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Generalized Ohm's law for relativistic plasmas

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We generalise the relativistic expression of Ohm's law by studying a multi-fluid system of charged species using the 1+3 covariant formulation of general relativistic electrodynamics. This is done by providing a fully relativistic, fully nonlinear propagation equation for the spatial component of the electric 4-current. Our analysis proceeds along the lines of the non-relativistic studies and extends previous relativistic work on cold plasmas. Exploiting the compactness and transparency of the covariant formalism, we provide a direct comparison with the standard Newtonian versions of Ohm's law and identify the relativistic corrections in an unambiguous way. The generalised expression of Ohm's law is initially given relative to an arbitrary observer and for a multi-component relativistic charged medium. Then, the law is written with respect to the Eckart frame and for a hot two-fluid plasma with zero total charge. Finally, we apply our analysis to a cold proton-electron plasma and recover the well known magnetohydrodynamic expressions. In every step, we discuss the approximations made and identify familiar effects, like the Biermann-battery and the Hall effects.

I. INTRODUCTION

Relativistic plasmas play a major role in high-energy phenomena, such as those associated with active galactic nuclei, black-hole magnetospheres, relativistic jets, the early universe, etc. Although a complete kinetic description of a relativistic plasma is desirable, it becomes less efficient when the large-scale bulk plasma motions are dominant. Thus, multi-fluid hydrodynamics is more appropriate in astrophysical systems characterized by those kind of motions, and in this framework it is desirable to coarsen the description to that of a one-fluid magnetohydrodynamics (MHD). The MHD equations, which comprise of the particle conservation law, the energy/momentum conservation laws and Maxwell's formulae, must be also complemented by Ohm's law, namely an equation relating the induced electric current with the electric field of the plasma. Ohm's law is directly involved in the magnetic induction equation, used in the description of bulk plasma dynamics. Despite its importance, however, there is still no general consensus on the form of Ohm's law for relativistic plasmas.

Some of the previous work towards obtaining a relativistic version of the generalised Ohm's law can be found in [1, 2, 3, 4, 5, 6]. In general, the authors start from a kinetic description and subsequently derive the required law in the form of an electric-current conservation equation. The approach of [4] assumes a multifluid description, but the study is then confined to a specific black-hole metric. Here, we will derive the relativistic version of Ohm's law for a multi-component plasma, a general metric and in the presence of an arbitrary electromagnetic field. We will do so by means of the 1 + 3 covariant approach to General Relativity and Electromagnetism (see [7, 8, 9, 10] for reviews and further references [21]). Ohm's law is given in the form of an evolution equation, along a Fermi-propagated frame, for the spacelike component of the electric 4-current. Exploiting the mathematical compactness and physical transparency of the covariant formalism, we arrive at an expression that allows for a direct comparison with the existing Newtonian version of Ohm's law and identifies the relativistic corrections in an unambiguous way. An additional advantage is that the covariant version of Ohm's law provided here goes beyond the cold plasma limit of the expressions obtained so far and applies to fully relativistic fluids.

The aim of the present, first, paper is primarily to set up the mathematical framework, discuss the physics and provide the relativistic expression of Ohm's law in its most general possible form. The latter means that our results can be applied to a wide variety of situations, ranging from relativistic plasmas to astrophysics, large-scale structure and cosmology. Thus, given any arbitrary "background" metric (astrophysical or cosmological), one can easily linearise our the equations around it. In addition, the 1+3 covariant formulation is built on irreducible kinematical and dynamical quantities, which assigns a clear physical interpretation to every variable in our equations. The latter should prove very advantageous when studying nonlinear effects, particularly those outside the ideal-MHD limit, like current-sheet formation, turbulent plasmas, magnetic-dynamo amplification and dissipative effects. Applications of this kind will be the subject of future work. Here, maintaining the generality of the metric and of the electromagnetic field, we focus on a two-component fluid and apply our multi-fluid expression of Ohm's law to a system of relativistic protons and electrons. Adopting the Eckart frame (e.g. see [12]), allows us to define the bulk quantities, such as the plasma velocity for example, in a way that closely resembles those of the non-relativistic treatments. This, in turn, helps us to discuss the common approximations in the use of the law and to explicitly show the relativistic counterparts of very well known Newtonian effects, like the Biermann battery and the Hall effect.

II. MULTI-COMPONENT FLUIDS

A. 1+3 kinematics

Consider a pseudo-Riemannian spacetime filled with metric g_{ab} of signature (+, +, +, -) and introduce a timelike velocity field u_a normalized so that $u_a u^a = -1$. The tensor

$$h_{ab} = g_{ab} + u_a u_b \,, \tag{1}$$

projects orthogonal to u_a and into the observers local rest-space. [22] The u_a -field defines the frame of our fundamental observers and, together with h_{ab} , achieves a unique 1+3 "threading" of the spacetime into time and space. The same two fields are also used to define the covariant time and spatial derivatives of any tensor field S_{ab} ...^{cd...} according to

$$\dot{S}_{ab\cdots}{}^{cd\cdots} = u^e \nabla_e S_{ab\cdots}{}^{cd\cdots} \qquad \text{and} \qquad D_e S_{ab\cdots}{}^{cd\cdots} = h_e{}^s h_a{}^f h_b{}^p h_q{}^c h_r{}^d S_{fp\cdots}{}^{qr\cdots}, \tag{2}$$

respectively (see [8, 10] for reviews on the covariant formalism and more details).

Covariantly, the kinematics of the fundamental observers are determined by decomposing the orthogonally projected gradient of their 4-velocity field into its irreducible parts. In particular, we have

$$D_b u_a = \frac{1}{3} \Theta h_{ab} + \sigma_{ab} + \omega_{ab} - \dot{u}_a u_b , \qquad (3)$$

where $\Theta = \nabla^a u_a = D^a u_a$ determines the average expansion (when positive) or contraction (when negative) of the volume element associated with u_a , $\sigma_{ab} = D_{\langle b} u_{a \rangle}$ indicates changes in its shape (under constant volume), $\omega_{ab} = D_{[b} u_{a]}$ measures the rotational behaviour of u_a and $\dot{u}_a = u^b \nabla_b u_a$ is the 4-acceleration. The latter indicates the presence of non-gravitational forces.

B. Tilted 4-velocities

Let us assume a mixture of fluids and define the 4-velocity $u_{(i)}^a$ of the *i*-th fluid component (with $u_{(i)}^a u_a^{(i)} = -1$) as the future directed timelike eigenvector of the associated Ricci tensor. Then, the tensor

$$h_{(i)}^{ab} = g^{ab} + u_{(i)}^a u_{(i)}^b \tag{4}$$

projects orthogonal to $u_{(i)}^a$. The relation between u^a and $u_{(i)}^a$ is determined by the hyperbolic angle $\beta_{(i)}$ between the two 4-velocity vectors. This 'tilt' angle, which also determines the peculiar motion of the *i*-th fluid component relative to u^a (see Eq. (6) below), is given by [13]

$$\cosh\beta_{(i)} = -u_a u^a_{(i)} \,, \tag{5}$$

with

$$u_{(i)}^{a} = \gamma_{(i)} \left(u^{a} + v_{(i)}^{a} \right)$$
(6)

and $v_{(i)}^a$ representing the peculiar velocity of the *i*-th species relative to the u_a -frame. Also, $\gamma_{(i)} = (1 - v_{(i)}^2)^{-1/2}$ is the Lorentz-boost factor and $v_{(i)}^2 = v_{(i)}^a v_a^{(i)}$. When the tilt angle is small (i.e. for $\beta_{(i)} \ll 1$) we have $v_{(i)} \simeq \beta_{(i)}$. This means non-relativistic peculiar velocities and $\gamma_{(i)} \simeq 1$.

C. Multi-component perfect fluids

A general material medium is described by its energy momentum tensor T^{ab} , its particle flux N^a and by the entropy flux vector S^a . The former two quantities are conserved (i.e., $\nabla_b T^{ab} = 0 = \nabla_a N^a$), while the last obeys the second law of thermodynamics (i.e., $\nabla_a S^a \ge 0$). When the strong energy condition holds, the energy momentum tensor of a fluid accepts a unique timelike eigenvector u_T^a , normalised so that $u_T^a u_a^T = -1$ [16]. One may also define a unitary timelike vector parallel to N^a by $u_N^a = N^a / \sqrt{-N_a N^a}$. Provided that the fluid is perfect (or in equilibrium), u_T^a , u_N^a and S^a are parallel and define a unique hydrodynamic 4-velocity vector, the rest-frame of the fluid flow (see e.g. [17]). This is also the only frame in which the energy momentum tensor of the matter assumes the perfect-fluid form. When dealing with an imperfect fluid, however, there is no longer a uniquely defined hydrodynamic 4-velocity.

Consider a mixture of perfect fluids, where the *i*-th component has energy density $\mu_{(i)}$, isotropic pressure $p_{(i)}$ and moves along the timelike 4-velocity field $u^a_{(i)}$. Relative to this frame, the energy momentum tensor of the individual species decomposes as

$$T_{(i)}^{ab} = \left(\mu_{(i)} + p_{(i)}\right) u_{(i)}^{a} u_{(i)}^{b} + p_{(i)} g^{ab} = \mu_{(i)} u_{(i)}^{a} u_{(i)}^{b} + p_{(i)} h_{(i)}^{ab},$$
(7)

with $h_{(i)}^{ab}$ given by (4). Also, the associated particle flux vector is given by

$$N_{(i)}^{a} = n_{(i)} u_{(i)}^{a} , (8)$$

where $n_{(i)}$ is the number density of each matter component in their own frame.

Substituting the transformation law (6) into Eq. (7) allows us to re-express the latter with respect to the u_a -frame. The result reads

$$T^{ab}_{(i)} = \hat{\mu}_{(i)} u^a u^b + \hat{p}_{(i)} h^{ab} + u^a \hat{q}^b_{(i)} + \hat{q}^a_{(i)} u^b + \hat{\pi}^{ab}_{(i)} , \qquad (9)$$

with h^{ab} given in (1). The above corresponds to the energy-momentum tensor of an imperfect fluid with

$$\hat{\mu}_{(i)} = \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)} \right) - p_{(i)} , \qquad (10)$$

$$\hat{p}_{(i)} = p_{(i)}, \tag{11}$$

$$\hat{q}^{a}_{(i)} = \gamma^{2}_{(i)} \left(\mu_{(i)} + p_{(i)} \right) v^{a}_{(i)}, \qquad (12)$$

$$\hat{\pi}_{(i)}^{ab} = \gamma_{(i)}^2 \left(\mu_{(i)} + p_{(i)} \right) v_{(i)}^a v_{(i)}^b \,. \tag{13}$$

We emphasise that the forms of both $\hat{p}_{(i)}$ and $\hat{\pi}_{(i)}^{ab}$ are different from those usually found in the standard literature (e.g. see [10, 13]). In particular, $\hat{\pi}_{(i)}^{ab}$ is no longer trace-free, which means that its does not only contain the purely anisotropic part of the system's pressure. For the same reason $\hat{p}_{(i)}$ is not the effective isotropic pressure of the total medium, since part of the latter is incorporated in the $\hat{\pi}_{(i)}^{ab}$ -tensor. Neverthelees, the adopted representation has serious technical advantages, which will allow us to go easily beyond the cold-plasma limit without compromising the physics of our discussion.

As with the energy-momentum tensor, we may also express the particle flux of the species relative to the u^a -frame. To be precise, combining Eqs. (6) and (8) we arrive at

$$N^{a}_{(i)} = \hat{n}_{(i)} u^{a} + \hat{\mathcal{N}}^{a}_{(i)} , \qquad (14)$$

with

$$\hat{n}_{(i)} = \gamma_{(i)} n_{(i)}$$
 and $\hat{\mathcal{N}}^a_{(i)} = \hat{n}_{(i)} v^a_{(i)}$, (15)

representing the associated number density and particle flux respectively. When dealing with non-relativistic species, $\gamma_{(i)} \simeq 1$ and we may ignore terms of quadratic (and higher) order in $v_{(i)}^a$. Then, according to (10)-(13), we have $\hat{\mu}_{(i)} \simeq \mu_{(i)}, \hat{p}_{(i)} \simeq p_{(i)}, \hat{q}_{(i)}^a \simeq (\mu_{(i)} + p_{(i)})v_{(i)}^a$ and $\hat{\pi}_{(i)}^{ab} \simeq 0$. Also, for $v_{(i)}^2 \ll 1$, relations (15) reduce to $\hat{n}_{(i)} \simeq n_{(i)}$ and $\hat{\mathcal{N}}_{(i)}^a \simeq n_{(i)}v_{(i)}^a$.

III. MULTI-COMPONENT CHARGED FLUIDS

A. Electromagnetic fields

The Maxwell field is covariantly characterised by the antisymmetric electromagnetic (Faraday) tensor F^{ab} . Relative to an observer moving with 4-velocity u^a , the later decomposes as (e.g. see [7, 9])

$$F^{ab} = 2u^{[a}E^{b]} + \epsilon^{abc}B_c \,, \tag{16}$$

where ϵ^{abc} is the permutation tensor orthogonal to u^a and $E^a = F^{ab}u_b$, $B^a = \epsilon^{abc}F_{bc}/2$ are respectively the electric and magnetic field measured by the fiducial observer. The evolution of these two fields is monitored by means of Maxwell's equations, which in covariant form read [7, 9]

$$\dot{E}^{\langle a \rangle} = -\frac{2}{3}\Theta E^a + (\sigma^a{}_b + \omega^a{}_b)E^b + \varepsilon^a{}_{bc}\dot{u}^bB^c + \operatorname{curl}B^a - \mathcal{J}^a, \qquad (17)$$

$$\dot{B}^{\langle a \rangle} = -\frac{2}{3} \Theta B^a + (\sigma^a{}_b + \omega^a{}_b) B^b - \varepsilon^a{}_{bc} \dot{u}^b E^c - \text{curl}E^a \,, \tag{18}$$

$$\mathcal{D}_a E^a = \rho - 2\omega_a B^a \tag{19}$$

and

$$\mathsf{D}_a B^a = 2\omega_a E^a \,. \tag{20}$$

Note that $\mathcal{J}^a = h^a{}_b J^b$ and $\rho = -u_a J^a$ are the spatial current and the charge density respectively, with J_a representing the 4-current.[23] Also, $\omega^a = \varepsilon^a{}_{bc}\omega^{bc}/2$ is the vorticity vector (with $\omega^{ab} = \varepsilon^{ab}{}_c\omega^c$) and curl $v^a \equiv \varepsilon^a{}_{bc}\mathrm{D}^b v^c$ for any orthogonally projected vector v^a .

The Faraday tensor obeys Maxwell's equations and also determines the energy-momentum tensor of the electromagnetic field by means of the familiar formula

$$T^{ab}_{(em)} = F^{ac} F^{b}{}_{c} - \frac{1}{4} F^{cd} F_{cd} g^{ab} .$$
⁽²¹⁾

The latter combines with Eq. (16) to facilitate the irreducible decomposition of $T^{ab}_{(em)}$ and a fluid description of the Maxwell field. Thus, relative to the u_a -frame,

$$T^{ab}_{(em)} = \frac{1}{2} \left(E^2 + B^2 \right) u^a u^b + \frac{1}{2} \left(E^2 + B^2 \right) h^{ab} + 2Q^{(a} u^{b)} + \Pi^{ab} , \qquad (22)$$

with $E^2 = E^a E_a$, $B^2 = B^a B_a$, $Q^a = \epsilon^{abc} E_b B_c$ and $\Pi^{ab} = -E^{\langle a} E^{b \rangle} - B^{\langle a} B^{b \rangle}$. In other words, the electromagnetic field can be treated as an imperfect medium with energy density $(E^2 + B^2)/2$, isotropic pressure $(E^2 + B^2)/2$, an energy flux represented by the Poynting vector Q^a and anisotropic stresses given by Π^{ab} [7, 9].

B. Conservation laws

Consider a multi-component system containing species of different nature (e.g. baryonic and non-baryonic matter, photons, etc) in the presence of an electromagnetic field. The charged particles are coupled to the Maxwell field and the mixture has a total energy-momentum tensor given by the sum $T_{(i)}^{ab} + T_{(em)}^{ab}$. The latter satisfies a conservation law of the form

$$\nabla_b T^{ab}_{(i)} - F^a{}_b J^b_{(i)} = \mathcal{G}^a_{(i)} , \qquad (23)$$

since $\nabla_b T^{ab}_{(em)} = -F^a{}_b J^b_{(i)}$, where $J^a_{(i)}$ is the electric 4-current density of the *i*-th (charged) species. The interaction term $\mathcal{G}^a_{(i)}$ represents forces other than electromagnetic. The latter, as a consequence of the conservation of the total energy-momentum tensor, obey the constraint

$$\sum_{i} \mathcal{G}^a_{(i)} = 0.$$
⁽²⁴⁾

The timelike and spacelike parts of expression (23) respectively provide the conservation laws of the energy density and of the momentum density of the *i*-th fluid component. Thus, by projecting (23) along u^a we arrive at the covariant form of the generalised continuity equation

$$\dot{\hat{\mu}}_{(i)} = -\left(\hat{\mu}_{(i)} + \hat{p}_{(i)}\right)\Theta - \mathcal{D}_a\hat{q}^a_{(i)} - 2\dot{u}_a\hat{q}^a_{(i)} - \left(\sigma_{ab} + \frac{1}{3}\Theta h_{ab}\right)\hat{\pi}^{ab}_{(i)} + E_a\mathcal{J}^a_{(i)} - \mathcal{G}_{(i)},$$
(25)

where $\mathcal{J}_{(i)}^a = h^a{}_b J^b_{(i)}$ and $\mathcal{G}_{(i)} = u_a \mathcal{G}_{(i)}^a$. Note the second last term in the right-hand side of the above, which describes alternations in the energy density of the *i*-th fluid due to the action of the electromagnetic field. This term may be seen as representing the familiar Joule heating effect in a covariant manner. Recall that $\hat{\pi}^{ab}_{(i)}$ is not generally a trace-free tensor and therefore the sum $h_{ab}\hat{\pi}^{ab}_{(i)}$ does not necessarily vanish. Similarly, projecting Eq. (23) orthogonal to u^a , gives the covariant form of the generalised Navier-Stokes equation

$$(\hat{\mu}_{(i)} + \hat{p}_{(i)}) \dot{u}^{a} = -\mathrm{D}^{a} \hat{p}_{(i)} - \dot{\hat{q}}_{(i)}^{\langle a \rangle} - \frac{4}{3} \Theta \hat{q}_{(i)}^{a} - (\sigma^{a}{}_{b} + \omega^{a}{}_{b}) \hat{q}_{(i)}^{b} - \mathrm{D}_{b} \hat{\pi}_{(i)}^{ab} - \hat{\pi}_{(i)}^{ab} \dot{u}_{b} + \left(\rho_{(i)} E^{a} + \epsilon^{a}{}_{bc} \mathcal{J}_{(i)}^{b} B^{c}\right) + \mathcal{G}_{(i)}^{\langle a \rangle} ,$$

$$(26)$$

with $\dot{q}_{(i)}^{\langle a \rangle} = h^a{}_b \dot{q}_{(i)}^b$ and $\rho_{(i)} = -u_a J^a_{(i)}$ representing the charge density of the *i*-th component. Turning to the particle numbers and assuming that the individual number densities are not necessarily conserved, we may write

$$\nabla_a N^a_{(i)} = \mathcal{Q}_{(i)} \,, \tag{27}$$

where $\mathcal{Q}_{(i)}$ indicates either an increase or a reduction. Particle-antiparticle annihilation, for example, will reduce the numbers, while $\sum_i Q_{(i)} = 0$ ensures overall particle conservation. Substituting (14) into the above and then using expressions (15), leads to

$$\dot{\hat{n}}_{(i)} = -\left(\Theta + \mathcal{D}_a v_{(i)}^a\right) \hat{n}_{(i)} - v_{(i)}^a \mathcal{D}_a \hat{n}_{(i)} - \dot{u}_a v_{(i)}^a \hat{n}_{(i)} + \mathcal{Q}_{(i)} , \qquad (28)$$

thus providing the propagation equation of the particle's number density relative to u_a . We close this section by pointing out that both Eq. (26) and Eq. (28) contain implicit derivatives (temporal and spatial) of $\gamma_{(i)}$. This is the consequence of our decomposition choice (reflected in the set (9)-(13)) and ensures that we can treat relativistic (hot) plasmas with minimal technical complexity.

C. Electric charges and currents

The 4-current density of each charged species is related to its associated particle-flux vector by $J^a_{(i)} = eZ_{(i)}N^a_{(i)}$, with e representing the fundamental electric charge and $Z_{(i)}$ the atomic number of the particles. Recalling that $N^a_{(i)} = \tilde{n}^{(i)}(u^a + v^a_{(i)})$, relative to the u^a frame – see Eqs. (14), (15), we have

$$J_{(i)}^{a} = eZ_{(i)}\hat{n}_{(i)} \left(u^{a} + v_{(i)}^{a} \right) .$$
⁽²⁹⁾

The timelike part of the above gives the charge density of the corresponding charged component in the fundamental frame, while its spacelike counterpart leads to the associated 3-current. In particular, projecting (29) along and orthogonal to u_a we arrive at

$$\rho_{(i)} = -u_a J^a_{(i)} = e Z_{(i)} \hat{n}_{(i)} \qquad \text{and} \qquad \mathcal{J}^a_{(i)} = J^{\langle a \rangle}_{(i)} = e Z_{(i)} \hat{n}_{(i)} v^a_{(i)} , \qquad (30)$$

respectively. Consequently, the total charge and the total 3-current are given by the sums

$$\rho = \sum_{i} \rho_{(i)} = e \sum_{i} Z_{(i)} \hat{n}_{(i)} \qquad \text{and} \qquad \mathcal{J}^{a} = \sum_{i} \mathcal{J}^{a}_{(i)} = e \sum_{i} Z_{(i)} \hat{n}_{(i)} v^{a}_{(i)} , \qquad (31)$$

with the former vanishing in the case of overall electrical neutrality. The latter is a good approximation on scales larger than the Debye length of the species, where the bulk properties of the plasma dominate. In that case $\sum_i Z_{(i)} \hat{n}_{(i)} = 0$.

IV. GENERALISED RELATIVISTIC OHM'S LAW

Relativistic expressions of the generalised Ohm's law, in the form of a propagation equation for the electric 3-current, appear in various versions in the literature [1, 3, 4, 5]. With the exception of [5], however, all of the aforementioned studies address cold plasmas only. Here, we use the 1+3 covariant approach to relativistic hydrodynamics and electrodynamics to express the generalised Ohm's law in terms of irreducible kinematical, dynamical and electrodynamical variables. The result is an expression that closely resembles the non-relativistic forms encountered in the standard plasma literature on one hand, while on the other it identifies the relativistic corrections in an unambiguous way.

Covariantly, the time evolution of any given quantity is monitored by the orthogonally projected time-derivative of the associated variable. When the latter is spacelike, as it happens in the case of the 3-current, the orthogonally projected time-derivative coincides with the familiar Fermi derivative. Thus, according to (31),

$$\dot{\mathcal{J}}^{\langle a \rangle} = e \sum_{i} Z_{(i)} \left(\dot{\hat{n}}_{(i)} v^{a}_{(i)} + \hat{n}_{(i)} \dot{v}^{\langle a \rangle}_{(i)} \right) \,. \tag{32}$$

To obtain the full expression, one needs to replace $\dot{\hat{n}}_{(i)}$ and $\dot{v}^a_{(i)}$. The evolution of $\hat{n}_{(i)}$ is readily given by Eq. (28). The one for $v^a_{(i)}$, on the other hand, is obtained by replacing (25) into Eq. (26). Then,

$$\dot{v}_{(i)}^{\langle a \rangle} = -\frac{1}{3} \Theta \left(1 - v_{(i)}^2 \right) v_{(i)}^a - (\sigma^a{}_b + \omega^a{}_b) v_{(i)}^b - v_{(i)}^b \mathcal{D}_b v_{(i)}^a + \dot{u}_b v_{(i)}^b v_{(i)}^a + \sigma_{bc} v_{(i)}^b v_{(i)}^c v_{(i)}^a - \dot{u}^a - \frac{1}{\hat{M}_{(i)}} \left\{ \left[\left(\dot{\hat{p}}_{(i)} + E_a \mathcal{J}_{(i)}^a - \mathcal{G}_{(i)} \right) v_{(i)}^a + \mathcal{D}^a \hat{p}_{(i)} - \left(\rho_{(i)} E^a + \varepsilon^a{}_{bc} \mathcal{J}_{(i)}^b B^c \right) \right] - \mathcal{G}_{(i)}^{\langle a \rangle} \right\},$$
(33)

where $\hat{M}_{(i)} = \hat{\mu}_{(i)} + \hat{p}_{(i)}$. Substituting the above result into expression (32), using (28) and employing some lengthy but relatively straightforward algebra, we arrive at

$$\dot{\mathcal{J}}^{\langle a \rangle} = -\frac{4}{3} \Theta \mathcal{J}^{a} - (\sigma^{a}{}_{b} + \omega^{a}{}_{b}) \mathcal{J}^{b} + e \sum_{i} Z_{(i)} \left(\frac{\hat{n}_{(i)}}{\hat{M}_{(i)}}\right) \rho_{(i)} E_{a} + e \sum_{i} Z_{(i)} \left(\frac{\hat{n}_{(i)}}{\hat{M}_{(i)}}\right) \varepsilon^{a}{}_{bc} \mathcal{J}^{b}_{(i)} B^{c}
-e \sum_{i} Z_{(i)} \left(\frac{\hat{n}_{(i)}}{\hat{M}_{(i)}}\right) \left(\dot{\hat{p}}_{(i)} v^{a}_{(i)} + D^{a} \hat{p}_{(i)}\right) - e \sum_{i} Z_{(i)} \hat{n}_{(i)} \dot{u}^{a} + e \sum_{i} Z_{(i)} \left(\frac{\hat{n}_{(i)}}{\hat{M}_{(i)}}\right) \mathcal{G}^{\langle a \rangle}_{(i)}
+e \sum_{i} Z_{(i)} \left[\mathcal{Q}_{(i)} + \left(\frac{\hat{n}_{(i)}}{\hat{M}_{(i)}}\right) \left(\mathcal{G}_{(i)} - E_{b} \mathcal{J}^{b}_{(i)}\right)\right] v^{a}_{(i)} - e \sum_{i} Z_{(i)} D_{b} \left(\hat{n}_{(i)} v^{b}_{(i)} v^{a}_{(i)}\right)
+e \sum_{i} Z_{(i)} \hat{n}_{(i)} \left(\frac{1}{3} \Theta v^{2}_{(i)} + \sigma_{bc} v^{b}_{(i)} v^{c}_{(i)}\right) v^{a}_{(i)}.$$
(34)

This monitors the evolution of the total 3-current, associated with a multi-component system of non-comoving charged perfect fluids. Given that Eq. (34) relates the total 3-current to the electric field, it also provides the fully relativistic version of the generalised Ohm's law. Recall that the "hat-variables" contain the full Lorentz-boost factors (i.e. the $\gamma_{(i)}$'s – see definitions (10)-(13) and also (15a)), which guarantees that our analysis applies to hot as well as to cold plasmas. We also note that, in principle, one can invert Eq. (34) to an expression for the electric field. Combined with (18), the latter can be used to study the magnetic component of the Maxwell field in detail.

Comparing the right-hand side of expression (34) to that of its Newtonian counterpart (see Eq. (3.5.9) in [20] for example), we can immediately see the analogies and also locate the relativistic corrections. Thus, the first two terms in the right-hand side of the above are due to the relative motion of the fundamental observers. Both terms also appear in Newtonian treatments and represent changes in the 3-current density triggered by inertial forces. The fourth term will lead to the familiar *Hall effect* (see § VC below), while the fifth comes from variations in the effective 'isotropic' pressure of the individual species. Note that both the spatial and the temporal pressure gradients are involved, with the latter treated as the relativistic correction to the Newtonian *Biermann battery effect*. The sixth term in the right-hand side of (34) vanishes when global electrical neutrality is imposed and the seventh is due to particle collisions that lead to momentum transfer between the species. The latter provide a measure of the electrical resistivity of the total medium. The first component of the eighth term comes from changes in the number density exchanges between the individual fluids and the Joule-heating effect (see Eq. (25) in § III B). The second last term in the right-hand side of (34) accounts for spatial inhomogeneities in the velocities and the number densities of the individual species, while the last is triggered by relative motion (inertial) effects. The former has a Newtonian counterpart (e.g. see (3.5.9) in [20]).

V. HOT TWO-FLUID PLASMAS

A. Eckart frame vs Landau frame

A mixture of relativistic ideal fluids does not behave as an ideal medium, which means that when considering the bulk motions of the total system we must first specify the reference frame we are working in. Traditionally, two choices are in order: the Eckart (or particle) frame [12] and the Landau (or energy) frame [14]. For our multi-component fluid, the 4-velocity associated with the total particle flux is

$$u_E^a = \frac{1}{\sum_i \sqrt{-N_a^{(i)} N_{(i)}^a}} \sum_i N_{(i)}^a = \frac{1}{\sum_i \hat{n}_{(i)}} \sum_i \left(\hat{n}_{(i)} u^a + \hat{\mathcal{N}}_{(i)}^a \right) , \tag{35}$$

with the second equality resulting from decomposition (14). On the other hand, using (9)-(13), we may write the 4-velocity associated with the total energy flux as

$$u_L^a = -\frac{1}{\sum_i \sqrt{-T_{(i)}^{ab} u_b T_{ac}^{(i)} u^c}} \sum_i T_{(i)}^{ab} u_b = \frac{1}{\sum_i \sqrt{\hat{\mu}_{(i)}^2 - \hat{q}_{(i)}^2}} \sum_i \left(\hat{\mu}_{(i)} u^a + \hat{q}_{(i)}^a \right) \,. \tag{36}$$

Observers following the Eckart frame see no particle flux, while for those in Landau coordinates the energy flux vanishes. Thus, in our case, the Eckart and Landau frames are defined by demanding that

$$\sum_{i} \hat{\mathcal{N}}^{a}_{(i)} = 0 \qquad \text{and} \qquad \sum_{i} \hat{q}^{a}_{(i)} = 0 \,, \tag{37}$$

respectively. The immediate consequence is that an observer in the Eckart frame detects a non-zero heat flux, while its Landau counterpart sees a particle drift. On these grounds, the bulk velocities of the plasma in the Eckart and the Landau frames are respectively defined by

$$v_E^a \equiv \frac{1}{\hat{M}} \sum_i \hat{q}_{(i)}^a \qquad \text{and} \qquad v_L^a \equiv \frac{1}{\hat{n}} \sum_i \hat{\mathcal{N}}_{(i)}^a \,, \tag{38}$$

where $\hat{M} = \sum_{i} \hat{M}_{(i)}$ and $\hat{n} = \sum_{i} \hat{n}_{(i)}$. Both frames are physically equivalent and choosing one against the other is a decision dictated by the particulars of the problem in hand. In Appendix A we discuss in more detail the definition of these frames and their non-relativist limits). Here, we shall work in the Eckart frame because there the definitions of the bulk quantities (e.g. the bulk velocity of the plasma and its energy density) closely resemble their non-relativistic associates. This will simplify our analysis and facilitate its physical interpretation.

B. Globally neutral plasmas in the Eckart frame

So far our analysis has been completely general, covering multi-component systems with species of different nature (i.e. baryons, non-baryons, photons, etc). Most physical plasmas, however, are treated as two-fluid mixtures of oppositely charged particles. In addition, Eqs. (38) can now be "inverted" to express the velocities of the individual species in terms of bulk variables only in two-component systems. For these reasons, and also for mathematical simplicity, we will from now on confine to plasmas containing one positively and one negatively charged species. All other possible constituents (as, e.g., photons) will be considered as external. Therefore Eq. (24) must be rewritten as

$$\mathcal{G}^a_+ + \mathcal{G}^a_- = \mathcal{G}^a_{ext} \tag{39}$$

where \mathcal{G}_{ext}^a represents interactions that now are external to our system as, for example, collisions with photons (i.e., Compton effect), anomalous resistivity due to scattering on turbulent flows, etc (see [3] for a short account of the possible external effects).

Assuming that v^a_+ and v^a_- are the associated velocities relative to the fundamental u^a -frame, the corresponding energy-flux vectors are

$$q_{\pm}^{a} = \hat{M}_{\pm} v_{\pm}^{a} \,, \tag{40}$$

with $\hat{M}_{\pm} = \hat{\mu}_{\pm} + \hat{p}_{\pm} = \gamma_{\pm}^2(\mu_{\pm} + p_{\pm})$ (see Eqs. (10)-(12)). Combining definition (38a) and expression (40) we deduce that, with respect to the Eckart frame, the bulk velocity of the system is

$$v_E^a = \frac{1}{\hat{M}} \left(\hat{M}_+ v_+^a + \hat{M}_- v_-^a \right) \,, \tag{41}$$

where $\hat{M} = \sum_{\pm} \hat{M}_{\pm}$. Also, following Eq. (31b), the total 3-current associated with our two-component system reads

$$\mathcal{J}^{a} = e \left(Z_{+} \hat{n}_{+} v_{+}^{a} + Z_{-} \hat{n}_{-} v_{-}^{a} \right) , \qquad (42)$$

with Z_{\pm} representing the atomic numbers of the positive and the negative charges respectively.

Proceeding in line with the non-relativistic studies, we will now express the velocities of the two charged components in terms of the bulk properties of the plasma. The first step towards this direction is to assume overall charge neutrality. This applies to scales larger than the Debye length of the species, where the bulk properties of the plasma dominate. In that case, $Z_{+}\hat{n}_{+} = -Z_{-}\hat{n}_{-} = \hat{n}_{E}$ – see definition (31a) – and expressions (30), (42) recast as

$$\rho_{\pm} = \pm e\hat{n}_E, \qquad \qquad \mathcal{J}^a_{\pm} = \pm e\hat{n}_E v^a_{\pm} \qquad \text{and} \qquad \qquad \mathcal{J}^a = e\hat{n}_E \left(v^a_+ - v^a_-\right), \qquad (43)$$

respectively. Then, using (43c) we can invert Eq. (41) and arrive at

$$v_{\pm}^{a} = v_{E}^{a} \pm \frac{\hat{M}_{\mp}}{e\hat{n}_{E}\hat{M}} \mathcal{J}^{a} , \qquad (44)$$

which substituted into Eq. (43b) gives

$$\mathcal{J}_{\pm}^{a} = \pm e \hat{n}_{E} v_{E}^{a} + \frac{\hat{M}_{\mp}}{\hat{M}_{+} + \hat{M}_{-}} \mathcal{J}^{a} \,. \tag{45}$$

These last two results express the velocities and the 3-currents of the individual species in terms of the corresponding bulk variables, provided global electric neutrality holds.

Returning to the Eckart frame, we recall that $\sum_i \mathcal{N}^a_{(i)} = 0$ there (see definition (37)). This immediately implies that $\hat{n}_+ v^a_+ = -\hat{n}_- v^a_-$. The latter combines with relations (44) and, given that the total charge density is zero, leads to

$$v_E^a = \frac{\hat{h}_+ Z_+ + \hat{h}_- Z_-}{e\hat{n}_E \hat{h}(Z_+ - Z_-)} \mathcal{J}_a \,. \tag{46}$$

Therefore, the bulk velocity of the plasma, as measured in the Eckart frame, is colinear to total current. Finally, substituting the above into Eqs. (44) and (45) leads to

$$v_{\pm}^{a} = \frac{Z_{\pm}}{e\hat{n}_{E}(Z_{+} - Z_{-})} \mathcal{J}_{a}, \quad \text{and} \quad \mathcal{J}_{\pm}^{a} = \pm \frac{Z_{\pm}}{Z_{+} - Z_{-}} \mathcal{J}_{a}, \quad (47)$$

respectively.

C. Ohm's law for hot two-fluid plasmas

The last two relations monitor the kinematics of the species in terms of the bulk quantities and with respect to the Eckart (particle) frame, provided overall electrical neutrality holds. That aside, no other restriction has been imposed and the individual charged species are completely general and fully relativistic. In what follows we will use the results of § VB to obtain Ohm's law for a relativistic two-fluid plasma. Thus, recalling that zero total charge

implies $Z_{+}\hat{n}_{+} = -Z_{-}\hat{n}_{-}$ and employing some straightforward algebra, Eq. (34) recasts to

$$\dot{\mathcal{J}}^{\langle a \rangle} = -\frac{4}{3} \Theta \mathcal{J}^{a} - (\sigma^{a}{}_{b} + \omega^{a}{}_{b}) \mathcal{J}^{b} + \frac{e^{2} \hat{n}_{E}^{2} (\dot{M}_{+} + \dot{M}_{-})}{\dot{M}_{+} \dot{M}_{-}} E^{a} + \frac{e^{2} \hat{n}_{E}^{2} (\dot{M}_{+} + \dot{M}_{-})}{\dot{M}_{+} \dot{M}_{-}} \varepsilon^{a}{}_{bc} v_{E}^{b} B^{c}
- \frac{e \hat{n}^{E} (\hat{M}_{+} - \dot{M}_{-})}{\dot{M}_{+} \dot{M}_{-}} \varepsilon^{a}{}_{bc} \mathcal{J}^{b} B^{c} - \frac{e \hat{n}_{E}}{\dot{M}_{+} \dot{M}_{-}} \left[\hat{M}_{-} \left(\dot{\hat{p}}_{+} v_{+}^{a} + D^{a} \hat{p}_{+} \right) - \dot{M}_{+} \left(\dot{\hat{p}}_{-} v_{-}^{a} + D^{a} \hat{p}_{-} \right) \right]
+ \frac{e \hat{n}_{E}}{\dot{M}_{+}} \mathcal{G}_{ext}^{\langle a \rangle} - \frac{e \hat{n}_{E} (\dot{M}_{+} + \dot{M}_{-})}{\dot{M}_{+} \dot{M}_{-}} \mathcal{G}_{-}^{\langle a \rangle} + \frac{e \hat{n}_{E}}{\dot{M}_{+}} \mathcal{G}_{ext} v_{+}^{a} - \frac{e \hat{n}_{E}}{\dot{M}_{+} \dot{M}_{-}} \mathcal{G}_{-} \left(\dot{M}_{-} v_{+}^{a} + \dot{M}_{+} v_{-}^{a} \right)
- e \mathcal{Q}_{-} \left(Z_{+} v_{+}^{a} - Z_{-} v_{-}^{a} \right) - \frac{e^{2} \hat{n}_{E}^{2}}{\dot{M}_{+} \dot{M}_{-}} E_{b} \left(\hat{M}_{-} v_{+}^{b} v_{+}^{a} + \dot{M}_{+} v_{-}^{b} v_{-}^{a} \right) - e D_{b} \left[\hat{n}_{E} \left(v_{+}^{a} v_{+}^{b} - v_{-}^{a} v_{-}^{b} \right) \right]
+ e \hat{n}_{E} \left[\left(\frac{1}{3} \Theta v_{+}^{2} + \sigma_{bc} v_{+}^{b} v_{+}^{c} \right) v_{+}^{a} - \left(\frac{1}{3} \Theta v_{-}^{2} + \sigma_{bc} v_{-}^{b} v_{-}^{c} \right) v_{-}^{a} \right].$$
(48)

For the physical interpretation of all the right-hand side terms and a comparison with their Newtonian counterparts, we refer the reader to the discussion given in § IV after Eq. (34). Relative to that expression, we have kept the velocities of the individual species and replaced the associated 3-currents by means of (43b). The only exception was when dealing with the fourth term in the right-hand of (34), where the 3-currents of the species were replaced using relation (45). The latter has allowed us to include the magnetic convection term and the Hall term in the right-hand side of Eq. (48) explicitly. Keeping the fluid velocities, on the other hand, will help obtain the non-relativistic limit of the above (see § VI below). Also note that the condition of global charge neutrality has removed the 4-acceleration term from (34). Finally, for the interactions we wrote (39) as $\mathcal{G}_{+}^{\langle a \rangle} + \mathcal{G}_{-}^{\langle a \rangle} = \mathcal{G}_{ext}^{\langle a \rangle}$, $\mathcal{G}_{+} + \mathcal{G}_{-} = \mathcal{G}_{ext}$ and assumed that $\mathcal{Q}_{+} + \mathcal{Q}_{-} = 0$. The first two conditions respectively guarantee momentum and energy-density conservation in absence of external interactions, while the last ensures that the overall particle number is preserved.

Expression (48) can be given in a variety of forms depending on the problem in hand. For example, using relations (47a) and (47b) we can recast the right-hand side of (34) in terms of the total 3-current. Alternatively, one can employ Eq. (46) to express everything in terms of the bulk velocity. Here we will do the former, while maintaining the condition of global charge neutrality. To compactify the results, we also introduce the auxiliary bulk variables

$$\hat{M} = \hat{M}_{+} + \hat{M}_{-}$$
 and $\hat{\Delta} = \hat{M}_{+} - \hat{M}_{-}$, (49)

which immediately imply that $4\hat{M}_+\hat{M}_- = \hat{M}^2 - \hat{\Delta}^2$. Employing these relations, Eqs. (34), (48) transform into

$$\begin{aligned} \dot{\mathcal{J}}^{\langle a \rangle} &= -\frac{4}{3} \Theta \mathcal{J}^{a} - (\sigma^{a}{}_{b} + \omega^{a}{}_{b}) \mathcal{J}^{b} + \frac{4e^{2} \hat{n}_{E}^{2} \hat{M}}{(\hat{M}^{2} - \hat{\Delta}^{2})} E^{a} + \frac{2e \hat{n}_{E} [\hat{M}(Z_{+} + Z_{-}) - \hat{\Delta}(Z_{+} - Z_{-})]}{(\hat{M}^{2} - \hat{\Delta}^{2})(Z_{+} - Z_{-})} \varepsilon^{a}{}_{bc} \mathcal{J}^{b} B^{c} \\ &- \frac{2[\hat{M}(\dot{\hat{p}}_{+} Z_{+} - \dot{\hat{p}}_{-} Z_{-}) - \hat{\Delta}(\dot{\hat{p}}_{+} Z_{+} + \dot{\hat{p}}_{-} Z_{-})]}{(\hat{M}^{2} - \hat{\Delta}^{2})(Z_{+} - Z_{-})} \mathcal{J}^{a} - \frac{2e \hat{n}_{E}}{\hat{M}^{2} - \hat{\Delta}^{2}} \left[\hat{M} \left(D^{a} \hat{p}_{+} - D^{a} \hat{p}_{-} \right) - \hat{\Delta} \left(D^{a} \hat{p}_{+} + D^{a} \hat{p}_{-} \right) \right] \right] \\ &+ \frac{2e \hat{n}_{E}}{\hat{M} + \hat{\Delta}} \mathcal{G}^{\langle a \rangle}_{ext} - \frac{4e \hat{n}_{E} \hat{M}}{\hat{M}^{2} - \hat{\Delta}^{2}} \mathcal{G}^{\langle a \rangle}_{-} + \frac{2Z_{+} \mathcal{G}_{ext}}{(Z_{+} - Z_{-})} \left(\hat{M} + \hat{\Delta} \right) \mathcal{J}_{a} - \frac{2\left[\hat{M} \left(Z_{+} + Z_{-} \right) - \hat{\Delta} \left(Z_{+} - Z_{-} \right) \right] \mathcal{G}_{-}}{(Z_{+} - Z_{-}) \left(\hat{M}^{2} - \hat{\Delta}^{2} \right)} \mathcal{J}_{a} \\ &- \frac{(Z_{+} + Z_{-}) \mathcal{Q}_{-}}{n_{E}} \mathcal{J}^{a} + \frac{2[\hat{\Delta}(Z_{+}^{2} - Z_{-}^{2}) - \hat{M}(Z_{+}^{2} + Z_{-}^{2})]}{(\hat{M}^{2} - \hat{\Delta}^{2})(Z_{+} - Z_{-})^{2}} E^{b} \mathcal{J}_{b} \mathcal{J}^{a} - \frac{Z_{+} + Z_{-}}{e(Z_{+} - Z_{-})} D^{b} \left(\frac{1}{\hat{n}_{E}} \mathcal{J}_{b} \mathcal{J}^{a} \right) \\ &+ \frac{Z_{+}^{3} - Z_{-}^{3}}{e^{2} \hat{n}_{E}^{2}(Z_{+} - Z_{-})^{2}} \left(\frac{1}{3} \Theta \mathcal{J}^{2} + \sigma^{bc} \mathcal{J}_{b} \mathcal{J}_{c} \right) \mathcal{J}^{a} , \tag{50}$$

with $\hat{M}^2 - \hat{\Delta}^2$, $Z_+ - Z_- \neq 0$. Relations (48) and (50) provide the relativistic (1+3 covariant) version of the generalised Ohm's law, with respect to the Eckart frame, when applied to a mixture of two hot and interacting charged fluids. We also remind the reader that these results have been obtained under the assumption of global electrical neutrality.

D. Relativistic particle-antiparticle plasmas

Expression (50) is particularly useful when dealing with particle-antiparticle pairs. An electron-positron plasma, for a example, has $Z_{\pm} = \pm 1$ and $M_{\pm} = M_{-}$. The latter means that $\hat{M} = 2\hat{M}_{\pm}$, $\hat{\Delta} = 0$ and $\hat{M}^2 - \hat{\Delta}^2 = \hat{M}^2 = 4\hat{M}_{\pm}^2$ (see

definitions (49)). In such an environment Eq. (50) simplifies to

$$\dot{\mathcal{J}}^{\langle a \rangle} = -\frac{4}{3} \Theta \mathcal{J}^a - (\sigma^a{}_b + \omega^a{}_b) \mathcal{J}^b + \frac{4e^2 \hat{n}_E^2}{\hat{M}} E^a - \frac{2\dot{p}}{\hat{M}} \mathcal{J}^a - \frac{4e\hat{n}_E}{\hat{M}} \mathcal{G}_-^{\langle a \rangle} -\frac{1}{\hat{M}} E^b \mathcal{J}_b \mathcal{J}^a + \frac{1}{2e^2 \hat{n}_E^2} \left(\frac{1}{3} \Theta \mathcal{J}^2 + \sigma^{bc} \mathcal{J}_b \mathcal{J}_c\right) \mathcal{J}^a,$$
(51)

and contains no Hall effect (since $\hat{\Delta} = 0$ in this case). Also, given that $p_+ = p_-$ for particle-antiparticle pairs, only part of Biermann-battery effect survives (the relativistic – carried by the fourth term in the right-hand side of the above). This and the Joule-heating term are the only purely relativistic corrections.

VI. OHM'S LAW FOR COLD PROTON-ELECTRON PLASMAS

A. Non-relativistic limit

Cold plasmas have components with non-relativistic relative velocities. Thus, at the low velocity limit, one can ignore terms of quadratic (and higher) order in $v_{(i)}^a$. As a result, $\gamma_{(i)} \simeq 1$, $\hat{n}_{(i)} \simeq n_{(i)}$, $\hat{\mu}_{(i)} \simeq \mu_{(i)}$ and $\hat{h}_{(i)} \simeq \mu_{(i)} + p_{(i)} \simeq \mu_{(i)}$. If, in addition, the plasma is a mixture of protons and electrons, we have $Z_{\pm} = \pm 1$. When the condition of overall electrical neutrality is also imposed, we may set $n_{\pm} = n$ and $\rho_{\pm} = \pm en$. Then, assuming that m_+ and m_- are the proton and the electron masses respectively (with $\mu_{\pm} = nm_{\pm}$ and $m_- \ll m_+$), we may write

$$\hat{M} = \hat{M}_{+} + \hat{M}_{-} \simeq nm_{+}$$
 and $\hat{\Delta} = \hat{M}_{+} - \hat{M}_{-} \simeq nm_{+}$. (52)

Note, however, that

$$\hat{M}^2 - \hat{\Delta}^2 = 4\hat{M}_+\hat{M}_- \simeq 4n^2m_+m_- \neq 0.$$
(53)

Applying the non-relativistic limit to (48) immediately removes the last four terms from the right-hand side of the latter (all quadratic in v_{\pm}^a). For cold species we may also ignore the \dot{p} – terms, though the spatial variations of the pressure are not necessarily negligible. When dealing with proton electron systems, the eighth (Q_+) term in the right-hand side of Eq. (48) vanishes (this is clearly shown in (50) – recall that $Z_{\pm} = \pm 1$). We may also ignore energy-density changes due to collisions between the cold components, and therefore remove the ninth term from the right-hand side of (48). Finally, when applied to our two-component fluid, the collisional ($\mathcal{G}_{-}^{\langle a \rangle}$) term in (48), which triggers changes in the momentum of the species, takes the more familiar form

$$-\frac{e\hat{n}_E(\dot{M}_+ + \dot{M}_-)}{\hat{M}_+ \hat{M}_-} \mathcal{G}_-^{\langle a \rangle} = -\overline{\nu} \mathcal{J}_a = -\eta \frac{e^2 n}{m_-} \mathcal{J}_a \,, \tag{54}$$

where $\overline{\nu}$ is the average collision frequency and $\eta = \overline{\nu}m_{-}/e^{2}n$ is the (scalar) electrical resistivity of the two-component medium (e.g. see [20]). Note that, although here we have adopted the common approximation of a scalar electrical resistivity, our analysis also applies to general fluids with anisotropic (tensor) resistivity. This can be done by using kinetic theory to specify the interaction terms. On these grounds and by using the auxiliary relations (52), (53), expression (48) reduces to

$$\dot{\mathcal{J}}^{\langle a \rangle} = -\frac{4}{3} \Theta \mathcal{J}^a - (\sigma^a{}_b + \omega^a{}_b) \mathcal{J}^b + \frac{e^2 n}{m_-} E_a + \frac{e^2 n}{m_-} \varepsilon^a{}_{bc} v^b_E B^c - \frac{e}{m_-} \varepsilon^a{}_{bc} \mathcal{J}^b B^c + \frac{e}{m_-} D^a p_- - \eta \frac{e^2 n}{m_-} \mathcal{J}^a ,$$
(55)

in agreement with the expressions found in the standard literature (e.g. compare to Eq. (3.5.9) in [20]). Alternatively, we may recast the above into the more familiar form

$$E^{a} + \varepsilon^{a}{}_{bc}v^{b}_{E}B^{c} - \eta \mathcal{J}^{a} = \frac{m_{-}}{e^{2}n} \left[\dot{\mathcal{J}}^{\langle a \rangle} + \frac{4}{3}\Theta \mathcal{J}^{a} + (\sigma^{a}{}_{b} + \omega^{a}{}_{b})\mathcal{J}^{b} \right] + \frac{1}{en}\varepsilon^{a}{}_{bc}\mathcal{J}^{b}B^{c} + \frac{1}{en}D^{a}p_{-},$$
(56)

which immediately shows the terms responsible for the Hall and the Biermann-battery effects – see the last two terms in the right-hand side. Either of these two expressions provides the 1+3 covariant form of the generalised Ohm's law for a cold proton-electron plasma.

B. Magnetohydrodynamic limits

Adopting the commonly used magnetohydrodynamic (MHD) approximations, namely ignoring all the right-hand side terms in Eq. (56), leads to the usual form of Ohm's law for an electrically resistive medium

$$E^a + \varepsilon^a{}_{bc} v^b_E B^c = \eta \mathcal{J}^a \,. \tag{57}$$

We note here that, instead of dropping the 3-current terms seen inside the square brackets of (56), one could in principle incorporate them within the resistive term in the left-hand side of that expression. Then, we will be referring to a "generalised resistivity" that accounts for changes in the electrical properties of the medium due to relative motion (i.e. inertial) effects. It is also clear that in the idealised case of a perfectly conducting medium, namely for $\eta \to 0$, Eq. (57) reduces to the ideal-MHD form of Ohm's law. The latter is given by the simple and well known formula

$$E^a = -\varepsilon^a{}_{bc}v^b_E B^c \,. \tag{58}$$

VII. SUMMARY

Hot plasmas are of major importance in a variety of physical phenomena, ranging from laboratory physics to astrophysics and cosmology. A key factor in determining the behaviour of plasmas is their electrical properties and these are theoretically monitored by means of Ohm's law. The latter appears in a number of different versions, which depend on the specifics of the problem in hand. Here, we are providing fully relativistic and fully nonlinear expressions for the generalised Ohm's law for plasmas, by deriving the 1+3 covariant propagation equation of the 3-current density associated with a multi-component fluid. Adopting a suitable definition for the irreducible variables of the matter fields (see Eqs. (10)-(13) and (15a)), we were able to address hot plasmas with a fluid-based approach and without the need of kinetic theory. The use of the covariant methods has also facilitated a mathematically compact and physically transparent presentation of the subject. As a result, our expressions allow for a direct comparison with the familiar Newtonian versions of Ohm's law, while identifying the relativistic corrections to them. We show, for example, that the relativistic analogue of the Biermann-battery effect has an additional contribution from the temporal pressure variations. Our main result is given in Eq. (34) and applies to any multi-component fluid, relativistic or not, which means that it can be adapted to address a great variety of physical problems. With the general form of Ohm's in hand, our next step was to introduce a particular reference frame. Identifying our fundamental observers with the Eckart frame, allowed us to follow on the steps of the non-relativistic studies and therefore considerably simplify the mathematics. Then, by confining to two-fluid systems and assuming overall charge neutrality, we expressed Ohm's law in terms of the properties of the bulk. Finally, we closed our discussion by considered a number of applications. These included hot plasmas of two oppositely charged fluids, hot electron-positron mixtures, cold electron-proton systems and also the resistive and the ideal-MHD limits of our results. In each case, we have discussed the physics of the situation, identified the familiar effects, like the Biermann-battery and the Hall effects, and pointed to the relativistic corrections were applicable.

The multi-fluid description adopted in the present paper is essential in almost every study of small-scale astrophysical plasmas. The same approach is also necessary when looking into the nonlinear regime of galaxy formation, when the proto-structure has decoupled from the background expansion and collapses. Then, one can use our equations to investigate the evolution of proto-galactic magnetic fields, in particular their amplification and dissipation, within and also outside the MHD limit. Moreover, the use of the irreducible variables, assigns an unambiguous physical interpretation to every variable in our equations and helps to isolate the physical effects under consideration. For example, vorticity terms are related to turbulence and dynamo-like mechanisms, while those involving the shear describe shape distortions and can play an important role during galaxy formation. In addition, going beyond the cold-plasma limit, makes our equations suitable for studies of relativistic plasmas, like those in hot interstellar clouds and in accretion discs around compact stars. Applications of this sort will be the subject of future work.

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APPENDIX A: THE LANDAU FRAME

As stated in § VA, an alternative frame choice is that of the Landau (or energy) frame. Both the particle and the energy frames are physically equivalent and choosing one against the other depends on the particulars of the problem

$$u_E^a u_a^L = \frac{1}{\sum_{i,j} n_{(i)} \sqrt{\hat{\mu}_{(j)}^2 - \hat{q}_{(j)}^2}} \sum_{i,j} \left[-\hat{n}_{(i)} \hat{\mu}_{(j)} + \mathcal{N}_{(i)}^a \hat{q}_a^{(j)} \right].$$
(A1)

When the velocities of the individual species are small, we can neglect terms proportional to the heat flux. This means that for cold plasmas the above reduces to $u_E^a u_a^L \simeq \sum_{i,j} \left[-\hat{n}_{(i)} \hat{\mu}_{(j)} \right] / \sum_{i,j} \left[\hat{n}_{(i)} \hat{\mu}_{(j)} \right] = -1$. Then, following (5), the Eckart

and the Landau frames (and their associated expressions of Ohm's law) effectively coincide.

- [1] H. Ardavan, Astrophys. J. 203, 226 (1976).
- [2] E. G. Blackman and G.B. Field, Phys. Rev. Lett. 71, 3481 (1993).
- [3] M. Gedalin, Phys. Rev. Lett. **76**, 3340 (1996).
- [4] R. Khanna, Mon. Not, R. Astron. Soc. 294, 673 (1998).
- [5] D. L. Meier, Astrophys. J. 605, 340 (2004).
- [6] G. M. Kremer and C. H. Patsko, Physica A, **322**, 329 (2003).
- [7] G.F.R. Ellis, in Cargèse Lectures in Physics, Vol. VI, Ed. E. Schatzman (Gordon & Breach, New York, 1973) p. 1.
- [8] G.F.R Ellis and H. van Elst, in *Theoretical and Observational Cosmology*, Ed. M. Lachièze-Rey (Kluwer, Dordrecht, 1998)
 p. 1
- [9] J.D. Barrow, R. Maartens and C.G Tsagas, Phys. Rep. 449, 131 (2007).
- [10] C.G. Tsagas, A. Challinor and R. Maartens, preprint: arXiv:0705.4397
- [11] K.S. Thorne and D. Macdonald, Mon. Not. R. Astron. Soc. 198, 339 (1982) (microfiche MN 198/1).
- [12] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (Wiley, New York, 1971).
- [13] A. R. King and G. F. R. Ellis, Comm. Math. Phys. 31, 209 (1973).
- [14] L.D. Landau and E.M. Lifshitz, Course of Theoretical Physics, Vol. 6: Fluid Mechanics (Butterworth & Heinemann, Oxford, 1998).
- [15] R.D. Hazeltine and S. M. Mahajan, Astrophys. J. 567, 1262 (2002).
- [16] J.L. Synge, Relativity: The Special Theory (North-Holland, Amsterdam, 1964).
- [17] W. Israel, Ann. Phys. 100, 310 (1976).
- [18] L. Spitzer Jr., Physics of Fully Ionized Gas (Dover, New York, 2006).
- [19] E. Priest and T. Forbes, Magnetic Reconnection: MHD Theory and Applications (Cambridge University Press, Cambridge, 2000).
- [20] N.A. Krall and A.W. Trivelpiece, Principles of Plasma Physics (McGraw-Hill, New York, 1973).
- [21] To a certain extend the analysis presented here resembles that given in [11].
- [22] Throughout the article we use geometrised units, where $\kappa = 8\pi G/c^4 = 1 = c$.
- [23] Angled brackets indicate the projected component of vectors (e.g. $\dot{E}^{\langle a \rangle} = h^a{}_b \dot{E}^b$ see Eq. (17)) and the orthogonally projected, symmetric and trace-free part of second-rank tensors (e.g. $B^{\langle a \rangle} = B^a B^b (B^2/3)h^{ab}$ see Eq. (22)).