \frac{h_0-1}{h_0}(\beta^2-h_0) \;&=\; \sum_{k=0}^{\infty}\frac{h_0\;\nu_k(1-h_{0*}/h_0)}{(h_0-1)^k} \;-\; h_0 \;+\; 1\\ &\geq\; 1 \;-\; h_{0*} \;+\; \frac{h_0\;\nu_1(1-h_{0*}/h_0)}{h_0-1} \;+\; \frac{h_0\;\nu_2(1-h_{0*}/h_0)}{(h_0-1)^2}.\end{aligned} \tag{2.39}$$

Lemma 2.9. *If $h_0 \geq 5$ then*

(a) $$1 \;-\; h_{0*} \;+\; \frac{h_0\;\nu_1(1-h_{0*}/h_0)}{h_0-1} \;>\; 2\log 2,$$

and

(b) $$\frac{h_0\;\nu_2(1-h_{0*}/h_0)}{(h_0-1)^2} \;>\; \frac{0.03}{\beta^2}$$

so

(c) $$\varepsilon(\beta) \;=\; \beta^2-\log(4\beta^2) \;-\; f_0 \;>\; \frac{0.03}{\beta^2}$$

Proof. (a) Now evaluating (2.36) for $k=1$,

$$\mu_1(\tau) \;=\; 2\log(1+\tau^{1/2}) \;-\; \tau^{1/2} \;+\; (\tau^{-1/2}-1)\log(1-\tau), \tag{2.40}$$

so we have

$$\nu_1(\tau) \;=\; \mu_0(\tau)\mu_1(\tau) \;=\; 2\tau^{1/2}\log(1+\tau^{1/2}) \;-\; \tau \;+\; (1-\tau^{1/2})\log(1-\tau). \tag{2.41}$$

We will estimate the terms on the right hand side. Since $\log(1-\tau) < 0$, we have

$$(1-\tau^{1/2})\log(1-\tau) \;>\; (1-\tau)\log(1-\tau),$$

and by the convexity of the logarithm we have $\log(1+x) > x\log 2$ for $0 < x < 1$, and so

$$\tau^{1/2}\log(1+\tau^{1/2}) \;>\; \tau\log 2.$$

Hence substituting these inequalities into (2.41),

$$\nu_1(\tau) \;>\; \tau(2\log 2 - 1) \;+\; (1-\tau)\log(1-\tau).$$

and writing $1-\tau = h_{0*}/h_0$ we have

$$\begin{aligned} 1 \;-\; h_{0*} \;+\; \frac{h_0\;\nu_1(1-h_{0*}/h_0)}{h_0-1} & \\ &> 1 \;-\; h_{0*} \;+\; \frac{h_0}{h_0-1}\left((2\log 2 \;-\; 1)\left(1 \;-\; \frac{h_{0*}}{h_0}\right) \;+\; \frac{h_{0*}}{h_0}\log\frac{h_{0*}}{h_0}\right) \\ &= 2\log 2 \;+\; \frac{(2\log 2 - 1) \;+\; 2h_{0*}(1-\log 2) \;-\; h_{0*}(h_0 + \log(h_0/h_{0*}))}{h_0 - 1}, \end{aligned}$$

and so (a) holds provided

$$2\log 2 - 1 \;+\; 2h_{0*}(1-\log 2) \;-\; h_{0*}(h_0+\log(h_0/h_{0*})) \;\geq\; 0, \qquad \text{for } h_0 > 5, \tag{2.42}$$

which certainly follows if we can show

$$h_{0*}(h_0 + \log h_0 + \log 1/h_{0*}) \;<\; (2\log 2 - 1), \qquad \text{for } h_0 > 5. \tag{2.43}$$

We first remark that

$$0.035 - \log 0.035 \;<\; 5 - \log 5,$$

and hence if $h_0 > 5$, then

$$h_{0*} \;<\; 0.035 \;<\; \exp(-1).$$

But then for $h_{0*} < 0.035$, we have that $h_{0*}\log(1/h_{0*})$ increases with h_{0*} and hence decreases with h_0. Moreover,

$$h_0 - 1 \;>\; 4 \;>\; 1 - h_{0*},$$

so the functions

$$h_0 \;\mapsto\; h_{0*}h_0$$

and

$$h_0 \;\mapsto\; h_{0*}\log h_0$$

are also decreasing with h_0, as can be checked by differentiating with respect to f_0. For example

$$\frac{d(h_{0*}h_0)}{df_0} \;=\; h_{0*}h_0\left(\frac{1}{h_0-1} - \frac{1}{1-h_{0*}}\right) \;<\; 0.$$

Hence the left hand side of (2.43) is decreasing with h_0, and so bounded above by

$$0.035(5+\log 5+\log 1/0.035) \;=\; 0.34866... \;<\; 0.386294.. \;=\; 2\log 2 - 1,$$

and (2.43) holds, so (a) holds.

(b) Now for $\tau > 0$,

$$\nu_2(\tau) \;=\; \frac{3\mu_0(\tau)\mu_2(\tau)}{4} \;+\; \frac{(\mu_1(\tau))^2}{4} \;>\; \frac{(\mu_1(\tau))^2}{4}.$$

Hence for $h_0 \geq 5$, we have

$$h_{0*} < 0.035$$

and

$$\frac{(\mu_1(1-h_{0*}/h_0))^2}{4} \;\geq\; \frac{(\mu_1(1-0.035/5))^2}{4} \;=\; 0.0340... \;>\; 0.03.$$

Hence

$$\frac{h_0\ \nu_2(1-h_{0*}/h_0)}{(h_0-1)^2} \;\geq\; \frac{0.03}{h_0-1} \;>\; \frac{0.03}{\beta^2}.$$

(c) Follows by substituting (a) and (b) into (2.39). $\square$

Proposition 2.10.

$$\varepsilon(\beta) \;=\; O(\beta^{-2}), \qquad\qquad as \qquad \beta \to \infty.$$

Proof. We will prove this by bounding the error when we approximate the series in (2.35) by the partial sums. Indeed, we show that there exists a constant $C(K)$ independent of h_0 such that

$$\left|\beta \;-\; \frac{h_0}{(h_0-1)^{1/2}} \sum_{k=0}^{K} \frac{\gamma_k\, \mu_k(1)}{(h_0-1)^k}\right| \;\leq\; \frac{C(K)}{h_0^{K+1/2}}, \qquad\qquad \text{for } h_0 > 2. \tag{2.44}$$

In fact, what we show is

$$\left|\beta \;-\; \frac{h_0}{(h_0-1)^{1/2}} \sum_{k=0}^{K} \frac{\gamma_k\, \mu_k(1)}{(h_0-1)^k}\right| \;\leq\; \frac{C(K)(\log h_0)^K}{h_0^{K-1/2}}, \qquad\qquad \text{for } h_0 > 2. \tag{2.45}$$

By applying (2.45) with K replaced by $K+2$, we get (2.44).

Notation. Suppose $h = (h_1, \ldots, h_p)$ and $k = (k_1, \ldots, k_q)$ are variables taking values in $U \subset \mathbb{R}^p$ and $V \subset R^q$ respectively, and suppose that F_1 and F_2 are two functions of (h,k). Then we write

$$F_1 \;\underset{k}{\leq}\; F_2,$$

if for every $k \in V$, there exists a constant $C(k) < \infty$, such that

$$F_1(h,k) \;\leq\; C(k)\ F_2(h,k) \qquad \text{for all } h \in U.$$

Now we prove (2.45). We first remark that

$$\left|(1-x)^{-1/2} \;-\; \sum_{k=0}^{K-1} \gamma_k x^k\right| \;\underset{K}{\leq}\; (1-x)^{-1/2}\, x^K, \qquad\qquad \text{for } 0 \leq x < 1.$$

Hence from (2.34), writing $t = 1 - h/h_0$, we have that for $h_0 > 2$,

$$(2.46)\qquad \left|\beta \;-\; \frac{h_0}{(h_0-1)^{1/2}} \sum_{k=0}^{K-1} \frac{\gamma_k\, \mu_k (1 - h_{0*}/h_0)}{(h_0-1)^k}\right|$$

$$\underset{K}{\leq} \quad \frac{h_0}{(h_0-1)^{K+1/2}} \int_0^{1-h_{0*}/h_0} t^{-1/2} \left(1 - \frac{-\log(1-t)-t}{(h_0-1)t}\right)^{-1/2} \left(\frac{-\log(1-t)-t}{t}\right)^K dt$$

$$= \frac{1}{(h_0-1)^K} \int_{h_{0*}}^{h_0} (h_0 - \log h_0 - (h - \log h))^{-1/2} \left(\frac{-\log(1-t)-t}{t}\right)^K dh.$$

We split into two cases. The function

$$\frac{-\log(1-t)-t}{t} \;=\; \sum_{k=1}^{\infty} \frac{t^k}{k+1}, \qquad\qquad 0 < t < 1,$$

is increasing with t, so decreasing with h. Hence for $h_0 > h$ and $h_0 > 2$, we have

$$\frac{-\log(1-t)-t}{t} \;=\; \frac{\log(h_0/h)}{1-h/h_0} \;-\; 1 \;<\; \begin{cases} \frac{8}{3}\log h_0 & h > 1/h_0, \\ -\frac{8}{3}\log h & h \leq 1/h_0. \end{cases}$$

Hence the right hand side of (2.46) is bounded up to a constant $C(K)$ by

$$\frac{(\log h_0)^K}{(h_0-1)^K} \int_{h_{0*}}^{h_0} (h_0 - \log h_0 - (h - \log h))^{-1/2}\; dh$$

$$(2.47)\qquad + \;\; \frac{1}{(h_0-1)^K} \int_{h_{0*}}^{1/h_0} (h_0 - \log h_0 - (h - \log h))^{-1/2} (-\log h)^K\, dh.$$

Using Corollary 2.5, for $h_0 > 2$, the first term in (2.47) is equal to

$$\frac{2(\log h_0)^K \beta}{(h_0-1)^K} \;\; \underset{K}{\leq} \;\; \frac{(\log h_0)^K}{(h_0-1)^K} \left(\frac{\pi}{2^{1/2}} + (f_0 - 1)^{1/2}\right) \;\; \underset{K}{\leq} \;\; \frac{(\log h_0)^K}{(h_0-1)^{K-1/2}}.$$

To bound the second term in (2.47), we change variables to $f = h - \log h$ to get the bound

$$(2.48)\qquad \frac{1}{(h_0-1)^K} \int_{\log h_0 + 1/h_0}^{f_0} (f_0 - f)^{-1/2} (-\log h_*(f))^K \frac{h_*(f)}{1 - h_*(f)}\, df.$$

But

$$-\log h_*(f) \;=\; f \;-\; h_*(f) \;<\; f \;+\; 1,$$

and for $f > \log(2/\log 2)$ we have

$$h_*(f) \;\leq\; 2e^{-f}.$$

Hence (2.48) is bounded up to a constant $C(K)$ by

$$\frac{1}{(h_0-1)^K} \int_0^{f_0} (f_0 - f)^{-1/2}\, e^{-f} f^K\, df.$$

But the integral here is uniformly bounded in f_0, so the second term in (2.47) is bounded up to $C(K)$ by

$$\frac{1}{(h_0-1)^K}.$$

So far we have bounded the left hand side of (2.46) by the right hand side of (2.45). To complete the proof of (2.45) we just have to show that for $h_0 > 2$,

$$|\mu_k(1-h_{0*}/h_0) - \mu_k(1)| \underset{k,\,K}{\leq} \frac{1}{h_0^K}.$$

However, the left hand side equals

$$\left| \frac{1}{2}\int_{1-h_{0*}/h_0}^{1} t^{-1/2}\left(\frac{-\log(1-t)-t}{t}\right)^k dt \right| \underset{k}{\leq} \int_0^{h_{0*}/h_0} |\log s|^k \, ds$$

$$\underset{k}{\leq} \frac{h_{0*}}{h_0}|\log h_0 - \log h_{0*}|^k = \frac{h_{0*}}{h_0}(h_0 - h_{0*})^k \underset{k}{\leq} e^{-f_0} h_0^{k-1} \underset{k,\,K}{\leq} \frac{1}{h_0^K}.$$

This completes the proof of (2.45). From this we get from this the asymptotic formula

$$\beta^2 \sim \frac{h_0^2}{h_0-1}\sum_{k=0}^{\infty}\frac{\nu_k(1)}{(h_0-1)^k},$$

where ν_k is defined in (2.37), in the sense that for $h_0 > 2$,

$$\left|\beta^2 - \frac{h_0^2}{h_0-1}\sum_{k=0}^{K}\frac{\nu_k(1)}{(h_0-1)^k}\right| \underset{K}{\leq} \frac{1}{h_0^K}.$$

Thus

$$\beta^2 = \frac{h_0^2}{h_0-1}\left(1 + \frac{2\log 2 - 1}{h_0-1}\right) + O(h_0^{-1}) = h_0 + 2\log 2 + O(h_0^{-1}). \tag{2.49}$$

From this we see that

$$\beta^2 - \log(\beta^2) - (h_0 - \log h_0) - 2\log 2 = O(h_0^{-1}) = O(\beta^{-2}).$$

This completes the proof of Proposition 2.10. □

Proposition 2.11.

$$M = -\frac{\beta^2}{3} + \log(4\beta^2) - 1 + O(\beta^{-2}) \qquad as \qquad \beta \to \infty.$$

Proof. From (2.29), we have

$$(2.50)\quad M \;=\; \frac{1}{4\beta}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{\phi(e^\phi+1)}{\sqrt{f_0-f(\phi)}}\,d\phi$$

$$= \frac{1}{3\beta}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{(\phi-\log h_0)e^\phi}{\sqrt{f_0-f(\phi)}}\,d\phi \;+\; \frac{\log h_0}{3\beta}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{e^\phi}{\sqrt{f_0-f(\phi)}}\,d\phi \;+\; \frac{1}{12\beta}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{\phi(3-e^\phi)}{\sqrt{f_0-f(\phi)}}\,d\phi$$

$$= \frac{1}{3\beta}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{(\phi-\log h_0)e^\phi}{\sqrt{f_0-f(\phi)}}\,d\phi \;+\; \frac{2\log h_0}{3} \;+\; \frac{1-f_0}{3}.$$

The third line here follows from (2.33) and the second equality in Proposition 2.4(b). Now we change variables to $h = e^\phi$ so $f = e^\phi - \phi = h - \log h$, and set $h_0 = h^*(f_0)$ and $h_{0*} = h_*(f_0)$. Then define

$$(2.51)\qquad N \;:=\; \frac{1}{2}\int_{\phi_*(f_0)}^{\phi^*(f_0)} \frac{(\phi-\log h_0)e^\phi}{\sqrt{f_0-f(\phi)}}\,d\phi \;=\; \frac{1}{2}\int_{h_{0*}}^{h_0} \frac{\log h - \log h_0}{\sqrt{f_0-f}}\,dh.$$

We follow the argument of (2.34)-(2.35) with β replaced by (2.51) to get

$$N \;=\; \frac{h_0}{2(h_0-1)^{1/2}}\int_0^{1-h_{0*}/h_0} (\log(1-t))\,t^{-1/2}\left(1-\frac{-\log(1-t)-t}{(h_0-1)t}\right)^{-1/2} dt$$

$$= \frac{h_0}{(h_0-1)^{1/2}}\sum_{k=0}^{\infty}\frac{\gamma_k\,\kappa_k(1-h_{0*}/h_0)}{(h_0-1)^k},$$

where

$$\kappa_k(\tau) \;=\; \frac{1}{2}\int_0^\tau (\log(1-t))\,t^{-1/2}\left(\frac{-\log(1-t)-t}{t}\right)^k dt.$$

Moreover, following the proof of (2.44)-(2.45), we conclude that for $h_0 > 2$,

$$\left| N \;-\; \frac{h_0}{(h_0-1)^{1/2}}\sum_{k=0}^{K-1}\frac{\gamma_k\,\kappa_k(1)}{(h_0-1)^k}\right| \;\underset{K}{\lesssim}\; \frac{1}{h_0^{K-1/2}}.$$

Now

$$\kappa_0(1) = 2\log 2 - 2,$$

and so in particular, using (2.49),

$$N \;=\; \kappa_0 h_0^{1/2} \;+\; O(h_0^{-1/2}) \;=\; (2\log 2-2)\beta \;+\; O(\beta^{-1}), \qquad \text{as } \beta\to\infty.$$

Substituting this into (2.50) and using (2.49), we see that as $\beta\to\infty$ we have

$$M \;=\; \frac{2(\log 4-2)}{3} + \frac{2\log(\beta^2)}{3} + \frac{-\beta^2+\log(4\beta^2)+1}{3} \;+\; O(\beta^{-2})$$

$$= -\frac{\beta^2}{3} \;+\; \log(4\beta^2) \;-\; 1 \;+\; O(\beta^{-2}).$$

Lemma 2.12. *Suppose that $C > 0$ and $\beta_1 > 1/\sqrt{2}$ are constants and that the formula*

$$-4\sum_{n=1}^{\infty}\log\left(1-e^{-2n\beta^2}\right) \;<\; \frac{C}{\beta^2}, \tag{2.52}$$

holds for $\beta = \beta_1$. Then it holds for all $\beta \geq \beta_1$.

Proof. Define

$$\omega(\beta) \;=\; -4\sum_{n=1}^{\infty}\log\left(1-e^{-2n\beta^2}\right), \qquad \beta > 0,$$

and

$$\psi(\beta) \;=\; \frac{C}{\beta^2} \;-\; \omega(\beta).$$

Then ω is positive and smooth, and

$$-\omega'(\beta) \;=\; 16\beta\sum_{n=1}^{\infty}\frac{n}{1-e^{-2n\beta^2}} \;>\; 16\beta\sum_{n=1}^{\infty}\frac{1}{1-e^{-2n\beta^2}} \;>\; 4\beta\omega(\beta).$$

Suppose that (2.52) fails, that is $\psi(\beta) \leq 0$, for some $\beta_2 > \beta_1$. Then we can choose $\beta_2 > \beta_1$ minimal such that this is the case, and clearly $\psi(\beta_2) = 0$. But then

$$\psi'(\beta_2) \;=\; \frac{2C}{\beta_2^3} \;-\; \omega'(\beta_2) \;=\; \frac{2\omega(\beta_2)}{\beta_2} \;-\; \omega'(\beta_2) \;\geq\; \omega(\beta_2)\left(\frac{-2}{\beta_2} \;+\; 4\beta_2\right).$$

But $\beta_2 > 1/\sqrt{2}$, so the right hand side is positive and so $\psi(\beta) < 0$ for some β with $\beta_1 < \beta < \beta_2$, which is a contradiction. □

Appendix. Explicit formulas for the flat torus and the round sphere.

Lemma A.1. *Let $T = \mathbb{C}/\Lambda$ be a torus of area 1, where Λ is a lattice, and let u and v be the generators of the dual lattice Λ^* and set $z = v/u$. Then for the flat metric g_0 on T,*

$$\operatorname{trace}\Delta_{g_0}^{-1} \;=\; -\frac{\log 2\pi}{2\pi} \;-\; \frac{\log(|\eta(z)|^4/|u|^2)}{4\pi}, \tag{A.1}$$

where the Dedekind eta function η is defined by

$$\eta(z) \;=\; e^{\pi iz/12}\prod_{n=1}^{\infty}(1-e^{2\pi inz}). \tag{A.2}$$

On the other hand,

$$\operatorname{trace}\Delta_{S^2,1}^{-1} \;=\; -\frac{\log\pi}{4\pi} \;-\; \frac{1}{4\pi}, \tag{A.3}$$

and so

$$\operatorname{trace}\Delta_{g_0}^{-1} \;-\; \operatorname{trace}\Delta_{S^2,1}^{-1} \;=\; \frac{1}{4\pi}\left(-\log(|\eta(z)|^4/|u|^2) \;-\; \log 4\pi \;+\; 1\right). \tag{A.4}$$

When Λ has generators $(1/b, a+bi)$ with $a, b \in \mathbb{R}$, we can choose $(u,v) = (-i/b, b-ai)$ and then (A.4) becomes (1.12).

Remark. . The quantity $\log(|\eta(z)|^4/|u|^2)$ was shown in [OsPS] to be maximized at the hexagonal torus, for which

$$\log(|\eta(z)|^4/|u|^2) \;=\; -1.0335...$$

Hence the hexagonal torus minimizes trace Δ^{-1} among flat tori of a given area.
Proof. Now

$$\Lambda^* \;=\; \{\mu \in \mathbb{C} : \Re(\bar{\mu}\lambda) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}.$$

The eigenfunctions of the Laplacian on $T = \mathbb{C}/\Lambda$ have the form

$$f(z) \;=\; e^{2\pi i \Re(\bar{\mu}\lambda)}, \qquad\qquad \text{for } \mu \in \Lambda^*.$$

The corresponding eigenvalue is $(2\pi)^2|\mu|^2$. Consider the Epstein zeta function

$$Z_T(s) \;=\; \sum_{\mu\in\Lambda^*-0} \frac{1}{(2\pi|\mu|)^{2s}}.$$

Kronecker's First Limit Formula states that

$$(2\pi)^{2s} Z_T(s) \;=\; \frac{\pi}{s-1} \;+\; 2\pi\left(-\Gamma'(1) \;-\; \log 2 \;-\; \log|\eta(z)|^2\right) \;+\; O(s-1).$$

Hence

$$Z_T(s) \;=\; \frac{1}{4\pi(s-1)} \;+\; Z_T^1 \;+\; O(s-1), \qquad\qquad Z_T^1 \;=\; \frac{1}{2\pi}\left(-\Gamma'(1) \;-\; \log(4\pi) \;-\; \log|\eta(z)|^2\right).$$

But Z_T^1 is a different regularization of the trace of Δ^{-1}, and it can be shown that this differs from our Green function regularization trace Δ^{-1} by a universal constant:

$$\text{(A.5)} \qquad\qquad \operatorname{trace}\Delta_{g_0}^{-1} \;=\; Z_T^1 \;+\; \frac{\log 2}{2\pi} \;+\; \frac{\Gamma'(1)}{2\pi}.$$

see [M2], [S1], [S2], or [O2] (A.6). Evaluating (A.5) we get (A.1).

Formula (A.3) is well known. Indeed, on the round 2-sphere of area 4π given by $x^2+y^2+z^2=1$, the Green function $G(p,q)$ can be written in terms of the distance r from p to q, as

$$G(p,q) \;=\; -\frac{1}{2\pi}\log|\sin r/2| \;-\; \frac{1}{4\pi}.$$

This gives the Robin mass

$$m_{S^2,4\pi} \;=\; \frac{\log 2}{2\pi} \;-\; \frac{1}{4\pi},$$

and combining this with (1.3) gives

$$m_{S^2,1} \;=\; m_{S^2,4\pi} \;-\; \frac{\log 4\pi}{4\pi} \;=\; -\frac{\log \pi}{4\pi} \;-\; \frac{1}{4\pi}.$$

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