Graphs with many copies of a given subgraph

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Abstract

Let $c>0$, and H be a fixed graph of order r. Every graph on n vertices containing at least cn^r copies of H contains a "blow-up" of H with $r-1$ vertex classes of size $\left\lfloor c^{r^2}\ln n\right\rfloor$ and one vertex class of size greater than $n^{1-c^{r-1}}$. A similar result holds for induced copies of H.

Keywords: *number of subgraphs; blow-up of a graph; induced subgraphs.*

Main results

This note is part of an ongoing project aiming to renovate some classical results in extremal graph theory, see, e.g., [**?**] and [3, 6].

Suppose that a graph G of order n contains cn^r copies of a given subgraph H on r vertices. How large "blow-up" of H must G contain? When H is an r-clique, this question was answered in [3]: G contains a complete r-partite graph with $r-1$ parts of size $\lfloor c^r \ln n\rfloor$ and one part larger than $n^{1-c^{r-1}}$.

The aim of this note is to answer this question for any subgraph H.

We first define precisely a "blow-up" of a graph: given a graph H of order r and positive integers $x_1,\ldots,x_r$, we write $H(x_1,\ldots,x_r)$ for the graph obtained by replacing each vertex $u\in V(H)$ with a set V_u of size x_u and each edge $uv\in E(H)$ with a complete bipartite graph with vertex classes V_u and V_v.

Theorem 1 *Let $r\geq 2$, $0<c\leq 1/2$, H be a graph of order r, and G be a graph of order n. If G contains more than cn^r copies of H, then $H(s,\ldots s,t)\subset G$, where $s=\left\lfloor c^{r^2}\ln n\right\rfloor$ and $t>n^{1-c^{r-1}}$.*

To state a similar theorem for induced subgraphs, we introduce a new concept: we say that a graph X *is of type* $H(x_1,\ldots,x_r)$, if X is obtained from $H(x_1,\ldots,x_r)$ by adding some edges within the sets V_u, $u\in V(H)$.

Theorem 2 *Let $r\geq 2$, $0<c\leq 1/2$, H be a graph of order r, and G be a graph of order n. If G contains more than cn^r induced copies of H, then G contains an induced subgraph of type $H(s,\ldots s,t)$, where $s=\left\lfloor c^{r^2}\ln n\right\rfloor$ and $t>n^{1-c^{r-1}}$.*

The proofs of Theorems 1 and 2 are almost identical; we shall present only the proof of Theorem 2, for it needs more care.

Our notation follows [1]. Thus $V(G)$ and $E(G)$ denote the vertex and edge sets of a graph G and $e(G) = |E(G)|$. The subgraph induced by $X \subset V(G)$ is denoted by $G[X]$.

Specific notation

Suppose G and H are graphs, and X is an induced subgraph of H.

We write $H(G)$ for the set of injections $h : H \to G$, such that $\{u, v\} \in E(H)$ if and only if $\{h(u), h(v)\} \in E(G)$.

We say that $P \in H(G)$ *extends* $R \in X(G)$, if $R = P|V(X)$.

Suppose $M \subset H(G)$.

We let

$$X(M) = \{R : (R \in X(G)) \ \& \ (\text{there exists } P \in M \text{ extending } R)\}.$$

For every $R \in X(M)$, we let

$$d_M(R) = |\{P : (P \in M) \ \& \ (P \text{ extends } R)\}|.$$

Suppose Y is a subgraph of G of type $H(s_1, \ldots, s_r)$ and $s = \min\{s_1, \ldots, s_r\}$.

We say that M *covers* Y if:

(a) for every edge ij going across vertex classes of Y, there exists $h \in K_2(M)$ mapping some edge of H onto ij;

(b) there exists $h_1, \ldots, h_s \in M$, such that $h_i(H) \cap h_j(H) = \varnothing$ for $i \neq j$, and for all $i \in [s]$, $h_i(H)$ intersects all vertex classes of Y.

We deduce Theorem 2 from the following technical statement.

Theorem 3 *Let $r \geq 2$, $0 < c \leq 1/2$, H be a graph of order r, and G be a graph of order n. If $M \subset H(G)$ and $|M| \geq cn^r$, then M covers an induced subgraph of type $H(s, \ldots s, t)$ with $s = \left\lfloor c^{r^2} \ln n \right\rfloor$ and $t > n^{1-c^{r-1}}$.*

The proof of Theorem 3 is based on the following routine lemma.

Lemma 4 *Let F be a bipartite graph with parts A and B. Let $|A| = m$, $|B| = n$, $r \geq 2$, $0 < c \leq 1/2$, and $s = \left\lfloor c^{r^2} \ln n \right\rfloor$. If $s \leq (c/2^r)m + 1$ and $e(F) \geq (c/2^{r-1})mn$, then F contains a $K_2(s,t)$ with parts $S \subset A$ and $T \subset B$ such that $|S| = s$ and $|T| = t > n^{1-c^{r-1}}$.*

Proof Let

$$t = \max\{x : \text{there exists } K_2(s, x) \subset F \text{ with part of size } s \text{ in } A\}.$$

For any $X \subset A$, write $d(X)$ for the number of vertices joined to all vertices of X. By definition, $d(X) \leq t$ for each $X \subset A$ with $|X| = s$; hence,

$$t\binom{m}{s} \geq \sum_{X \subset A, |X| = s} d(X) = \sum_{u \in B} \binom{d(u)}{s}. \tag{1}$$

Following [2], p. 398, set

$$f\left(x\right)=\left\{\begin{array}{ll}\binom{x}{s} & \text{if } x\geq s-1\\ 0 & \text{if } x<s-1,\end{array}\right.$$

and note that $f\left(x\right)$ is a convex function. Therefore,

$$\sum_{u\in B}\binom{d\left(u\right)}{s}=\sum_{u\in B}f\left(d\left(u\right)\right)\geq nf\left(\frac{1}{n}\sum_{u\in B}d\left(u\right)\right)=n\binom{e\left(F\right)/n}{s}\geq n\binom{cm/2^{r-1}}{s}.$$

Combining this inequality with (1), and rearranging, we find that

$$\begin{aligned}t &\geq n\frac{\left(cm/2^{r-1}\right)\left(cm/2^{r-1}-1\right)\cdots\left(cm/2^{r-1}-s+1\right)}{m\left(m-1\right)\cdots\left(m-s+1\right)}>n\left(\frac{cm/2^{r-1}-s+1}{m}\right)^{s}\\ &\geq n\left(\frac{c}{2^{r}}\right)^{s}\geq n\left(e^{\ln\left(c/2^{r}\right)}\right)^{c^{r^{2}}\ln n}=n^{1+c^{r^{2}}\ln\left(c/2^{r}\right)}=n^{1+2^{r}c^{r^{2}-1}\left(c/2^{r}\right)\ln\left(c/2^{r}\right)}.\end{aligned}$$

Since $c/2^{r}\leq1/8<1/e$ and $x\ln x$ is decreasing for $0<x<1/e$, we see that

$$t>n^{1+2^{r}c^{r^{2}-1}\left(c/2^{r}\right)\ln\left(c/2^{r}\right)}\geq n^{1-\left(r+1\right)c^{r^{2}-1}\left(1/2\right)\ln2}>n^{1-c^{r-1}\left(r+1\right)2^{-r^{2}+r}\left(1/2\right)\ln2}.$$

Now, $t>n^{1-c^{r-1}}$ follows, in view of

$$\frac{2^{r^{2}-r}}{r+1}\geq\frac{1}{2\ln2},$$

completing the proof. □

Proof of Theorem 3 Let $M\subset H\left(G\right)$ satisfy $\left|M\right|\geq cn^{r}$. We shall use induction on r to prove that M covers an induced subgraph of type $H\left(s,\ldots s,t\right)$ with $s=\left\lfloor c^{r^{2}}\ln n\right\rfloor$ and $t>n^{1-c^{r-1}}$.

Assume $r=2$ and let A and B be two disjoint copies of $V\left(G\right)$. We can suppose that $H=K_{2}$, as otherwise we apply the subsequent argument to the complement of G.

Define a bipartite graph F with parts A and B, joining $u\in A$ to $v\in B$ if $uv\in M$. Set $s=\left\lfloor c^{4}\ln n\right\rfloor$ and note that $s\leq\left(c/4\right)n+1$. Since $e\left(F\right)=\left|M\right|\geq cn^{2}>\left(c/2\right)n^{2}$, Lemma 4 implies that F contains a $K_{2}\left(s,t\right)$ with $t>n^{1-c}$. Hence M covers an induced graph of type $K_{2}\left(s,t\right)$, proving the assertion for $r=2$. Assume the assertion true for $2\leq r'<r$.

Let $V\left(H\right)=\left\{v_{1},\ldots,v_{r}\right\}$ and $H'=H\left[\left\{v_{1},\ldots,v_{r-1}\right\}\right]$.

We first show that there exists $L\subset M$ with $\left|L\right|>\left(c/2\right)n^{r}$ such that $d_{L}\left(R\right)>\left(c/2\right)n$ for all $R\in H'\left(L\right)$. Indeed, set $L=M$ and apply the following procedure.

While *there exists an* $R\in H'\left(L\right)$ *with* $d_{L}\left(R\right)\leq\left(c/2\right)n$ **do**

Remove from L *all members extending* R.

When this procedure stops, we have $d_{L}\left(R\right)>\left(c/2\right)n$ for all $R\in H'\left(L\right)$, and also

$$\left|M\right|-\left|L\right|\leq cn\left|H'\left(M\right)\right|<\frac{c}{2}n\cdot n^{r-1},$$

giving $\left|L\right|>\left(c/2\right)n^{r}$, as claimed.

Since $H'(L) \subset H'(G)$ and

$$|H'(L)| \geq |L|/n > (c/2)\, n^r/n = (c/2)\, n^{r-1},$$

the induction assumption implies that $H'(L)$ covers an induced subgraph $Z \subset G$ of type $H'(p, \ldots, p)$ with $p = \left\lfloor c^{(r-1)^2} \ln n \right\rfloor$. Here we use the inequalities

$$n^{1-c^{r-2}} \geq n^{1-c} \geq n^{1/2} > 2^{-4} \ln n \geq c^{(r-1)^2} \ln n.$$

Since $H'(L)$ covers Z, there exist $R_1, \ldots, R_p \in H'(L)$ such that $R_1(H'), \ldots, R_p(H')$ are disjoint subgraphs of Z intersecting all its vertex classes. For every $i \in [p]$, let

$$W_i = \{v : (\text{there exists } P \in L \text{ extending } R_i) \;\&\; (P(v_r) = v)\}.$$

Write d for the degree of v_r in H and note that each $v \in W_i$ is joined to exactly d vertices of $R_i(H')$. Since, by our selection, $d_L(R_i) \geq (c/2)\, n$ for all $i \in [p]$, there is a set $X_i \subset W_i$ with

$$|X_i| \geq (cn/2) \Big/ \binom{r-1}{d} \geq cn/2^{r-1}$$

such that the vertices of X_i have the same neighbors in $R_i(H')$. Let $Y_i \subset [r-1]$ be the set of classes of Z containing the neighbors of the vertices of X_i.

Next, set $m = \lceil p/2^{r-2} \rceil$, and note that there is a set $A \subset [r-1]$ with $|A| = m$ such that all sets Y_i, $i \in A$, are the same.

Define a bipartite graph F with parts A and $B = V(G)$, joining $i \in A$ to $v \in B$ if $v \in X_i$. Since $|X_i| > (c/2^{r-1})\, n$ for all $i \in A$, we have

$$e(F) > \left(c/2^{r-1}\right) mn.$$

Also, setting $s = \left\lfloor c^{r^2} \ln n \right\rfloor$, we find that

$$\begin{aligned} s &\leq c^{r^2} \ln n = c^{2r-1} c^{(r-1)^2} \ln n \leq \left(c/2^{2r-2}\right) \left\lfloor c^{(r-1)^2} \ln n \right\rfloor + 1 \\ &\leq (c/2^r)\, p/2^{r-2} + 1 \leq (c/2^r)\, m + 1. \end{aligned}$$

Therefore, by Lemma 4, there exists $K_2(s,t) \subset F$ with parts $S \subset A$ and $T \subset B$ such that $|S| = s$ and $|T| = t > n^{1-c^{r-1}}$.

Let $G' = G[\cup_{i \in S} R_i(H')]$ and $G'' = G[\cup_{i \in S} R_i(H') \cup T]$. Note that $G' \subset Z$, and clearly G' is of type $H'(s, \ldots, s)$. Since each vertex $v \in T$ is joined to exactly the same vertices of $\cup_{i \in S} R_i(H')$, we see that G'' is of type $H(s, \ldots, s, t)$.

To finish the proof, we show that L covers G''. First, we see that, for every edge ij going across vertex classes of G'', there exists $h \in K_2(L)$ mapping some edge of H onto ij. Finally, taking s distinct vertices $u_1, \ldots, u_s \in T$, by the construction of T, for every $i \in S$, there exists $P_i \in L$ with $P_i|V(X) = R_i$ and $P_i(v_r) = u_i$. Hence, L covers G'', completing the induction step and the proof. $\square$

Concluding remarks

Using random graphs, it is easy to see that most graphs on n vertices contain substantially many copies of any fixed graph, but contain no $K_2(s,s)$ for $s \gg \log n$. Hence, Theorems 1, 2, and 3 are essentially best possible.

Finally, a word about the project mentioned in the introduction: in this project we aim to give wide-range results that can be used further, adding more integrity to extremal graph theory.

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