The Magueijo-Smolin model of Deformed Special Relativity from five dimensions

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Abstract

It is known that the space of momenta of DSR can be identified with de Sitter space. In this paper, we discuss the relation of the noncanonical phase space of the Magueijo-Smolin model of DSR with the canonical 5-dimensional phase space in which the de Sitter space of momenta is embedded. We suggest that in analogy with the momentum variables, also the position variables should be constrained to lie on a null hypersurface of the five-dimensional space.

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The purpose of doubly special relativity (DSR) theories is to give an effective description of quantum gravity effects on particle dynamics at energies near the Planck scale, by postulating a nonlinear (deformed) action of the Lorentz group on momentum space, such that the Planck energy κ is left invariant [1-4].

Since the deformation is not uniquely defined, one can obtain many different realizations of the theory. An advance in the understanding of this problem was given by the observation that the momentum space of DSR models can be thought as a four-dimensional hyperboloid embedded in a five dimensional target space [5]. The choice of coordinates on the hyperboloid corresponds to different DSR models.

Although this picture is very suggestive, no convincing interpretation has been advanced for the dual five-dimensional position space. It must be noticed however that already in a four-dimensional setting the realization of position space in DSR theories is ambiguous, since it is not determined in a natural way by the DSR postulates (see e.g. [6] and references therein). For example, although a realization in terms of noncommutative coordinates appears more natural, it is also possible to adopt standard noncommuting coordinates [10].

Moreover, the physical meaning of a fifth position coordinate is unclear. In [7] a proposal was advanced based on the existence of a linear realization of the deformed Lorentz group in five dimensional momentum space. The fifth position coordinate was identified with the evolution parameter of the field equations in a commutative spacetime. A different interpretation based on the hamiltonian formalism was put forward in [8]. In that case the fifth coordinate is related to an arbitrary choice of gauge, but its relation with the physical observables is unclear.

In this paper we use some recent results [9] to identify the correspondence between the physical coordinates of the MS model [4], which is probably the simplest realization of DSR from an algebraic point of view, and the 5-dimensional target space variables of [5]. Following [7], we also identify a formal fifth spacetime coordinate with the invariant evolution parameter which appears in the field equations. In this way we constrain the 5-dimensional position coordinates to form a null vector, so that only four coordinates are independent, as required by physics.

We use the following conventions: $A = 0, \dots, 4$ are target space indices, $\mu = 0, \dots, 3$ are spacetime indices, $i = 1, \dots, 3$ are spatial indices. We always use lower indices, which are summed by means of the flat metric $\eta_{\mu\nu} = \text{diag}\ (1, -1, -1, -1)$. The modulus of a 4-vector A_μ is denoted by $A \equiv \sqrt{A_\mu A_\mu}$.

The MS model is defined by nontrivial transformation laws of the physical momenta p_μ under boosts [4]. For infinitesimal boosts in the i-th direction, the momenta transform as

$$\delta_i p_0 = (1 - p_0/\kappa) p_i, \qquad \delta_i p_j = \delta_{ij} p_0 - p_i p_j/\kappa, \tag{1}$$

while they transform in the standard way under rotations. The quantity

$$\frac{p^2}{(1 - p_0/\kappa)^2} \tag{2}$$

is invariant under the deformed transformations (1).

The MS model does not specify the transformation law under boosts for the position coordinates x_μ. The most natural choice is based on the requirement that they transform covariantly with respect to the momenta [10]:

$$\delta_i x_0 = x_i + p_i x_0/\kappa, \qquad \delta_i x_j = \delta_{ij} x_0 + p_i x_j/\kappa. \tag{3}$$

This can be achieved if the phase space coordinates satisfy the Poisson brackets [10,11]

$$\begin{aligned} &\{x_0, x_i\} = x_i/\kappa, \quad \{p_0, p_i\} = 0, \quad \{x_0, p_0\} = 1 - p_0/\kappa, \\ &\{x_i, p_j\} = -\delta_{ij}, \quad \{x_0, p_i\} = -p_i/\kappa, \qquad \{x_i, p_0\} = 0, \end{aligned} \tag{4}$$

which are typical of DSR models. Notice that one is forced to use noncommutative coordinates. This definition also yields the classical transformation law for the velocity and fixes the speed of light as the limit velocity [10,11]. The main peculiarity of (3) is the momentum dependence of the transformations of the spacetime coordinates under boosts.

In a recent paper [9], it was shown that one can define auxiliary variables *

$$P_\mu = \frac{p_\mu}{1 - p_0/\kappa}, \qquad X_\mu = (1 - p_0/\kappa)\, x_\mu, \tag{5}$$

that satisfy canonical Poisson brackets

$$\{X_\mu, P_\nu\} = \eta_{\mu\nu}, \qquad \{X_\mu, X_\nu\} = \{P_\mu, P_\nu\} = 0. \tag{6}$$

These coordinates are unphysical, but are helpful in order to convert the results of special relativity to those of the MS model. In particular, the physical quantities have the standard expression in terms of the auxiliary coordinates X_μ, P_μ. To derive their expression in the MS model it is then sufficient to write them in terms of the physical coordinates x_μ, p_μ.

For example, the generators $J_{\mu\nu}$ of the Lorentz transformations take the form $J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$. Using (5), one easily deduces that the Lorentz generators have the standard form $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ also in terms of the MS coordinates [9]. From the Poisson brackets (4) one can then recover the deformed Lorentz transformations for x_μ and p_μ.

The purpose of this letter is to relate the results of [9] to the five-dimensional formalism of [5] and to give an interpretation of the fifth coordinate as attempted in [7,8].

In [5] it was shown that one can identify the space of momenta with an hyperboloid of equation $\pi^2 - \pi_4^2 = -\kappa^2$ in a 5-dimensional momentum space of coordinates π_A and metric $\eta_{AB} = (1, -1, -1, -1, -1)$. The choice of different coordinates $\tilde{P}_\mu$ on the hyperboloid gives rise to different realizations of DSR. For the Snyder basis, for example,

$$\tilde{P}_\mu = \frac{\kappa}{\pi_4}\, \pi_\mu. \tag{7}$$

* This result had been independently anticipated in [11].

Introducing the 5-dimensional position variables ξ_A canonically conjugate to π_A, one can write down the variables $\tilde{X}_\mu$ canonically conjugated to $\tilde{P}_\mu$ as

$$\tilde{X}_\mu = \frac{\pi_4}{\kappa}\,\xi_\mu. \tag{8}$$

We identify the variables $\tilde{P}_\mu$ and $\tilde{X}_\mu$ with P_μ and X_μ defined above. From (5) and (7)-(8) then follows

$$p_\mu = \frac{\kappa}{\pi_0+\pi_4}\,\pi_\mu, \qquad x_\mu = \frac{\pi_0+\pi_4}{\kappa}\,\xi_\mu. \tag{9}$$

At this point one might apply the hamiltonian formalism introduced in [8] for the Snyder model. Denoting with a bar the variables of [8], they can be written in terms of X_A and P_A as $\bar{p}_\mu = P_\mu$, $\bar{x}'_\mu = X_\mu - X_A P_A\, P_\mu/\kappa^2$, $\bar{T} = X_A P_A$. Then our variables X_μ and P_μ are analogous to y_μ and q_μ in eq. (30) of ref. [8].

Instead, we prefer to introduce a different interpretation of 5-dimensional spacetime, closer to that proposed in [7]. In particular, we wish to constrain the 5-dimensional coordinates so that the fifth coordinate coincides with the evolution time τ that parametrizes the trajectories of the particles. The relation of τ with the spacetime coordinates, first derived in [11], has often been overlooked, but is important, at least in a lagrangian formalism. Since it is by definition invariant under the deformed Lorentz transformations, $d\tau^2$ can be identified with the line element of the 4-dimensional spacetime [10,11]. Although in this context the introduction of the coordinate τ is purely formal, it gives a more elegant formulation of the action principle.

From (7) we may define the fifth component of P_μ as $P_4 = \kappa$. We also define the fifth component of X_μ by imposing that X_A be a null vector, i.e. $X_4^2 = X^2$. Then,

$$X_4 = X = \frac{\pi_4}{\kappa}\,\xi. \tag{10}$$

It is easy to check that X_4 is left invariant by the deformed Lorentz transformations generated by $J_{\mu\nu}$. By the previous definition, X_4 coincides with the invariant affine parameter τ which parametrizes the trajectories of point particles [10,11]. In analogy with [7], we consider therefore a 5-dimensional spacetime with coordinates (x_μ, τ). One may interpret this choice as constraining the motion to the hypersurface $\xi_A^2 = 0$, in analogy with the constraint $\pi_A^2 = -\kappa^2$ on momentum space. This appears to be the most natural condition to be imposed in order to reduce to four the number of independent position coordinates.

From

$$\{X_\mu, X_4\} = 0, \qquad \{P_\mu, X_4\} = \frac{X_\mu}{X_4}, \tag{11}$$

follows

$$\{x_\mu, \tau\} = (1-p_0/\kappa)^2\ \frac{x_0 x_\mu/\kappa}{\tau}, \qquad \{p_\mu, \tau\} = (1-p_0/\kappa)^2\ \frac{x_\mu - x_0 p_\mu/\kappa}{\tau}.$$

The dynamics of a free particle of mass m can be obtained by varying the action

$$\begin{aligned} I &= \int d\tau \left[\dot{X}_\mu P_\mu - \frac{\lambda}{2}(P^2 - m^2)\right] \\ &= \int d\tau \left[\frac{p_\mu}{1-p_0/\kappa}\ \frac{d}{d\tau}\left[(1-p_0/\kappa)x_\mu\right] - \frac{\lambda}{2}\left(\frac{p^2}{(1-p_0/\kappa)^2} - m^2\right)\right], \end{aligned} \tag{12}$$

with τ defined above. This is in contrast with other formulations where τ is an external parameter whose properties are not specified. The action is equivalent up to total derivatives to those given in refs. [9,10], where the field equations following from (12) are discussed.

To conclude, we consider the extension from the Lorentz to the Poincaré algebra. This is achieved by adding the translations generators T_μ to the Lorentz algebra. It must be noticed, however, that the generators T_μ are not determined uniquely. Usually they are identified (at least implicitly) with the momentum coordinates p_μ. With this definition, they act linearly on the space of momenta, but the Poincaré algebra is deformed. In particular, the boost generators N_i have nonlinear Poisson brackets with the translation generators,

$$\{N_i, T_0\} = T_i - T_0 T_i/\kappa, \qquad \{N_i, T_j\} = \delta_{ij} T_0 - T_i T_j/\kappa. \tag{13}$$

The action of the translations on the coordinates following from this definition is

$$\begin{aligned} &\{T_0, x_0\} = -(1 - p_0/\kappa) \qquad \{T_0, x_i\} = 0, \\ &\{T_i, x_0\} = p_i/\kappa, \qquad \{T_i, x_j\} = \delta_{ij}. \end{aligned} \tag{14}$$

An alternative possibility [9] is to define $T_\mu = P_\mu$. In this case the standard form of the Poincaré algebra is preserved, but the translation operator acts nontrivially on the momenta. The action of the translations on the coordinates takes a neater form,

$$\begin{aligned} &\{T_0, x_0\} = \frac{-1}{1 - p_0/\kappa} \qquad \{T_0, x_i\} = 0, \\ &\{T_i, x_0\} = 0, \qquad \{T_i, x_j\} = \frac{\delta_{ij}}{1 - p_0/\kappa}. \end{aligned} \tag{15}$$

Its effect is a sort of momentum-dependent dilation of time and lengths under translations.

At the classical level, the difference between the two representations of translations is not great, since in both cases $\{T_\mu, p_\nu\} = 0$, but at the quantum level is more evident. In particular, the natural law of addition of momenta, arising from translation invariance, is linear in the first case, while in the second case coincides with that proposed in [12], which implies the existence of a maximum energy κ. The most natural choice in the context of DSR for both the addition law of momenta and the transformation of coordinates seems to be that determined by $T_\mu = P_\mu$. However, since the one-particle action (12) is invariant under the action of any generator T_μ which is function only of p_μ, at this stage one could choose T_μ arbitrarily. This posibility may be of help in the solution of the so-called soccer ball problem. In any case, the correct addition law can be derived only after the general form of the interaction between particles has been established.

References

[1] G. Amelino-Camelia, Int. J. Mod. Phys. **D11**, 35 (2002), Phys. Lett. **B510**, 255 (2001).

[2] J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. Phys. **243**, 90 (1995).

[3] J. Kowalski-Glikman, Mod. Phys. Lett. **A17**, 1 (2002).

[4] J. Magueijo and L. Smolin, Phys. Rev. Lett. **88**, 190403 (2002).

[5] J. Kowalski-Glikman, Phys. Lett. **B547**, 291 (2002); J. Kowalski-Glikman, S. Nowak, Class. Quantum Grav. **20**, 4799 (2003).

[6] P. Galán, G. Mena Marugan, Int. J. Mod. Phys. **D16**, 1133 (2007).

[7] A.A. Deriglazov, B.F. Rizzuti, Phys. Rev. **D71**, 123515 (2005).

[8] F. Girelli, T. Konopka, J. Kowalski-Glikman, E.R. Livine, Phys. Rev. **D73**, 045009 (2006).

[9] S. Ghosh, P. Pal, Phys. Rev. **D75**, 105021 (2007).

[10] S. Mignemi, Phys. Rev. **D68**, 065029 (2005).

[11] A. Granik, `hep-th/0207113`.

[12] S. Judes and M. Visser, Phys. Rev. **D68**, 045001 (2003).