Coframe geometry and gravity

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Abstract

The possible extensions of GR for description of fermions on a curved space, for supergravity and for loop quantum gravity require a richer set of 16 independent variables. These variables can be assembled in a coframe field, i.e., a local set of four linearly independent 1-forms. In this chapter we study the gravity field models based on a coframe variable alone. We give a short review of the coframe gravity. This model has the viable Schwarzschild solutions even being alternative to the standard GR. Moreover, the coframe model treating of the gravity energy may be preferable to the ordinary GR where the gravity energy cannot be defined at all. A principle problem that the coframe gravity does not have any connection to a specific geometry even being constructed from the geometrical meaningful objects. A geometrization of the coframe gravity is an aim of this chapter. We construct a complete class of the coframe connections which are linear in the first order derivatives of the coframe field on an n dimensional manifolds with and without a metric. The subclasses of the torsion-free, metric-compatible and flat connections are derived. We also study the behavior of the geometrical structures under local transformations of the coframe. The remarkable fact is an existence of a subclass of connections which are invariant when the infinitesimal transformations satisfy the Maxwell-like system of equations. In the framework of the coframe geometry construction, we propose a geometrical action for the coframe gravity. It is similar to the Einstein-Hilbert action of GR, but the scalar curvature is constructed from the general coframe connection. We show that this geometric Lagrangian is equivalent to the coframe Lagrangian up to a total derivative term. Moreover there is a family of coframe connections which Lagrangian does not include the higher order terms at all. In this case, the equivalence is complete.

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0 Introduction. Why do we have to go beyond Riemannian geometry?

General relativity (GR) is, probably, the best of the known theories of gravity. From mathematical and aesthetic points of view, it can be used as a standard of what a physical theory has to be. Up to this day, the Einstein theory is in a very good agreement with the observation data. Probably the main idea of Einstein's GR is that the physical properties of the gravitational field are in one-to-one correspondence with the geometry of the base manifold. The standard GR is based on a Riemannian geometry with a unique metric tensor and a unique Levi-Civita connection constructed from this tensor. Hence, the gravity field equations of GR predicts a unique (up to diffeomorphism transformations) metric tensor and consequently a unique geometry. Therefore any physical field except of gravity can not have an intrinsic geometrical sense in the Riemannian geometry.

After the classical works of Weyl, Cartan and others, we know that the Riemannian construction is not a unique possible geometry. A most general geometric framework involves independent metric and independent connection. A gravity field model based on this general geometry (Metric-affine gravity) was studied intensively, see [1]— [17] and the references given therein. Probably a main problem of this construction is a huge number of geometrical fields which do not find their physical partner.

In this chapter we study a much more economical construction based on a unique geometrical object — coframe field. Absolute (teleparallel) frame/coframe variables (repèr, vierbein, ...) were introduced in physics by Einstein in 1928 with an aim of a unification of gravitational and electromagnetic fields (for classical references, see [18]). The physical models for gravity based on the coframe variable are well studied, see [19]— [42]. In some aspects such models are even preferable from the standard GR. In particular, they involve a meaningful definition of the gravitational energy, which is in a proper correspondence with the Noether procedure. Moreover some problems inside and beyond Einstein's gravity require a richer set of 16 independent variables of the coframe. In the following issues of gravity, the coframe is not only a useful tool but often it cannot even be replaced by the standard metric variable: (i) Hamiltonian formulation [43], [44]; (ii) positive energy proofs [45]; (iii) fermions on a curved manifold [46], [47]; (iv) supergravity [48]; (v) loop quantum gravity [49].

Unfortunately, in the coframe gravity models, the proper connection between physics and the underlying geometry is lost. In this chapter, we propose a way of geometrization of the coframe gravity. In particular, we study which geometric structure can be constructed from the vierbein (frame/coframe) variables and which gravity field models can be related to this geometry.

The organization of the chapter is as follows:

In the first section, we give a brief account of the gravity field model based on the coframe field instead of the pure metrical construction of GR. We discuss the following features: (i) The coframe gravity is described by a 3-parametric set of models; (ii) All the coframe models are derivable from a Yang-Mills-type Lagrangian; (iii) The coframe field equations are well defined for all values of the parameters. Only for the pure GR case, the system id degenerated to 10 equations for 16 variables; (iv) The energy-momentum tensor of the coframe field is well defined for all models except GR. In the latter case the tensor nature of the energy-momentum expression is lost; (v) There is a subset of viable fields with a unique spherical symmetric solution, which corresponds to Schwarzschild metric; (vi) The same subset is derived by the requirement of the free field limit approximation. All these positive properties make the coframe gravity a relevant subject of investigation.

In section 2, we construct a geometrical structure based on a coframe variable as unique building block. In an addition to the coframe volume element and metric, we present a most general coframe connection. The Levi-Civita and flat connections are special cases of it. The torsion and nonmetricity tensors of the general coframe connection are calculated. We identify the subclasses of symmetric (torsion-free) connections and of metric-compatible connections. The unique symmetric metric-compatible connection is of Levi-Civita. We study the transformations of the coframe field and identify a subclass of connections which are invariant under restricted coframe transformations. Quite remarkable that restriction conditions are approximated by a Maxwell-type system.

In section 3, we are looking for a geometric representation of the gravity coframe model. The main result is that the free-parametric gravity coframe Lagrangian can be replaced by a standard Einstein-Hilbert Lagrangian, when the curvature scalar is calculated on a general coframe connection. The standard GR Lagrangian contains a second order derivative term which appears in the form of the total derivative. This term does not influence the field equation, but it cannot be consistently removed. We show that there is a set of coframe connections which Einstein-Hilbert Lagrangian does not involve the second order derivative term at all.

In the last section, some proposals of possible developments of a geometrical coframe construction and its applications to gravity are presented.

1 Coframe gravity

Let us give a brief account of gravity field models based on a coframe field. We refer to such models as *coframe gravity*. This is instead of the Einsteinian *metric gravity* based on a metric tensor field. We will use here mostly the notations accepted in [33].

1.1 Coframe Lagrangian

Consider a smooth, non-degenerated coframe field $\{\vartheta^\alpha,\ \alpha=0,1,2,3\}$ defined on a $4D$ smooth differential manifold M. The 1-forms ϑ^α are declared to be pseudo-orthonormal. Thus a metric on M is defined by

$$g=\eta_{\alpha\beta}\vartheta^\alpha\otimes\vartheta^\beta\,,\qquad \eta_{\alpha\beta}=(-1,1,1,1)\,. \tag{1.1}$$

So, the coframe field ϑ^α is considered as a basic dynamical variable while the metric g is treated as a secondary structure.

The coframe field is defined only up to *global pseudo-rotations*, i.e. $SO(1,3)$ transformations. Consequently, the truly dynamical variable is an equivalence class of coframes $[\vartheta^\alpha]$, while the global pseudo-rotations produce an equivalence relation on this class. Hence, in addition to the invariance under the diffeomorphic transformations of the manifold M, the basic geometric structure has to be global (rigid) $SO(1,3)$ invariant.

Gravity is described by differential invariants of the coframe structure. There is an important distinction between the diffeomorphic invariants of the metric and of the coframe structures. Since the metric invariants of the first order are trivial, the metric structure admits diffeomorphic invariants only of the second order or greater. A unique invariant of the second order is the scalar curvature. This expression is well known to play the key role of an integrand in the Einstein-Hilbert action. The coframe structure admits diffeomorphic and rigid $SO(1,3)$ invariants even of the first order. A simple example is the expression $e_\alpha\rfloor d\vartheta^\alpha$, see Appendix for notations and basic definitions. The operators, which are diffeomorphic invariants and global covariants, can contribute to a general coframe field equation. A rich class of such equations is constructed in [27]. A requirement of derivability of the field equations from a Lagrangian strictly restricts the variety of possible options.

We restrict the consideration to odd, quadratic (in the first order derivatives of the coframe field ϑ^α), diffeomorphic, and global $SO(1,3)$ invariant Lagrangians. A general Lagrangian of such a type is represented by a linear combination of three 4-forms which are referred to as the Weitzenböck invariants. Consider the exterior differentials of the basis 1-forms $d\vartheta^\alpha$ and introduce the coefficients of their expansion in the basis of even 2-forms $\vartheta^{\alpha\beta}$

$$d\vartheta^\alpha=\vartheta^\alpha_{i,j}dx^i\wedge dx^j=\frac{1}{2}C^\alpha{}_{\beta\gamma}\vartheta^{\beta\gamma}\,. \tag{1.2}$$

We use here the abbreviation $\vartheta^{\alpha\beta\cdots}=\vartheta^\alpha\wedge\vartheta^\beta\wedge\cdots$. By definition, the coefficients $C^\alpha{}_{\beta\gamma}$ are antisymmetric, $C^\alpha{}_{\beta\gamma}=-C^\alpha{}_{\gamma\beta}$. Their explicit expression can be given by the differential form notations (see Appendix)

$$C^\alpha{}_{\beta\gamma}=e_\gamma\rfloor(e_\beta\rfloor d\vartheta^\alpha)\,. \tag{1.3}$$

The symmetric form of a general second order coframe Lagrangian is given by [25]

$${}^{(\texttt{cof})}L=\frac{1}{2\ell^2}\sum_{i=1}^{3}\rho_i\ {}^{(i)}L\,, \tag{1.4}$$

where ℓ denotes the Planck length constant, while ρ_i are dimensionless parameters. The partial Lagrangian expressions are

$$\begin{aligned} {}^{(1)}L &= d\vartheta^\alpha \wedge *d\vartheta_\alpha = \frac{1}{2} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma} * 1\,, && (1.5)\\ {}^{(2)}L &= (d\vartheta_\alpha \wedge \vartheta^\alpha) \wedge * \left(d\vartheta_\beta \wedge \vartheta^\beta\right) = \frac{1}{2} C_{\alpha\beta\gamma} \left(C^{\alpha\beta\gamma} + C^{\beta\gamma\alpha} + C^{\gamma\alpha\beta}\right) * 1\,, && (1.6)\\ {}^{(3)}L &= (d\vartheta_\alpha \wedge \vartheta^\beta) \wedge * (d\vartheta_\beta \wedge \vartheta^\alpha) = \frac{1}{2} \left(C_{\alpha\beta\gamma} C^{\alpha\beta\gamma} - 2C^\alpha{}_{\alpha\gamma} C_\beta{}^{\beta\gamma}\right) * 1\,. && (1.7) \end{aligned}$$

The 1-forms ϑ^α are assumed to carry the dimension of length, while the coefficients ρ_i are dimensionless. Hence the total Lagrangian ${}^{(\mathtt{cof})}L$ is dimensionless. In order to simplify the formulas below we will use the Lagrangian $L = \ell^2 {}^{(\mathtt{cof})}L$ which dimension is length square. In other worlds the geometrized units system with $G = c = \hbar = 1$ is applied.

Every term of the Lagrangian (1.4) is independent of a specific choice of a coordinate system and invariant under a global (rigid) $SO(1,3)$ transformation of the coframe field. Thus, different choices of the free parameters ρ_i yield different rigid $SO(1,3)$ and diffeomorphic invariant classical field models. Some of them are known to be applicable for description of gravity.

Let us rewrite the coframe Lagrangian in a compact form

$$ {}^{(\mathtt{cof})}L = \frac{1}{4} C_{\alpha\beta\gamma} C_{\alpha'\beta'\gamma'} \lambda^{\alpha\beta\gamma\alpha'\beta'\gamma'} * 1\,, \tag{1.8}$$

where the constant symbols

$$\begin{aligned} \lambda^{\alpha\beta\gamma\alpha'\beta'\gamma'} &= (\rho_1+\rho_2+\rho_3)\eta^{\alpha\alpha'}\eta^{\beta\beta'}\eta^{\gamma\gamma'} + \rho_2(\eta^{\alpha\beta'}\eta^{\beta\gamma'}\eta^{\gamma\alpha'} + \eta^{\alpha\gamma'}\eta^{\beta\alpha'}\eta^{\gamma\beta'})\\ &\quad -2\rho_3\eta^{\alpha\gamma}\eta^{\alpha'\gamma'}\eta^{\beta\beta'} \end{aligned} \tag{1.9}$$

are introduced. It can be checked, by straightforward calculation, that these λ-symbols are invariant under a transposition of the triplets of indices:

$$\lambda^{\alpha\beta\gamma\alpha'\beta'\gamma'} = \lambda^{\alpha'\beta'\gamma'\alpha\beta\gamma}\,. \tag{1.10}$$

We also introduce an abbreviated notation

$$F^{\alpha\beta\gamma} = \lambda^{\alpha\beta\gamma\alpha'\beta'\gamma'} C_{\alpha'\beta'\gamma'}\,. \tag{1.11}$$

The total Lagrangian (1.4) reads now as

$${}^{(\mathtt{cof})}L = \frac{1}{4} C_{\alpha\beta\gamma} F^{\alpha\beta\gamma} * 1\,. \tag{1.12}$$

This form of the Lagrangian will be used in sequel for the variation procedure. The Lagrangian (1.12) can also be rewritten in a component free notations. Define one-indexed 2-forms: a *field strength form*

$$\mathcal{C}^\alpha := \frac{1}{2} C^{\alpha\beta\gamma}\vartheta_{\beta\gamma} = d\vartheta^\alpha\,, \tag{1.13}$$

and a *conjugate field strength form* $\mathcal{F}^\alpha := \frac{1}{2}F^{\alpha\beta\gamma}\vartheta_{\beta\gamma}$

$$\mathcal{F}^\alpha = (\rho_1+\rho_3)\mathcal{C}^\alpha + \rho_2 e^\alpha \rfloor (\vartheta^\mu \wedge \mathcal{C}_\mu) - \rho_3 \vartheta^\alpha \wedge (e_\mu \rfloor \mathcal{C}^\mu)\,. \tag{1.14}$$

Another form of $\mathcal{F}^\alpha$ can be given via the irreducible (under the Lorentz group) decomposition of the 2-form $\mathcal{C}^\alpha$ (see [5], [4]). Write

$$\mathcal{C}^\alpha = {}^{(1)}\mathcal{C}^\alpha + {}^{(2)}\mathcal{C}^\alpha + {}^{(3)}\mathcal{C}^\alpha, \tag{1.15}$$

where

$${}^{(2)}\mathcal{C}^\alpha = \frac{1}{3}\vartheta^\alpha \wedge (e_\mu \rfloor \mathcal{C}^\mu)\,, \qquad {}^{(3)}\mathcal{C}^\alpha = \frac{1}{3}e^\alpha \rfloor (\vartheta_\mu \wedge \mathcal{C}^\mu)\,, \tag{1.16}$$

while ${}^{(1)}\mathcal{C}^\alpha$ is the remaining part. Substitute (1.16) into (1.14) to obtain

$$\mathcal{F}^\alpha = (\rho_1 + \rho_3)^{(1)}\mathcal{C}^\alpha + (\rho_1 - 2\rho_3)^{(2)}\mathcal{C}^\alpha + (\rho_1 + 3\rho_2 + \rho_3)^{(3)}\mathcal{C}^\alpha. \tag{1.17}$$

The coefficients in (1.17) coincide with those calculated in [25].
The 2-forms $\mathcal{C}^\alpha$ and $\mathcal{F}^\alpha$ do not depend on a choice of a coordinate system. They change as vectors by global $SO(1,3)$ transformations of the coframe. Using (1.13) the coframe Lagrangian can be rewritten as

$${}^{(\texttt{cof})}L = \frac{1}{2}\mathcal{C}_\alpha \wedge *\mathcal{F}^\alpha\,. \tag{1.18}$$

Observe that the Lagrangian (1.18) is of the same form as the standard electromagnetic Lagrangian ${}^{(\texttt{cof})}L = \frac{1}{2}F \wedge *F$. Observe, however, that the coframe Lagrangian involves the vector valued 2-forms of the field strength, while the electromagnetic Lagrangian is constructed of the the scalar valued 2-forms.

1.2 Variation of the Lagrangian

The Lagrangian (1.18) depends on the coframe field ϑ^a and on its first order derivatives only. Thus the first order variation formalism guarantee the corresponding Euler-Lagrange equation to be at most of the second order. Consider the variation of the coframe Lagrangian (1.12) with respect to small independent variations of the 1-forms ϑ^α. The λ-symbols (1.9) are constants and obey the symmetry property (1.10). Thus

$$C_{\alpha\beta\gamma}\delta F^{\alpha\beta\gamma} = C_{\alpha\beta\gamma}\lambda^{\alpha\beta\gamma\alpha'\beta'\gamma'}\delta C_{\alpha'\beta'\gamma'} = \delta C_{\alpha\beta\gamma}F^{\alpha\beta\gamma}\,. \tag{1.19}$$

Consequently the variation of the Lagrangian (1.12) takes the form

$$\delta L = \frac{1}{2}\delta C_{\alpha\beta\gamma}F^{\alpha\beta\gamma} * 1 - L * \delta(*1)\,. \tag{1.20}$$

The variation of the volume element is

$$\delta(*1) = -\delta(\vartheta^{0123}) = -\delta\vartheta^0 \wedge \vartheta^{123} - \cdots = -\delta\vartheta^0 \wedge *\vartheta^0 - \cdots = \delta\vartheta^\alpha \wedge *\vartheta_\alpha\,.$$

Thus the second term of (1.20) is given by

$$L * \delta(*1) = (\delta\vartheta^\alpha \wedge *\vartheta_\alpha) * L = -\delta\vartheta^\alpha \wedge (e_\alpha \rfloor L)\,. \tag{1.21}$$

As for the variation of the C-coefficients, we calculate them by equating the variations of the two sides of the equation (1.2)

$$\delta d\vartheta_\alpha = \frac{1}{2}\delta C_{\alpha\mu\nu}\vartheta^{\mu\nu} + C_{\alpha\mu\nu}\delta\vartheta^\mu \wedge \vartheta^\nu\,. \tag{1.22}$$

Use the formulas (A.12) and (A.13) to derive

$$\begin{aligned}\delta d\vartheta_\alpha\wedge *\vartheta_{\beta\gamma} &= \frac{1}{2}\delta C_{\alpha\mu\nu}\vartheta^{\mu\nu}\wedge *\vartheta_{\beta\gamma}+C_{\alpha\mu\nu}\delta\vartheta^\mu\wedge\vartheta^\nu\wedge *\vartheta_{\beta\gamma}\\ &= -\frac{1}{2}\delta C_{\alpha\mu\nu}\vartheta^\mu\wedge *(e^\nu\rfloor\vartheta_{\beta\gamma})-C_{\alpha\mu\nu}\delta\vartheta^\mu\wedge *(e^\nu\rfloor\vartheta_{\beta\gamma})\\ &= \delta C_{\alpha\beta\gamma} *1-2\delta\vartheta^\mu\wedge C_{\alpha\mu[\beta} *\vartheta_{\gamma]}\,.\end{aligned}$$

Therefore

$$\delta C_{\alpha\beta\gamma} *1=\delta(d\vartheta_\alpha)\wedge *\vartheta_{\beta\gamma}+2\delta\vartheta^\mu\wedge C_{\alpha\mu[\beta} *\vartheta_{\gamma]}\,. \tag{1.23}$$

After substitution of (1.21–1.23) into (1.20) the variation of the Lagrangian takes the form

$$\delta L=\frac{1}{2}F^{\alpha\beta\gamma}\Big(\delta(d\vartheta_\alpha)\wedge *\vartheta_{\beta\gamma}+2\delta\vartheta^\mu\wedge C_{\alpha\mu[\beta} *\vartheta_{\gamma]}\Big)+\delta\vartheta^\mu\wedge(e_\mu\rfloor L)\,.$$

Extract the total derivatives to obtain

$$\delta L=\frac{1}{2}\delta\vartheta_\mu\wedge\Big(d(*F^{\mu\beta\gamma}\vartheta_{\beta\gamma})+2F^{\alpha\beta\gamma}C_{\alpha\mu[\beta} *\vartheta_{\gamma]}+2e_\mu\rfloor L\Big)+\frac{1}{2}d\Big(\delta\vartheta_\alpha\wedge *F^{\alpha\beta\gamma}\vartheta_{\beta\gamma}\Big)\,. \tag{1.24}$$

The variation relation (1.24) plays a basic role in derivation of the field equation and of the conserved current. We rewrite it in a compact form by using the 2-forms (1.13) and (1.14). The terms of the form $F\cdot C$ can be rewritten as

$$F^{\alpha\beta\gamma}C_{\alpha\mu[\beta} *\vartheta_{\gamma]}=(F^{\alpha\beta\gamma}-F^{\alpha\beta\gamma})C_{\alpha\mu[\beta} *\vartheta_{\gamma]}=C_{\alpha\mu\beta} *(e^\beta\rfloor\mathcal{F}^\alpha)=-(e_\mu\rfloor\mathcal{C}_\alpha)\wedge *\mathcal{F}^\alpha\,.$$

Hence, (1.24) takes the form

$$\delta L\;=\;\delta\vartheta^\mu\wedge\Big(d(*\mathcal{F}_\mu)-(e_\mu\rfloor\mathcal{C}_\alpha)\wedge *\mathcal{F}^\alpha+e_\mu\rfloor L\Big)+d(\delta\vartheta^\mu\wedge\mathcal{F}_\mu)\,. \tag{1.25}$$

Collect now the quadratic terms into a differential 3-form

$$\mathcal{T}_\mu:=(e_\mu\rfloor\mathcal{C}_\alpha)\wedge *\mathcal{F}^\alpha-e_\mu\rfloor L\,. \tag{1.26}$$

Consequently, the variational relation (1.24) results in a compact form

$$\delta L=\delta\vartheta^\mu\wedge\Big(d*\mathcal{F}_\mu-\mathcal{T}_\mu\Big)+d\Big(\delta\vartheta^\mu\wedge\mathcal{F}_\mu\Big)\,. \tag{1.27}$$

1.3 The coframe field equations

We are ready now to write down the field equations. Consider independent free variations of a coframe field vanishing at infinity (or at the boundary of the manifold ∂M). The variational relation (1.27) yields *the coframe field equation*

$$d*\mathcal{F}^\mu=\mathcal{T}^\mu\,. \tag{1.28}$$

Observe that the structure of coframe field equation is formally similar to the structure of the standard electromagnetic field equation $d*F=J$. Namely, in both equations, the left hand sides are the exterior derivative of the dual field strength while the right hand sides are odd 3-forms. Thus the 3-forms $\mathcal{T}^\mu$ serves as a source for the field strength $\mathcal{F}^\mu$, as well as the 3-form of electromagnetic current J is a source for the electromagnetic field F. There are, however, some important distinctions: (i) The coframe field current $\mathcal{T}_\mu$ is a vector-valued 3-form while the electromagnetic current

J is a scalar-valued 3-form. (ii) The field equation (1.28) is nonlinear. (iii) The electromagnetic current J depends on an exterior matter field, while the coframe current $\mathcal{T}^\mu$ is interior (depends on the coframe itself).

The exterior derivation of the both sides of field equation (1.28) yields the conservation law

$$d\mathcal{T}^\mu = 0. \tag{1.29}$$

Note, that this equation obeys all the symmetries of the coframe Lagrangian. It is diffeomorphism invariant and global $SO(1,3)$ covariant. Thus we obtain a conserved total 3-form (1.26) which is constructed from the first order derivatives of the field variables (coframe). It is local and covariant. The 3-form $\mathcal{T}_\mu$ is our candidate for the coframe energy-momentum current.

1.4 Conserved current and Noether charge

The current $\mathcal{T}_\mu$ is obtained directly, i.e., by separation of the terms in the field equation. In order to identify the proper nature of this conserved 3-form we have to answer the question: *What symmetry this conserved current can be associated with?*

Return to the variational relation (1.27). On shell, for the fields satisfying the field equations (1.28), it takes the form

$$\delta L = d(\delta\vartheta^\alpha \wedge *\mathcal{F}_\alpha)\,. \tag{1.30}$$

Consider the variations of the coframe field produced by the Lie derivative taken relative to a smooth vector field X, i.e.,

$$\delta\vartheta^\alpha = \mathcal{L}_X\vartheta^\alpha = d(X\rfloor\vartheta^\alpha) + X\rfloor d\vartheta^\alpha\,. \tag{1.31}$$

The Lagrangian (1.12) is a diffeomorphic invariant, hence its variation is produced by the Lie derivative taken relative to the same vector field X, i.e.,

$$\delta L = \mathcal{L}_X L = d(X\rfloor L)\,. \tag{1.32}$$

Thus the relation (1.30) takes a form of a conservation law $d\Theta(X)$ for the Noether 3-form

$$\Theta(X) = \Big(d(X\rfloor\vartheta^\alpha) + X\rfloor\mathcal{C}^\alpha\Big) \wedge *\mathcal{F}_\alpha - X\rfloor L\,. \tag{1.33}$$

This quantity includes the derivatives of an arbitrary vector field X. Such a non-algebraic dependence of the conserved current is an obstacle for definition of an energy-momentum tensor. This problem is solved merely by using the canonical form of the current. Let us take $X = e_\alpha$. The first term of (1.33) vanishes identically. Thus

$$\Theta(e_\mu) = (e_\mu\rfloor\mathcal{C}^\alpha) \wedge *\mathcal{F}_\alpha - e_\mu\rfloor L\,. \tag{1.34}$$

Observe that the right hand side of the equation (1.34) is exactly the same expression as the source term of the field equation (1.28):

$$\Theta(e_\mu) = \mathcal{T}_\mu\,. \tag{1.35}$$

Thus the conserved current $\mathcal{T}_\mu$ defined in (1.26) is associated with the diffeomorphism invariance of the Lagrangian. Consequently the vector-valued 3-form (1.26) represents the *energy-momentum current of the coframe field*.

Let us look for an additional information incorporated in the conserved current (1.33). Extract the total derivative to obtain

$$\Theta(X) = d\Big((X\rfloor\vartheta^\alpha) * \mathcal{F}_\alpha\Big) - (X\rfloor\vartheta^\alpha)(d * \mathcal{F}_\alpha - \mathcal{T}_\alpha)\,. \tag{1.36}$$

Thus, up to the field equation (1.28), the current $\mathcal{T}(X)$ represents a total derivative of a certain 2-form $\Theta(X) = dQ(X)$. This result is a special case of a general proposition due to Wald [54] for a diffeomorphic invariant Lagrangians. The 2-form

$$Q(X) = (X\rfloor\vartheta^\alpha) * \mathcal{F}_\alpha\,. \tag{1.37}$$

can be referred to as the *Noether charge for the coframe field*. Consider $X = e_\alpha$ and denote $Q_\alpha := Q(e_\alpha)$. From (1.37) we obtain that this canonical Noether charge of the coframe field coincides with the dual of the conjugate strength

$$Q_\alpha = Q(e_\alpha) = *\mathcal{F}_\alpha\,. \tag{1.38}$$

In this way, the 2-form $\mathcal{F}_\alpha$, which was used above only as a technical device for expressing the equations in a compact form, obtained now a meaningful description. Note, that the Noether charge plays an important role in Wald's treatment of the black hole entropy [54].

1.5 Energy-momentum tensor

In this section we construct an expressions for the energy-momentum tensor for the coframe field. Let us first introduce the notion of the energy-momentum tensor via the differential-form formalism. We are looking for a second rank tensor field of a type $(0,2)$. Such a tensor can always be treated as a bilinear map $T:\ \mathcal{X}(M)\times\mathcal{X}(M)\rightarrow\mathcal{F}(M)$, where $\mathcal{F}(M)$ is the algebra of C^∞-functions on M while $\mathcal{X}(M)$ is the $\mathcal{F}(M)$-module of vector fields on M. The unique way to construct a scalar from a 3-form and a vector is is to take the Hodge dual of the 3-form and to contract the result by the vector. Consequently, we define the energy-momentum tensor as

$$T(X,Y) := Y\rfloor * \mathcal{T}(X)\,. \tag{1.39}$$

Observe that this quantity is a tensor if and only if the 3-form current $\mathcal{T}$ depends linearly (algebraic) on the vector field X. Certainly, $T(X,Y)$ is not symmetric in general. The antisymmetric part of the energy-momentum tensor is known from the Poincaré gauge theory [1] to represent the spinorial current of the field. The canonical form of the energy-momentum $T_{\alpha\beta} := T(e_\alpha, e_\beta)$ tensor is

$$T_{\alpha\beta} = e_\beta\rfloor * \mathcal{T}_\alpha\,. \tag{1.40}$$

Another useful form of this tensor can be obtained from (1.40) by applying the rule (A.13)

$$T_{\alpha\beta} = *(\mathcal{T}_\alpha\wedge\vartheta_\beta)\,. \tag{1.41}$$

The familiar procedure of rising the indices by the Lorentz metric $\eta^{\alpha\beta}$ produces two tensors of a type $(1,1)$

$$T_\alpha{}^\beta = *(\mathcal{T}_\alpha\wedge\vartheta^\beta), \quad \text{and} \quad T^\alpha{}_\beta = *(\mathcal{T}^\alpha\wedge\vartheta_\beta)\,, \tag{1.42}$$

which are different, in general. By applying the rule (A.13) the first relation of (1.42) is converted into

$$\mathcal{T}_\alpha = T_\alpha{}^\beta * \vartheta_\beta\,. \tag{1.43}$$

Thus, the components of the energy-momentum tensor are regarded as the coefficients of the current $\mathcal{T}_\alpha$ in the dual basis $*\vartheta^\alpha$ of the vector space Ω^3 of odd 3-forms.

In order to show that (1.43) conforms with the intuitive notion of the energy-momentum tensor let us restrict to a flat manifold and represent the 3-form conservation law as a tensorial expression. Take a closed coframe $d\vartheta^\alpha = 0$, thus $d * \vartheta_\beta = 0$. From (1.43) we derive

$$d\mathcal{T}_\alpha = dT_\alpha{}^\beta\wedge *\vartheta_\beta = -T_\alpha{}^\beta{}_{,\beta} * 1\,.$$

Hence, in this approximation, the differential-form conservation law $d\mathcal{T}_\alpha = 0$ is equivalent to the tensorial conservation law $T_\alpha{}^\beta{}_{,\beta} = 0$.

Apply now the definition (1.40) to the conserved current (1.26) for the coframe field. The energy-momentum tensor $T_{\mu\nu} = e_\nu \rfloor * \mathcal{T}_\mu$ is derived in the form

$$T_{\mu\nu} = e_\nu \rfloor * \Big((e_\mu \rfloor \mathcal{C}_\alpha) \wedge *\mathcal{F}^\alpha - \frac{1}{2} e_\mu \rfloor (\mathcal{C}_\alpha \wedge *\mathcal{F}^\alpha) \Big) . \tag{1.44}$$

Using (A.13) we rewrite the first term in (1.44) as

$$e_\nu \rfloor * \Big((e_\mu \rfloor \mathcal{C}_\alpha) \wedge *\mathcal{F}^\alpha \Big) = - * \Big((e_\mu \rfloor \mathcal{C}_\alpha) \wedge *(e_\nu \rfloor \mathcal{F}^\alpha) \Big) .$$

As for the second term in (1.44) it takes the form

$$-\frac{1}{2} e_\nu \rfloor * \Big(e_\mu \rfloor (\mathcal{C}_\alpha \wedge *\mathcal{F}^\alpha) \Big) = \frac{1}{2} \eta_{\mu\nu} * (\mathcal{C}_\alpha \wedge *\mathcal{F}^\alpha) .$$

Consequently the energy-momentum tensor for the coframe field is

$$T_{\mu\nu} = - * \Big((e_\mu \rfloor \mathcal{C}_\alpha) \wedge *(e_\nu \rfloor \mathcal{F}^\alpha) \Big) + \frac{1}{2} \eta_{\mu\nu} * (\mathcal{C}_\alpha \wedge *\mathcal{F}^\alpha) . \tag{1.45}$$

Observe that this expression is formally similar to the known expression for the energy-momentum tensor of the Maxwell electromagnetic field:

$${}^{(\mathtt{em})}T_{\mu\nu} = - * \Big((e_\mu \rfloor F) \wedge *(e_\nu \rfloor F) \Big) + \frac{1}{2} \eta_{\mu\nu} * (F \wedge *F) . \tag{1.46}$$

The form (1.46) is no more than an expression of the electromagnetic energy-momentum tensor in arbitrary frame. In a specific coordinate chart $\{x^\mu\}$ it is enough to take the coordinate basis vectors $e_a = \partial_\alpha$ and consider $T_{\alpha\beta} := {}^{(e)}T(\partial_\alpha, \partial_\beta)$ to obtain the familiar expression

$${}^{(\mathtt{em})}T_{\alpha\beta} = -F_{\alpha\mu} F_\beta{}^\mu + \frac{1}{4} \eta_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} . \tag{1.47}$$

The electromagnetic energy-momentum tensor is obviously traceless. The same property holds also for the coframe field tensor. In fact, the coframe energy-momentum tensor defined by (1.45) is traceless for all models described by the Lagrangian (1.4), i.e., for all values of the parameters ρ_i. Indeed, compute the trace $T^\mu{}_\mu = T_{\mu\nu}\eta^{\mu\nu}$ of (1.45):

$$\begin{aligned} {}^{(\mathtt{cof})}T^\mu{}_\mu &= - * \Big((e_\mu \rfloor \mathcal{C}_\alpha) \wedge *(e^\mu \rfloor \mathcal{F}^\alpha) \Big) + 2 * (\mathcal{C}_\alpha \wedge *\mathcal{F}^\alpha) \\ &= - * \Big(\vartheta^\mu \wedge (e_\mu \rfloor \mathcal{C}_\alpha) \wedge *\mathcal{F}^\alpha \Big) + 2 * (\mathcal{C}_\alpha \wedge *\mathcal{F}^\alpha) = 0 . \end{aligned}$$

It is well known that the traceless of the energy-momentum tensor is associated with the scale invariance of the Lagrangian. The rigid (λ is a constant) scale transformation $x^i \to \lambda x^i$, is considered acting on a matter field as $\phi \to \lambda^d \phi$, where d is the dimension of the field. The transformation does not act, however, on the components of the metric tensor and on the frame (coframe) components. It is convenient to shift the change on the metric and on the frame (coframe) components, i.e., to consider

$$g_{ij} \to \lambda^2 g_{ij} , \qquad \vartheta^\alpha{}_i \to \lambda \vartheta^\alpha{}_i , \qquad \text{and} e_\alpha{}^i \to \lambda^{-1} \vartheta_\alpha{}^i \tag{1.48}$$

with no change of coordinates. In the coordinate free formalism the difference between two approaches is neglected and the transformation is

$$g \to \lambda^2 g\,, \qquad \vartheta^\alpha \to \lambda\vartheta^\alpha\,, \qquad \text{and} \qquad e_\alpha \to \lambda^{-1} e_\alpha\,. \tag{1.49}$$

The transformation law of the coframe Lagrangian is simple to obtain from the component-wise form (1.8). Under the transformation (1.49) the volume element changes as $*1 \to \lambda^4 * 1$. As for the C-coefficients, they transform due to (1.3) as $C^a{}_{bc} \to \lambda^{-1} C^a{}_{bc}$. Consequently, by (1.5), the transformation law of the Lagrangian 4-form is $L \to \lambda^2 L$, which is the same as for the Hilbert-Einstein Lagrangian $L_{HE} = R\sqrt{-g}d^4x \to \lambda^2 L_{HE}$. After rescaling the Planck length the scale invariance is reinstated. Hence, for the pure coframe field model the energy-momentum tensor have to be traceless in accordance with the proposition above.

1.6 The field equation for a general system

The coframe field equation have been derived for a pure coframe field. Consider now a general minimally coupled system of a coframe field ϑ^α and a matter field ψ. The matter field can be a differential form of an arbitrary degree and can carry arbitrary number of exterior and interior indices. Take the total Lagrangian of the system to be of the form ($\ell =$ Planck length)

$$L = \frac{1}{\ell^2}\,{}^{(\mathtt{cof})}L(\vartheta^\alpha, d\vartheta^\alpha) + {}^{(\mathtt{mat})}L(\vartheta^\alpha, \psi, d\psi)\,, \tag{1.50}$$

where the coframe Lagrangian ${}^{(\mathtt{cof})}L$, defined by (1.4), is of dimension length square. The matter Lagrangian ${}^{(\mathtt{mat})}L$ is dimensionless.

The minimal coupling means here the absence of coframe derivatives in the matter Lagrangian. Take the variation of (1.50) relative to the coframe field ϑ^α to obtain

$$\delta L = \frac{1}{\ell^2}\delta\vartheta^\alpha \wedge \left(d * \mathcal{F}_\alpha - {}^{(\mathtt{cof})}\mathcal{T}_\alpha - \ell^{2\,(\mathtt{mat})}\mathcal{T}_\alpha\right), \tag{1.51}$$

where the 3-form of coframe current is defined by (1.28). The 3-form of matter current is defined via the variation derivative of the matter Lagrangian taken relative to the coframe field ϑ^α:

$${}^{(\mathtt{mat})}\mathcal{T}_\alpha := -\frac{\delta}{\delta\vartheta^\alpha}{}^{(\mathtt{mat})}L\,. \tag{1.52}$$

Introduce the total current of the system ${}^{(\mathtt{tot})}\mathcal{T}_\alpha = {}^{(\mathtt{cof})}\mathcal{T}_\alpha + \ell^{2\,(\mathtt{mat})}\mathcal{T}_\alpha$, which is of dimension length (mass). Consequently, the field equation for the general system (1.50) takes the form

$$d * \mathcal{F}_\alpha = {}^{(\mathtt{tot})}\mathcal{T}_\alpha\,. \tag{1.53}$$

Using the energy-momentum tensor (1.43) this equation can be rewritten in a tensorial form

$$e_\beta \rfloor * d * \mathcal{F}_\alpha = {}^{(\mathtt{tot})}T_{\alpha\beta}\,, \tag{1.54}$$

or equivalently

$$\vartheta_\beta \wedge d * \mathcal{F}_\alpha = {}^{(\mathtt{tot})}T_{\alpha\beta} * 1\,. \tag{1.55}$$

The conservation law for the total current $d\mathcal{T}_\alpha = 0$ is a straightforward consequence of the field equation (1.53). The form (1.53) of the field equation looks like the Maxwell field equation for the electromagnetic field $d * F = J$. Observe, however, an important difference. The source term in the right hand side of the electromagnetic field equation depends only on external fields. In the absence of the external sources $J = 0$, the electromagnetic strength $*F$ is a closed form. As a consequence, its cohomology class interpreted as a charge of the source. The electromagnetic field itself is uncharged.

As for the coframe field strength $\mathcal{F}^\alpha$ its source depends on the coframe and of its first order derivatives. Consequently, the 2-form $*\mathcal{F}^\alpha$ is not closed even in absence of the external sources. Hence the gravitational field is massive (charged) itself.

On the other hand the tensorial form (1.54) of the coframe field equation is similar to the Einstein field equation for the metric tensor $G_{\alpha\beta} = 8\pi^{(\mathtt{mat})}T_{\alpha\beta}$. Indeed, the left hand side in both equations are pure geometric quantities. Again, the source terms in the field equations are different. The source of the Einstein gravity is the energy-momentum tensor only of the matter fields. The conservation of this tensor is a consequence of the field equation. Thus even if some meaningful conserved energy-momentum current for the metric field existed it would have been conserved regardless of the matter field current. Consequently, any redistribution of the energy-momentum current between the matter and gravitational fields is forbidden in the framework of the traditional Einstein gravity.

As for the coframe field equation, the total energy-momentum current plays a role of the source of the field. Consequently the coframe field is completely "self-interacted" - the energy-momentum current of the coframe field produces an additional field. The conserved current of the coframe-matter system is the total energy-momentum current, not only the matter current. Thus in the framework of general coframe construction the redistribution of the energy-momentum current between the matter field and the coframe field is possible, in principle.

1.7 Spherically symmetric solution

Let us look for a static spherically symmetric solution to the field equation (1.28). We will use the isotropic coordinates $\{x^{\hat{i}}\,,\ \hat{i} = 1, 2, 3\}$ with the isotropic radius ρ. Denote

$$s = \rho^2 = \delta_{\hat{i}\hat{j}} x^{\hat{i}} x^{\hat{j}} = x^2 + y^2 + z^2\,. \tag{1.56}$$

Recall that we identify the gravity variable with the coframe field defined up to an infinitesimal Lorentz transformation. It is equivalent to the metric field. So it is enough to look for a coframe solution of a "diagonal" form [29]

$$\vartheta^0 = f(s)\,dx^0\,, \qquad \vartheta^{\hat{i}} = g(s)\,dx^{\hat{i}}\,. \tag{1.57}$$

Although this ansatz is not the most general one, it is enough because (1.57) corresponds to a most general static spherical symmetric metric

$$ds^2 = e^{2f(s)}dt^2 - e^{2g(s)}(dx^2 + dy^2 + dz^2)\,. \tag{1.58}$$

Substitution of (1.57) into the field equation (1.28) we obtain an over-determined system of three second order ODE for two independent variables $f(s)$ and $g(s)$

$$\begin{cases} \rho_1\Big(2f''s + 3f' + 2f'g's - 2(g')^2s + (f')^2s\Big) + 2\rho_3\Big(2g''s + 3g' + (g')^2s\Big) = 0 \\ \rho_1\Big(2g'' + 2f'g' - 2(f')^2 - 2(g')^2\Big) + 2\rho_3\Big(f'' + g'' + (f')^2 - 2f'g' - (g')^2\Big) = 0 \\ \rho_1\Big(4g''s + 4g' + 4f'g's - 2(f')^2s\Big) + 2\rho_3\Big(2f''s + 2f' + 2g''s + 2g' + 2(f')^2s\Big) = 0. \end{cases} \tag{1.59}$$

This system has a solutions with the Newtonian behavior on infinity $f \sim 1 - C/\rho$ only if the parameter ρ_1 is equal to zero. In this case, the system (1.59) has a unique solution

$$f = \ln \frac{1 - \frac{1}{c\rho}}{1 + \frac{1}{c\rho}}, \qquad g = 2\ln\left(1 + \frac{1}{c\rho}\right). \tag{1.60}$$

By taking the parameter of integration to be inversely proportional to the mass of the central body $c = \frac{2}{m}$ we obtain the coframe field in the form

$$\vartheta^0 = \frac{1 - \frac{m}{2\rho}}{1 + \frac{m}{2\rho}} dt, \qquad \vartheta^i = \left(1 + \frac{m}{2\rho}\right)^2 dx^i, \qquad i = 1, 2, 3\,. \tag{1.61}$$

This coframe field yields the Schwarzschild metric in isotropic coordinates

$$ds^2 = \left(\frac{1 - \frac{m}{2\rho}}{1 + \frac{m}{2\rho}}\right)^2 dt^2 - \left(1 + \frac{m}{2\rho}\right)^4 (dx^2 + dy^2 + dz^2)\,. \tag{1.62}$$

Note that the values of the parameters ρ_2, ρ_3 are not determined via the "diagonal" ansatz. Thus the Schwarzschild metric is a solution for a family of the coframe field equations which defined by the parameters:

$$\rho_1 = 0\,, \quad \rho_2, \rho_3 - \text{arbitrary}. \tag{1.63}$$

The ordinary GR is extracted from this family by requiring of the *local* $SO(1, 3)$ invariance, which is realized by an additional restriction of the parameters:

$$\rho_1 = 0\,, \quad 2\rho_2 + \rho_3 = 0\,. \tag{1.64}$$

1.8 Weak field approximation

Linear approximation of coframe models was usual applied for study the deviation from the standard GR, and for comparison with the observation data, see [20], [21], [23]. We will use this approach to study the meaning of the condition $\rho_1 = 0$, see [39]. Recall that this condition guarantees the existence of viable solutions.

To study the approximate solutions to (1.37), we start with a trivial exact solution, a *holonomic coframe*, for which

$$d\vartheta^a = 0\,. \tag{1.65}$$

Consequently, $\mathcal{F}^a = \mathcal{C}^a = 0$, so both sides of Eq. (1.37) vanish. By Poincaré's lemma, the solution of (1.65) can be locally expressed as $\vartheta^a = d\tilde{x}^a(x)$, where $\tilde{x}^a(x)$ is a set of four smooth functions defined in a some neighborhood U of a point $x \in \mathcal{M}$. The functions $\tilde{x}^a(x)$, being treated as the components of a coordinate map $\tilde{x}^a : U \to \mathbb{R}^4$, generate a local coordinate system on U. The metric tensor reduces, in this coordinate chart, to the flat Minkowski metric $g = \eta_{ab} d\tilde{x}^a \otimes d\tilde{x}^b$. Thus the holonomic coframe plays, in the coframe background, the same role as the Mankowski metric in the (pseudo-)Riemannian geometry. Moreover, a manifold endowed with a (pseudo-)orthonormal holonomic coframe is flat. The weak perturbations of the basic solution $\vartheta^a = dx^a$ are

$$\vartheta^a = dx^a + h^a = (\delta^a_b + h^a{}_b)\, dx^b\,. \tag{1.66}$$

The indices in $h^a{}_b$ can be lowered and raised by the Mankowski metric

$$h_{ab} := \eta_{am} h^m{}_b\,, \qquad h^{ab} := \eta^{bm} h^a{}_m\,. \tag{1.67}$$

The first operation is exact (covariant to all orders of approximations), while the second is covariant only to the first order, when $g^{ab} \approx \eta^{ab}$. The symmetric and the antisymmetric combinations of the perturbations

$$\theta_{ab} := h_{(ab)} = \frac{1}{2}(h_{ab} + h_{ba}), \qquad \text{and} \qquad w_{ab} := h_{[ab]} = \frac{1}{2}(h_{ab} - h_{ba})\,. \tag{1.68}$$

as well as the trace $\theta := h^m{}_m = \theta^m{}_m$ are covariant to the first order. The components of the metric tensor, in the linear approximation, involve only the symmetric combination of the coframe perturbations

$$g_{ab} = \eta_{ab} + 2\theta_{ab}\,. \tag{1.69}$$

When the decomposition

$$h_{ab} = \theta_{ab} + w_{ab} \tag{1.70}$$

is applied, the field strength is splitted to a sum of two independent strengths — one defined by the symmetric field θ_{ab} and the second one defined by the antisymmetric field w_{ab}

$$\mathcal{F}_a(\theta_{mn}, w_{mn}) = {}^{(\mathtt{sym})}\mathcal{F}_a(\theta_{mn}) + {}^{(\mathtt{ant})}\mathcal{F}_a(w_{mn})\,, \tag{1.71}$$

where

$${}^{(\mathtt{sym})}\mathcal{F}_a = -\Big[(\rho_1 + \rho_3)\theta_{a[b,c]} + \rho_3\eta_{a[b}\theta_{c]m}{}^{,m} - \rho_3\eta_{a[b}\theta_{,c]}\Big]\,\vartheta^b \wedge \vartheta^c\,, \tag{1.72}$$

and

$${}^{(\mathtt{ant})}\mathcal{F}_a = -\Big[(\rho_1 + \rho_3)w_{a[b,c]} + 3\rho_2 w_{[ab,c]} - \rho_3\eta_{a[b}w_{c]m}{}^{,m}\Big]\,\vartheta^b \wedge \vartheta^c\,. \tag{1.73}$$

Hence, for arbitrary values of the parameters ρ_i, the field strengths of the fields θ_{ab} and w_{ab} are independent.

The linearized field equation takes the form

$$(\rho_1+\rho_3)\big(\Box\,\theta_{ab} - \theta_{am,b}{}^{,m}\big) + \rho_3\big(-\eta_{ab}\Box\,\theta - \theta_{mb}{}^{,m}{}_{,a} + \theta_{,a,b} + \eta_{ab}\theta_{mn}{}^{,m,n}\big) +$$
$$(\rho_1 + 2\rho_2 + \rho_3)\big(\Box\,w_{ab} - w_{am,b}{}^{,m}\big) + (2\rho_2+\rho_3)w_{bm,a}{}^{,m} = 0\,. \tag{1.74}$$

Proposition 1: *For the case $\rho_1 = 0$, the linearized coframe field equation (1.74), splits, in arbitrary coordinates, into two independent systems*

$${}^{(\mathtt{sym})}\mathcal{E}_{(ab)}(\theta_{mn}) = \Box\,\theta_{ab} = 0\,, \qquad \textit{and} \qquad {}^{(\mathtt{ant})}\mathcal{E}_{[ab]}(w_{mn}) = \Box\,w_{ab} = 0\,.$$

If $\rho_1 \neq 0$, Eq.(1.74) does not split in any coordinate system.

Consequently, for $\rho_1 = 0$ and for generic values of the parameters ρ_2, ρ_3, the field equation of the coframe field is splitted to two independent field equations for two independent field variables. This splitting emerges also for the Lagrangian and the energy-momentum current.

Proposition 2: *For $\rho_1 = 0$, the Lagrangian of the coframe field is reduced, up to a total derivative term, to the sum of two independent Lagrangians*

$$\mathcal{L}(\theta_{ab}, w_{ab}) = {}^{(\mathtt{sym})}\mathcal{L}(\theta_{ab}) + {}^{(\mathtt{ant})}\mathcal{L}(w_{ab})\,. \tag{1.75}$$

Moreover, the coframe energy-momentum current is reduced, on shell, in the first order approximation, as

$$\mathcal{T}_a(\theta_{mn}, w_{mn}) = {}^{(\mathtt{sym})}\mathcal{T}_a(\theta_{mn}) + {}^{(\mathtt{ant})}\mathcal{T}_a(w_{mn})\,, \tag{1.76}$$

up to a total derivative.

The result of our analysis is as following: In the linear approximation the field variable is splitted to a sum of two independent fields. These fields do not interact only in the case $\rho_1 = 0$. Remarkable that this condition coincides with the viable condition (1.63), which is necessary for Schwarzschild metric.

2 Coframe geometry

The coframe gravity represented above is not related to a certain specific geometric structure. In this section we are looking for a geometry that can be constructed from the coframe field. It is well known that, on a Riemannian manifold there exists a unique linear connection of Levi-Civita [50]. Already this statement indicates that when we want to deal with some other connection, for instance with the flat one, we have to use some other non-Riemannian geometric structure. In this section, we define a geometry based on a coframe field. It is instead of the of the standard Riemannian geometry based on a metric tensor field.

2.1 Coframe manifold. Definitions and notations

Our construction will repeat the main properties of the Riemannian structure. Let us start with the basic definitions.

Differential manifold. Let M be a smooth $n+1$ dimensional differentiable manifold, which is locally (in an open set $U \subset M$) parametrized by a coordinate chart $\{x^i;\, i=0,1,\dots n\}$. The set of $n+1$ differentials dx^i provides a coordinate basis for the module of the differential forms on U. Similarly, the set of $n+1$ vector fields $\partial_i = \partial/\partial x^i$ forms the coordinate basis for the module of the vector fields on U. Arbitrary smooth transformations of the coordinates $x^i \to y^i(x^j)$ are admissible. Under these transformations, the elements of the coordinate bases transform by the tensorial law

$$dx^i \to dy^i = \frac{\partial y^i}{\partial x^j}\, dx^j\,, \qquad \frac{\partial}{\partial x^i} \to \frac{\partial}{\partial y^i} = \frac{\partial x^j}{\partial y^i}\, \frac{\partial}{\partial x^j}\,. \tag{2.1}$$

The Jacobian matrix $\partial y^i/\partial x^j$ is assumed to be smooth and invertible. The coordinate bases dx^i and $\partial_i = \partial/\partial x^i$ are referred to as *holonomic bases*. They satisfy the relations $d(dx^i) = 0$ and $[\partial_i, \partial_j] = 0$.

For a compact representation of geometric quantities, it is useful to have an alternative description via *nonholonomic bases*. Denote by θ^a a generic nonholonomic basis of the module of the 1-forms on U. Its dual f_a is a basis of of the module of the vector fields on U. In general, $d\theta^a \neq 0$ and $[f_a, f_b] \neq 0$. Relative to the coordinate bases, the elements of the nonholonomic bases are locally expressed as

$$\theta^a = \theta^a{}_i\, dx^i\,, \qquad f_a = f_a{}^i\, \partial_i\,. \tag{2.2}$$

Here the matrices $\theta^a{}_i$ and $f_a{}^i$ are the inverse to each-other, i.e.,

$$\theta^a{}_i\, f_a{}^j = \delta_i^j\,, \qquad \theta^a{}_i\, f_b{}^i = \delta_b^a\,. \tag{2.3}$$

Arbitrary smooth pointwise transformations of the nonholonomic bases

$$\theta^a \to A^a{}_b(x)\theta^b\,, \qquad f_a \to A_a{}^b(x) f_b\,. \tag{2.4}$$

are admissible. Here $A_a{}^b$ denotes, as usual, the matrix inverse to $A^a{}_b$.

Although the basis indices change in the same range $a, b, \dots = \{0, 1, \dots, n\}$ they are distinguished from the coordinate indices $i, j, \cdots$. In particular, the contraction of the indices in the quantities $\theta^a{}_i$ or $f_a{}^i$ is forbidden since the result of such an action is not a scalar. The base transformations (2.4) are similar to the coordinate transformations (2.1). Note that the basis θ^a can be changed to an arbitrary other basis, for instance to the coordinate one. Indeed, the formulas (2.2) can be treated as certain transformations of the bases. Consequently, θ^a cannot be given any intrinsic geometrical sense. In particular, it cannot be used as a model of a physical field.

Coframe field. Let the manifold M be endowed with a smooth nondegenerate coframe field ϑ^α. It comes together with its dual — the frame field e_α. In an arbitrary chart of local coordinates $\{x^i\}$, these fields are expressed as

$$\vartheta^\alpha = \vartheta^\alpha{}_i dx^i\,, \qquad e_\alpha = e_\alpha{}^i \partial_i\,, \tag{2.5}$$

i.e., by two nondegenerate matrices $\vartheta^\alpha{}_i$ and $e_\alpha{}^i$ which are the inverse to each-other. In other words, we are considering a set of n^2 independent smooth functions on M. Also the coframe indices change in the same range $\alpha, \beta, \ldots = 0, \ldots, n$ as the coordinate indices $i, j, \ldots$ and the basis indices $a, b, \ldots$. They all however have to be strictly distinguished. In particular, the indices in $\vartheta^\alpha{}_i$ or $e_\alpha{}^i$ cannot be contracted.

Coframe transformation. For most physical models based on the coframe field, this field is defined only up to global transformations. It is natural to consider a wider class of coframe fields related by local pointwise transformations

$$\vartheta^\alpha \to L^\alpha{}_\beta(x)\vartheta^\beta\,, \qquad e_\alpha \to L_\alpha{}^\beta(x) e_\beta\,. \tag{2.6}$$

Here $L^\alpha{}_\beta(x)$ and $L_\alpha{}^\beta(x)$ are inverse to each-other at arbitrary point x. Denote the group of matrices $L^\alpha{}_\beta(x)$ by G. Note two specially important cases: (i) G is a group of global transformations with a constant matrix $L^\alpha{}_\beta$; (ii) G is a group of arbitrary local transformations such that the entries of $L^\alpha{}_\beta$ are arbitrary functions of a point. In the latter case, the difference between the coframe field ϑ^α and the reference basis θ^a is completely removed and the coframe structure is trivialized.

Consequently we involve an additional element of the coframe structure — *the coframe transformations group*

$$G = \Big\{ L^\alpha{}_\beta(x) \in GL(n+1, \mathbb{R});\ \text{for every}\, x \in M \Big\}\,. \tag{2.7}$$

On this stage, we only require the matrices $L^\alpha{}_\beta(x)$ to be invertible at an arbitrary point $x \in M$. The successive specializations of the coframe transformation matrix will be involved in sequel.

Coframe field volume element. We assume the coframe field to be non-degenerate at an arbitrary point $x \in M$. Consequently, a special $n+1$-form, *the coframe field volume element*, is defined and nonzero. Define

$$\text{vol}(\vartheta^\alpha) = \frac{1}{n!}\,\varepsilon_{\alpha_o \cdots \alpha_n} \vartheta^{\alpha_o} \wedge \cdots \wedge \vartheta^{\alpha_n}\,, \tag{2.8}$$

where $\varepsilon_{\alpha_o \cdots \alpha_n}$ is the Levi-Civita permutation symbol normalized by $\varepsilon_{01\cdots n} = 1$. Treating the coframe volume element as one of the basic elements of the coframe geometric structure, we apply the following invariance condition.

> *Volume element invariance postulate:* Volume element $\text{vol}(\vartheta^\alpha)$ is assumed to be invariant under pointwise transformations of the coframe field
>
> $$\text{vol}\,(\vartheta^\alpha) = \text{vol}\left(L^\alpha{}_\beta \vartheta^\beta\right)\,. \tag{2.9}$$
>
> This condition is satisfied by matrices with unit determinant. Consequently, the coframe transformation group (2.7) is restricted to
>
> $$G = \Big\{ L^\alpha{}_\beta(x) \in SL(n+1, \mathbb{R});\ \text{for every}\, x \in M \Big\}\,. \tag{2.10}$$

Metric tensor. For a meaningful physical field model, it is necessary to have a metric structure on M. Moreover, the metric tensor has to be of the Lorentzian signature. In a coordinate basis and in an arbitrary reference basis, a generic metric tensor is written correspondingly as

$$g = g_{ij}dx^i \otimes dx^j\,, \qquad g = g_{ab}\theta^a \otimes \theta^b\,, \tag{2.11}$$

where the components g_{ij} and g_{ab} are smooth functions of a point $x \in M$.

On a coframe manifold, a metric tensor is not an independent quantity. Instead, we are looking for a metric explicitly constructed from a given coframe field, $g = g(\vartheta^\alpha)$. We assume the metric tensor to be quadratic in the coframe field components and independent of its derivatives. Moreover, it should be of the Lorentzian type, i.e., should be reducible at a point to the Lorentzian metric $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, \cdots, 1)$. These requirements are justified by an almost flat approximation: for an almost holonomic coframe, ${\vartheta^\alpha}_i \approx \delta^\alpha_i$, we have to reach the flat Lorentzian metric. With these restrictions, we come to a definition of the *coframe field metric tensor*

$$g = \eta_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta\,, \qquad g_{ij} = \eta_{\alpha\beta}{\vartheta^\alpha}_i{\vartheta^\beta}_j\,. \tag{2.12}$$

Note that the equations (2.12) often appear as a definition of a (non unique) orthonormal basis of reference for a given metric. Another interpretation treats (2.12) as an expression of a given metric in a special orthonormal basis of reference, as in (2.11). In our approach, (2.12) has a principle different meaning. It is a definition of the metric tensor field via the coframe field. Certainly the form of the metric $\eta_{\alpha\beta}$ in the tangential vector space T_xM is an additional axiom of our construction. With an aim to define an invariant coframe geometric structure we require:

> *Metric tensor invariance postulate:* Metric tensor is assumed to be invariant under pointwise transformations of the coframe field, i.e.,
>
> $$g\left(\vartheta^\alpha\right) = g\left({L^\alpha}_\beta\vartheta^\beta\right)\,. \tag{2.13}$$
>
> This condition is satisfied by pseudo-orthonormal matrices,
>
> $$\eta_{\mu\nu}{L^\mu}_\alpha L^\nu_\beta = \eta_{\alpha\beta}. \tag{2.14}$$
>
> Consequently, the invariance of the coframe metric restricts the coframe transformation group to
>
> $$G = \Big\{{L^\alpha}_\beta(x) \in O(1, n, \mathbb{R});\ \text{for every}\, x \in M\Big\}\,. \tag{2.15}$$
>
> In order to have simultaneously a metric and a volume element structures both constructed from the coframe field, we have to assume a successive restriction of the coframe transformation group:
>
> $$G = \Big\{{L^\alpha}_\beta(x) \in SO(1, n, \mathbb{R});\ \text{for every}\, x \in M\Big\}\,. \tag{2.16}$$

Topological restrictions. A global smooth coframe field may be defined only on a parallelizable manifold, i.e., on a topological manifold of a zero second Whitney class. This topological restriction is equivalent to existence of a spinorial structure on M. In this chapter, we restrict ourselves to a local consideration, thus the global definiteness problems will be neglected. Moreover, we assume the coframe field to be smooth and nonsingular only in a "weak" sense. Namely, the components ${\vartheta^\alpha}_i$ and ${e_\alpha}^i$ are required to be differentiable and linearly independent at almost all points of M, i.e., except of a zero measure set. So, in general, the coframe field can degenerate at singular points, on singular lines (strings), or even on singular submanifolds (p-branes). This assumption leaves a room for the standard singular solutions of the physics field equations such as the Coulomb field, the Schwarzschild metric, the Kerr metric etc..

2.2 Coframe connections

From the geometrical point of view, a differential manifold endowed with a coframe field is a rather poor structure. In particular, we can not determine if two vectors attached at distance points are parallel to each-other or not. In order to have a meaningful geometry and, consequently, a meaningful geometrical field model for gravity, we have to consider a reacher structure. In this section we define a coframe manifold with a linear coframe connection. The connection 1-form $\mathit{\Gamma}_a{}^b$ will not be an independent variable, as in the Cartan geometry or in MAG [5]. Alternatively in our construction the connection will be explicitly constructed from the coframe field and its first order derivatives. Thus we are dealing with a category of *coframe manifolds with a linear coframe connection:*

$$\Big\{M\,,\vartheta^\alpha,G\,,\Gamma_a{}^b(\vartheta^\alpha)\Big\}\,. \tag{2.17}$$

We start with a coframe manifold without an addition metric structure. Metric contributions to the connection will be considered in sequel.

Affine connection. Recall the main properties of a generic linear affine connection on an $(n+1)$ dimensional differential manifold. Relative to a local coordinate chart x^i, a connection is represented by a set of $(n+1)^3$ independent functions $\Gamma^k{}_{ij}(x)$ — *the coefficients of the connection.* The only condition these functions have to satisfy is to transform, under a change of coordinates $x^i\mapsto y^i(x)$, by an inhomogeneous linear rule:

$$\Gamma^i{}_{jk}\mapsto\left(\Gamma^l{}_{mn}\frac{\partial y^m}{\partial x^j}\frac{\partial y^n}{\partial x^k}+\frac{\partial^2 y^l}{\partial x^j\partial x^k}\right)\frac{\partial x^i}{\partial y^l}\,. \tag{2.18}$$

When an arbitrary reference basis $\{\theta^a\,,f_b\}$ is involved, the coefficients of the connection are arranged in a $GL(n,\mathbb{R})$-valued *connection 1-form*, which is defined as [50]

$$\mathit{\Gamma}_a{}^b=f_a{}^k\left(\theta^b{}_i\Gamma^i{}_{jk}-\theta^b{}_{k,j}\right)dx^j\,. \tag{2.19}$$

In a holonomic coordinate basis, we can simply use the identities $\theta^a{}_i=\delta^a_i$ and $f_a{}^i=\delta^i_a$. Consequently, in a coordinate basis, the derivative term is canceled out and (2.19) reads

$$\mathit{\Gamma}_j{}^i=\Gamma^i{}_{jk}\,dx^k\,. \tag{2.20}$$

Due to (2.18), this quantity transforms under the coordinate transformations as

$$\mathit{\Gamma}_j{}^i\to\left[\Gamma_m{}^l\frac{\partial y^m}{\partial x^j}+d\left(\frac{\partial y^l}{\partial x^j}\right)\right]\frac{\partial x^i}{\partial y^l}\,. \tag{2.21}$$

Alternatively, the connection 1-form (2.19) is invariant under smooth transformations of coordinates. The inhomogeneous linear behavior is shifted here to the transformations of $\mathit{\Gamma}_a{}^b$ under a linear local map of the reference basis (θ^a,f_a) given in (2.6):

$$\mathit{\Gamma}_a{}^b\mapsto\left(\Gamma_c{}^dA_a{}^c+dA_a{}^d\right)A^b{}_d\,. \tag{2.22}$$

On a manifold with a given coframe field ϑ^α, the connection 1-form (2.19), can also be referred to this field. We denote this quantity by $\mathit{\Gamma}_\alpha{}^\beta$. It is defined similarly to (2.19):

$$\mathit{\Gamma}_\alpha{}^\beta=\left(\vartheta^\beta{}_i\Gamma^i{}_{jk}-\vartheta^\beta{}_{k,j}\right)e_\alpha{}^k\,dx^j\,. \tag{2.23}$$

This quantity can be treated as an expression of a generic connection (2.19) in a special basis. Note an essential difference between two very similar equations (2.19) and (2.23). In (2.19), we must be able to apply arbitrary pointwise linear transformations of the basis. The coefficients of the connection $\Gamma^i{}_{jk}$ are independent on the basis (θ^a, f_a) used in (2.19). On the other hand, in (2.23), we permit only the transformations of the coframe field ϑ^α that are restricted by some invariance requirements. Moreover, we will require the connection $\Gamma^i{}_{jk}$ to be constructed explicitly from the derivatives of the coframe field itself.

Linear coframe connections. We restrict ourselves to the quasi-linear $\Gamma^i{}_{jk}(\vartheta^\alpha)$, i.e., we consider a connection constructed as a linear combination of the first order derivatives of the coframe field. The coefficients in this linear expression may depend on the frame/coframe components. In other words, we are looking for a coframe analog of an ordinary Levi-Civita connection.

Let us assist ourselves with a similar construction from the Riemannian geometry. So let us look now for a most general connection that can be constructed from the metric tensor components. Consider a general linear combination of the first order derivatives of the metric tensor:

$$g^{mi}(\alpha_1 g_{mj,k} + \alpha_2 g_{mk,j} + \alpha_3 g_{jk,m})\,. \tag{2.24}$$

Although this expression has the same index content as $\Gamma^i{}_{jk}$, it is a connection only for some special values of the parameters $\alpha_1, \alpha_2.\alpha_3$. Indeed, any two connections are differ by a tensor. Thus an arbitrary connection can be expressed as a certain special connection plus a tensor

$$\Gamma^i{}_{jk} = \overset{*}{\Gamma}{}^i{}_{jk} + K^i{}_{jk}\,. \tag{2.25}$$

Use for $\overset{*}{\Gamma}{}^i{}_{jk}$ the Levi-Civita connection

$$\overset{*}{\Gamma}{}^i{}_{jk} = \frac{1}{2} g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m})\,. \tag{2.26}$$

However in Riemannian geometry, does not exist a tensor constructed from the first order derivatives of the metric. Therefore $K^i{}_{jk} = 0$, thus the Levi-Civita connection is a unique connection that can be constructed from the first order derivatives of the metric tensor. It is evidently symmetric and metric compatible.

In an analogy to this construction, we will look for a most general coframe connection of the form

$$\Gamma^i{}_{jk}(\vartheta^\alpha) = \overset{\circ}{\Gamma}{}^i{}_{jk}(\vartheta^\alpha) + K^i{}_{jk}(\vartheta^\alpha)\,. \tag{2.27}$$

Here $\overset{\circ}{\Gamma}{}^i{}_{jk}$ is a certain special connection, while $K^i{}_{jk}$ is a tensor. To start with, we need a certain analog of the Levi-Civita connection, i.e., a special connection constructed from the coframe field.

The flat Weitzenböck connection. On a bare differentiable manifold M, without any additional structure, the notion of parallelism of two vectors attached to distance points depends on a curve joint the points. Oppositely, on a coframe manifold $\{M\,, \vartheta^\alpha\}$, a certain type of the parallelism of distance vectors may be defined in an absolute (curve independent) sense [52]. Namely, two vectors $u(x_1)$ and $v(x_2)$ may be declared parallel to each other, if, being referred to the local elements of the coframe field $u(x_1) = u_\alpha(x_1)\vartheta^\alpha(x_1)$ and $v(x_2) = v_\alpha(x_2)\vartheta^\alpha(x_2)$, they have the proportional components $u_\alpha(x_1) = Cv_\alpha(x_2)$. This definition is independent on the coordinates used on the manifold and on the nonholonomic frame of reference. It do depends on the coframe field. Since, by local transformations, the coframes at distance points change differently, only rigid linear coframe transformations preserve such type of a parallelism.

This geometric picture may be reformulated in term of a special connection. The elements of the coframe field attached to distinct points have to be assumed parallel to each other. It means that a special connection $\overset{\circ}{\Gamma}{}^i{}_{jk}$ exists such that the corresponding covariant derivative of the coframe field components is zero:

$$\vartheta^\alpha{}_{j;k} = \vartheta^\alpha{}_{j,k} - \overset{\circ}{\Gamma}{}^i{}_{jk}\vartheta^\alpha{}_i = 0\,. \tag{2.28}$$

Multiplying by $e_\alpha{}^i$, we have an explicit expression

$$\overset{\circ}{\Gamma}{}^i{}_{jk} = e_\alpha{}^i\vartheta^\alpha{}_{k,j}\,. \tag{2.29}$$

Under a smooth transform of coordinates, this expression is transformed in accordance with the inhomogeneous linear rule (2.18). Consequently, (2.29) indeed gives the coefficients of a special connection which is referred to as the *Weitzenböck flat connection*. This connection is unique for a class of coframes related by rigid linear transformations.

In an arbitrary nonholonomic reference basis (θ^a, f_a), we have correspondingly a unique Weitzenböck's connection 1-form which is constructed by (2.19) from (2.29)

$$\Gamma_a{}^b = f_a{}^k\left(\theta^b{}_i\,\overset{\circ}{\Gamma}{}^i{}_{jk} - \theta^b{}_{k,j}\right)dx^j\,. \tag{2.30}$$

Substituting the coframe field ϑ^α instead of the nonholonomic basis θ^a we have

$$\overset{\circ}{\Gamma}{}_\alpha{}^\beta = \left(-\vartheta^\beta{}_{i,j} + \vartheta^\beta{}_k e_\alpha{}^k\vartheta^\alpha{}_{i,j}\right)e_\alpha{}^i\,dx^j = 0\,. \tag{2.31}$$

Thus the Weitzenböck connection 1-form is zero, when it is referred to the coframe field $(\vartheta^\alpha, e_\alpha)$ itself. Certainly, this property is only a basis related fact. It yields, however, vanishing of the curvature of the Weitzenböck connection, which is a basis independent property.

General coframe connections. Recall that we are looking for a general coframe connection constructed from the first order derivatives of the coframe field components. In the Riemannian geometry, the analogous construction yields an unique connection of Levi-Civita. In the coframe geometry, however, the situation is different.

Proposition 3: *The general linear connection constructed from the first order derivatives of the coframe field is given by a 3-parametric family:*

$$\Gamma^i{}_{jk} = \overset{\circ}{\Gamma}{}^i{}_{jk} + \alpha_1 C^i{}_{jk} + \alpha_2 C_j\delta^i_k + \alpha_3 C_k\delta^i_j\,. \tag{2.32}$$

Proof: The difference of two connections is a tensor of a type $(1,2)$, so an arbitrary connection can be expressed as the Weitzenböck connection plus a tensor

$$\Gamma^i{}_{jk} = \overset{\circ}{\Gamma}{}^i{}_{jk} + K^i{}_{jk}\,. \tag{2.33}$$

Since $\overset{\circ}{\Gamma}{}^i{}_{jk}$ is already a linear combination of the first order derivatives, the additional tensor also has to be of the same form. Observe that $K^i{}_{jk}$ involves only coordinate indices, while the partial derivatives $\vartheta^\alpha{}_{j,i}$ have a coframe index α. This coframe index has to be suppressed. Hence the first order derivatives of the coframe components may appear in $K^i{}_{jk}$ only by the expressions $e_\alpha{}^i\vartheta^\alpha{}_{j,k}$. Notice that this quantity coincides with the coefficients of Weitzenböck's connection (2.29), which is not a tensor. Since the matrix of the frame field components $e_\alpha{}^j$ is the inverse of $\vartheta^\alpha{}_i$, the derivatives of the frame field $e_\alpha{}^j{}_{,k}$ are linear combinations of $\vartheta^\alpha{}_{j,k}$. Thus we do not need to involve additional derivatives of the frame field into $K^i{}_{jk}$. Consequently, the components of the tensor $K^i{}_{jk}$ have to be linear in $\overset{\circ}{\Gamma}{}^i{}_{jk}$. Write a general expression of such a type:

$$K^i{}_{jk} = \frac{1}{2}\,\chi_{jkl}{}^{imn}\,\overset{\circ}{\Gamma}{}^l{}_{mn}\,. \tag{2.34}$$

Since, under a transformation of coordinates, the connection $\overset{\circ}{\Gamma}{}^l{}_{mn}$ changes by inhomogeneous rule, it can appear in the tensor $K^i{}_{jk}$ only in the antisymmetric combination. Thus the most general expression for this tensor is

$$K^i{}_{jk} = \frac{1}{2}\,\chi_{jkl}{}^{imn}\ \overset{\circ}{\Gamma}{}^l{}_{[mn]} = \frac{1}{2}\,\chi_{jkl}{}^{imn}C^l{}_{mn}\,. \tag{2.35}$$

Hence, the symmetry relation $\chi_{jkl}{}^{imn} = \chi_{jkl}{}^{i[mn]}$ holds. The coefficients $\chi_{jkl}{}^{imn}$ have to be constructed from the components of the absolute basis $\vartheta^\alpha{}_m$ and $e_\alpha{}^m$. Again, since $\chi_{jkl}{}^{imn}$ involves only coordinate indices, it has to be constructed from the traced products of the frame and the coframe components. However all such products are equal to the Kronecker symbol. Thus $\chi_{jkl}{}^{imn}$ has to be a tensor expressed only by the Kronecker symbols. Consequently, the general expression for $\chi_{jkl}{}^{imn}$ can be written as

$$\chi_{jkl}{}^{imn} = \alpha_1\delta_j^{[m}\delta_k^{n]}\delta_l^i + \alpha_2\delta_l^{[m}\delta_k^{n]}\delta_j^i + \alpha_3\delta_l^{[m}\delta_j^{n]}\delta_k^i\,. \tag{2.36}$$

Substituting into (2.35) we have

$$K^i{}_{jk} = \alpha_1 C^i{}_{jk} + \alpha_2 C_j\delta_k^i + \alpha_3 C_k\delta_j^i\,. \tag{2.37}$$

Consequently (2.32) is proved. ■

By (2.19), the connection 1-form corresponded to the coefficients (2.32), being referred to a nonholonomic basis, takes the form

$$\mathit{\Gamma}_a{}^b = f_a{}^k\left(-\theta^b{}_{k,m} + \theta^b{}_l\ \overset{\circ}{\Gamma}{}^l{}_{mk} + K^l{}_{mk}\theta^b{}_l\right)dx^m\,. \tag{2.38}$$

When this quantity is referred to the coframe field itself, the first two terms are canceled. In this special basis, the expression is simplified to

$$\mathit{\Gamma}_\alpha{}^\beta = K^i{}_{jk}e_\alpha{}^k\vartheta^\beta{}_i dx^j\,, \tag{2.39}$$

where $K^i{}_{jk}$ is given in (2.37). Since the 1-form (2.39) depends only on antisymmetric combinations of the first order derivatives, it can be expressed by the exterior derivative of the coframe:

$$\mathit{\Gamma}_\alpha{}^\beta = \left(\alpha_1 C^\beta{}_{\gamma\alpha} + \alpha_2 C_\gamma\delta_\alpha^\beta + \alpha_3 C_\alpha\delta_\gamma^\beta\right)\vartheta^\gamma\,. \tag{2.40}$$

Also a components free expression is available

$$\mathit{\Gamma}_\alpha{}^\beta = -\frac{1}{2}\left[\alpha_1 e_\alpha\rfloor d\vartheta^\beta + \alpha_2(e_\alpha\rfloor\mathcal{A})\vartheta^\beta + \alpha_3\delta_\alpha^\beta\mathcal{A}\right]\,. \tag{2.41}$$

Metric-coframe connection. Consider a manifold endowed with the coframe metric tensor (2.12). Again, we are looking for a most general coframe connection that can be constructed from the first order derivatives of the coframe field. We will refer to it as the *metric-coframe connection*. Thus we are deal with a category of *coframe manifolds with a coframe metric and a linear coframe connection:*

$$\left\{M\,,\vartheta^\alpha, G\,, g(\vartheta^a), \Gamma_a{}^b(\vartheta^\alpha)\right\}. \tag{2.42}$$

Now the connection expression will involve some additional terms which depend on the metric tensor (2.12). To describe all possible combinations of the metric tensor components and frame/coframe components it is useful to pull down all the indices. Define:

$$\Gamma_{ijk} = g_{im}\Gamma^m{}_{jk}\,, \qquad C_{ijk} = g_{im}C^m{}_{jk}\,. \tag{2.43}$$

Proposition 4: *The most general metric-coframe connection constructed from the first order derivatives of the coframe field is represented by a 6-parametric family:*

$$\Gamma_{ijk} = \overset{\circ}{\Gamma}_{ijk} + \alpha_1 C_{ijk} + \alpha_2 g_{ik} C_j + \alpha_3 g_{ij} C_k + \beta_1 g_{jk} C_i + \beta_2 C_{jki} + \beta_3 C_{kij}\,. \tag{2.44}$$

Proof: Similarly to the case of a pure coframe connection, a metric-coframe connection can be represented as the Weitzenböck connection plus an arbitrary tensor. So we can write

$$\Gamma_{ijk} = \overset{\circ}{\Gamma}_{ijk} + K_{ijk}\,. \tag{2.45}$$

The tensor K_{ijk} has to be proportional to the derivatives of the coframe field $\vartheta^\alpha{}_{i,j}$. Repeating the consideration given above we come to the same conclusion: the first order derivatives of the coframe field can appear in the tensor K_{ijk} only via the antisymmetric combination of the flat connection $\overset{\circ}{\Gamma}_{l[mn]} = C_{lmn}$. Consequently we have a relation

$$K_{ijk} = \frac{1}{2}\,\chi_{ijk}{}^{lmn} C_{lmn}\,. \tag{2.46}$$

The tensor $\chi_{ijk}{}^{lmn}$ may involve now the components of the metric tensor in addition to the Kronecker symbol. Using the symmetry relation $\chi_{ijk}{}^{lmn} = \chi_{ijk}{}^{l[mn]}$ we construct a most general expression of such a type

$$\begin{aligned}\chi_{ijk}{}^{lmn} \quad = \quad & \alpha_1 \delta^l_i \delta^{[m}_j \delta^{n]}_k + \beta_2 \delta^l_j \delta^{[m}_k \delta^{n]}_i + \beta_3 \delta^l_k \delta^{[m}_i \delta^{n]}_j + \\ & \alpha_2 g_{ik} g^{l[m} \delta^{n]}_j + \alpha_3 g_{ij} g^{l[m} \delta^{n]}_k + \beta_1 g_{jk} g^{l[m} \delta^{n]}_i\,. \end{aligned} \tag{2.47}$$

Consequently, the additional tensor takes the required form

$$K_{ijk} = \alpha_1 C_{ijk} + \alpha_2 g_{ik} C_j + \alpha_3 g_{ij} C_k + \beta_1 g_{jk} C_i + \beta_2 C_{jki} + \beta_3 C_{kij}\,. \tag{2.48}$$

■

The expression (2.44) can be rewritten in a

$$\Gamma^i{}_{jk} = \overset{\circ}{\Gamma}{}^i{}_{jk} + \alpha_1 C^i{}_{jk} + \alpha_2 \delta^i_k C_j + \alpha_3 \delta^i_j C_k + \beta_1 g^{il} g_{jk} C_l + \beta_2 g^{il} C_{jkl} + \beta_3 g^{il} C_{klj}\,. \tag{2.49}$$

In fact, this expression is a proper form of the coefficients of the coframe connection. Here we can identify two groups of terms: (i) The terms with the coefficient α_i that do not depend on the metric; (ii) The terms with the coefficient β_i that can be constructed only by use of the metric tensor.

With respect to a nonholonomic basis (f_a, θ^a), the coefficients of a connection (2.49) correspond to a connection 1-form (2.19)

$$\mathit{\Gamma}_a{}^b = \overset{\circ}{\mathit{\Gamma}}_a{}^b + K^i{}_{jk} f_a{}^k \theta^b{}_i dx^j\,. \tag{2.50}$$

When (2.50) is referred to the coframe field itself, it is simplified to

$$\mathit{\Gamma}_\alpha{}^\beta = K^i{}_{jk} e_\alpha{}^k \vartheta^\beta{}_i dx^j\,. \tag{2.51}$$

This expression depends only on the antisymmetric combinations of the first order derivatives of the coframe components. So it can be expressed by the exterior derivative of the coframe. We have

$$\mathit{\Gamma}_\alpha{}^\beta = \left(\alpha_1 C^\beta{}_{\gamma\alpha} + \alpha_2 C_\gamma \delta^\beta_\alpha + \alpha_3 C_\alpha \delta^\beta_\gamma + \beta_1 C^\beta \eta_{\alpha\gamma} + \beta_2 C_{\gamma\alpha\nu}\eta^{\beta\nu} + \beta_3 C_{\alpha\nu\gamma}\eta^{\beta\nu}\right)\vartheta^\gamma\,, \tag{2.52}$$

or, equivalently,

$$\begin{aligned} \mathit{\Gamma}_\beta{}^\alpha &= -\frac{1}{2}\Big[\alpha_1 e_\beta\rfloor d\vartheta^\alpha + \alpha_2\vartheta^\alpha(e_\beta\rfloor\mathcal{A}) + \alpha_3\delta^\alpha_\beta\mathcal{A} + \beta_1(e^\alpha\rfloor A)\vartheta_\beta + \\ &\qquad \beta_2 e^\alpha\rfloor(e_\beta\rfloor d\vartheta_\mu)\vartheta^\mu + \beta_3 e^\alpha\rfloor d\vartheta_\beta\Big]\,. \end{aligned} \tag{2.53}$$

2.3 Torsion of the coframe connection

Torsion tensor and torsion 2-form. Definitions. Consider a connection 1-form $\Gamma_b{}^a$ referred to an arbitrary basis (θ^a, f_a). For a tensor valued p-form of a representation type $\rho\left(A_a{}^b\right)$, the *covariant exterior derivative* operator $D:\Omega^p(\mathcal{M})\to\Omega^{p+1}(\mathcal{M})$ is defined as [24], [5]

$$D = d + \mathit{\Gamma}_b{}^a\rho\left(A_a{}^b\right)\wedge\,. \tag{2.54}$$

In particular, the covariant exterior derivative of a scalar-valued form ϕ is $D\phi = d\phi$. For a vector-valued form ϕ^a, it is given by $D\phi^a = d\phi^a + \mathit{\Gamma}_b{}^a\wedge\phi^b$, etc.

For a connection 1-form $\mathit{\Gamma}_a{}^b$ written with respect to a nonholonomic basis, the *torsion 2-form* $\mathcal{T}^a$ is defined as

$$\mathcal{T}^a = D\theta^a = d\theta^a + \mathit{\Gamma}_b{}^a\wedge\theta^b\,. \tag{2.55}$$

On a D dimensional manifold, this covector valued 2-form has $D(D^2-D)/2$ independent components. Substituting (2.19) into (2.55), we observe that the coframe derivative term $d\vartheta^a$ cancels out. Hence,

$$\mathcal{T}^a = \Gamma^i{}_{jk}\theta^a{}_i dx^j\wedge dx^k = \Gamma^i{}_{[jk]}\theta^a{}_i dx^j\wedge dx^k\,. \tag{2.56}$$

In a coordinate coframe, this expression is simplified to

$$\mathcal{T}^i = \Gamma^i{}_{[jk]}dx^j\wedge dx^k\,. \tag{2.57}$$

Consequently, the torsion 2-form $\mathcal{T}^a$ is completely determined by an antisymmetric combination of the coefficients of the connection. Observe that such combination is a tensor. Thus, the torsion 2-form is completely equivalent to a $(1,2)$-rank *torsion tensor* which is defined as

$$T^i{}_{jk} = 2\Gamma^i{}_{[jk]}\,. \tag{2.58}$$

In a holonomic and a nonholonomic bases, the torsion 2-form is expressed respectively as

$$\mathcal{T}^i = \frac{1}{2}T^i{}_{jk}dx^j\wedge dx^k\,,\qquad \mathcal{T}^a = \frac{1}{2}T^i{}_{jk}\theta^a{}_i dx^j\wedge dx^k\,. \tag{2.59}$$

It is useful to define also a quantity

$$\mathcal{T}^\alpha = \frac{1}{2}T^i{}_{jk}\vartheta^\alpha{}_i dx^j\wedge dx^k\,. \tag{2.60}$$

Observe that this set of 2-forms cannot be regarded as a vector-valued form since the transformations of the coframe field ϑ^α are restricted. However, the proper vector valued torsion 2-forms (2.59) are related to the quantity (2.60) by the following simple equations

$$\mathcal{T}^i = e_\alpha{}^i\mathcal{T}^\alpha\,,\qquad \mathcal{T}^a = \theta^a{}_i e_\alpha{}^i\mathcal{T}^\alpha\,. \tag{2.61}$$

With respect to the coframe field, the torsion 2-form of the Weitzenböck connection (2.60) reads

$$\overset{\circ}{\mathcal{T}}{}^\alpha = d\vartheta^\alpha\,. \tag{2.62}$$

Torsion of the metric-coframe connection. For the metric-coframe connection (2.44), the covariant components $T_{ijk}=2g_{im}\Gamma^m{}_{[jk]}$ of the torsion tensor take the form

$$T_{ijk}=2(1+\alpha_1)C_{ijk}+(\alpha_2-\alpha_3)(g_{ik}C_j-g_{ij}C_k)+(\beta_2+\beta_3)(C_{jki}+C_{kij})\,. \tag{2.63}$$

The corresponded torsion 2-form is expressed in the coordinate basis as

$$T^i=\Big[(1+\alpha_1)C^i{}_{jk}+(\alpha_2-\alpha_3)C_j\delta^i_k+(\beta_2+\beta_3)g^{im}C_{jkm}\Big]dx^j\wedge dx^k\,. \tag{2.64}$$

In term of the differential forms $\mathcal{A}$ and $\mathcal{B}$ (see Appendix) we derive

$$\mathcal{T}^\alpha=(1+\alpha_1)\,d\vartheta^a-\frac{1}{2}\,(\alpha_2-\alpha_3)\vartheta^\alpha\wedge\mathcal{A}-\frac{1}{2}\,(\beta_2+\beta_3)\Big(d\vartheta^a-e^\alpha\rfloor\mathcal{B}\Big)\,. \tag{2.65}$$

Irreducible decomposition of the torsion. On a manifold of a dimension $D\geq 3$ endowed with a metric tensor, the torsion 2-form admits an irreducible decomposition into three independent pieces [5]

$$\mathcal{T}^a={}^{(1)}\mathcal{T}^a+{}^{(2)}\mathcal{T}^a+{}^{(3)}\mathcal{T}^a\,. \tag{2.66}$$

Here *the trator* and *the axitor* parts [5] are defined correspondingly as

$${}^{(2)}\mathcal{T}^a=\frac{1}{n-1}\,\theta^a\wedge(f_b\rfloor\mathcal{T}^b)\,,\qquad {}^{(3)}\mathcal{T}^a=\frac{1}{3}f^a\rfloor(\theta^b\wedge\mathcal{T}_\beta)\,. \tag{2.67}$$

The remainder ${}^{(1)}\mathcal{T}^a$ is referred to as a *tentor part*. The irreducible decomposition means that the different pieces transform independently by the same tensorial rule as the total quantity. Particularly, we can check straightforwardly that for every part of the torsion tensor

$${}^{(p)}\mathcal{T}^a={}^{(p)}\mathcal{T}^\alpha\theta^a{}_ie_\alpha{}^i\,,\qquad p=1,2,3. \tag{2.68}$$

So it is enough to provide the calculations of the irreducible pieces with respect to the coframe field itself. We have the second piece of the torsion as

$${}^{(2)}\mathcal{T}^\alpha=\frac{1}{n-1}\,\vartheta^\alpha\wedge(e_\beta\rfloor\mathcal{T}^\beta)=\frac{\tau_2}{2(n-1)}\,\vartheta^\alpha\wedge\mathcal{A}\,, \tag{2.69}$$

where

$$\tau_2=2(1+\alpha_1)-(\beta_2+\beta_3)-(\alpha_2-\alpha_3)(n-1)\,. \tag{2.70}$$

The third piece of torsion is given by

$${}^{(3)}\mathcal{T}^\alpha=\frac{1}{3}e^\alpha\rfloor(\vartheta^\beta\wedge\mathcal{T}_\beta)=\frac{\tau_3}{3}\,e^\alpha\rfloor\mathcal{B}\,, \tag{2.71}$$

where

$$\tau_3=(1+\alpha_1)+(\beta_2+\beta_3)\,. \tag{2.72}$$

The first part takes the form

$${}^{(1)}\mathcal{T}^\alpha\ =\ \mathcal{T}^\alpha-{}^{(2)}\mathcal{T}^\alpha-{}^{(3)}\mathcal{T}^\alpha=\tau_1\left(d\vartheta^a-\frac{1}{n-1}\,\vartheta^\alpha\wedge\mathcal{A}-\frac{1}{3}\,e^\alpha\rfloor\mathcal{B}\right)\,, \tag{2.73}$$

where

$$\tau_1=(1+\alpha_1)-\frac{1}{2}(\beta_2+\beta_3)\,. \tag{2.74}$$

Torsion-free metric-coframe connection. Let us look for which values of the parameters the torsion of the metric-coframe connection is identically zero. The corresponded connection is called *the symmetric or torsion-free connection*. It is clear from (2.63) that the metric-coframe connection is symmetric if

$$\alpha_1 = -1\,, \qquad \alpha_2 = \alpha_3\,, \qquad \beta_2 = -\beta_3\,. \tag{2.75}$$

The necessity of this condition can be derived from the irreducible decomposition. Indeed, since the three pieces of the torsion are mutually independent, they have to vanish simultaneously. Hence we have a condition $\tau_1 = \tau_2 = \tau_3 = 0$ which is equivalent to (2.75). Note that this requirement is necessary only for a manifold of the dimension $D \geq 3$. On a two-dimensional manifold, the metric-coframe connection is symmetric under a weaker condition

$$2(1+\alpha_1) + (\alpha_2 - \alpha_3) - (\beta_2 + \beta_3) = 0\,. \tag{2.76}$$

On a curve, every connection is unique and symmetric.

Thus on a manifold of the dimension $D \geq 3$ there exists a 3-parametric family of the symmetric (torsion-free) connections:

$$\Gamma^i{}_{jk} = \overset{\circ}{\Gamma}{}^i{}_{jk} - C^i{}_{jk} + \alpha_2\left(\delta^i_k C_j + \delta^i_j C_k\right) + \beta_1 g_{jk} g^{im} C_m + \beta_2 g^{im}\left(C_{jkm} - C_{kmj}\right)\,. \tag{2.77}$$

2.4 Nonmetricity of the metric-coframe connection

Nonmetricity tensor and nonmetricity 2-form. Definition. When Cartan's manifold is endowed with a metric tensor, the connection generates an additional tensor field called *the nonmetricity tensor*. It is expressed as a covariant derivative of the metric tensor components. For a metric given in a local system of coordinates as $g = g_{ij}dx^i \otimes dx^j$, the nonmetricity tensor is defined as

$$Q_{kij} = -\nabla_k g_{ij} = -g_{ij,k} + \Gamma^m{}_{ik} g_{mj} + \Gamma^m{}_{jk} g_{im}\,, \tag{2.78}$$

or,

$$Q_{kij} = -g_{ij,k} + \Gamma_{jik} + \Gamma_{ijk}\,. \tag{2.79}$$

Evidently, this tensor is symmetric in the last pair of indices $Q_{kij} = Q_{kji}$. Hence, on a D dimensional manifold, the nonmetricity tensor has $D(D^2+D)/2$ independent components.

For the exterior form representation, it is useful to define *the nonmetricity 1-form*. In a coordinate basis, it is given by

$$Q_{ij} = Q_{kij}dx^k = -dg_{ij} + \Gamma_{ij} + \Gamma_{ji}\,. \tag{2.80}$$

In an arbitrary reference basis $(f_a\,, \theta^a)$, the metric tensor is expressed as $g = g_{ab}\theta^a \otimes \theta^b$. Correspondingly, the nonmetricity 1-form reads

$$Q_{ab} = -dg_{ab} + \Gamma_{ab} + \Gamma_{ba}\,. \tag{2.81}$$

With respect to the coframe field ϑ^α, the components of the metric are constants $\eta_{\alpha\beta}$, thus the nonmetricity is merely the symmetric combination of the connection 1-form components

$$Q_{\alpha\beta} = \Gamma_{\alpha\beta} + \Gamma_{\beta\alpha}\,. \tag{2.82}$$

Note, that this expression is not a usual tensorial quantity. In fact, it is an expression of a tensor-valued 1-form of nonmetricity with respect to a special class of bases. Its relation to a proper tensorial valued 1-form (2.80) is, however, very simple. By a substitution of (2.23) into (2.82) we have

$$Q_{ij} = Q_{\alpha\beta}\vartheta^\alpha{}_i\vartheta^\beta{}_j\,. \tag{2.83}$$

The following generalization of the Levi-Civita theorem from the Riemannian geometry provides a decomposition of an arbitrary affine connection [53]. Its simple proof is instructive for our construction.

Proposition 5: *Let a metric g on a manifold M be fixed and two tensors T_{ijk} and Q_{ijk} with the symmetries*

$$T_{ijk} = -T_{ikj}\,, \qquad Q_{kij} = Q_{kji}\,. \tag{2.84}$$

be given. A unique connection Γ_{ijk} exists on M such that T_{ijk} is its torsion and Q_{ijk} is its nonmetricity. Explicitly,

$$\Gamma_{ijk} \quad = \quad \overset{*}{\Gamma}{}_{ijk} - \frac{1}{2}\big(Q_{ijk} - Q_{jki} - Q_{kij}\big) + \frac{1}{2}\big(T_{ijk} + T_{jki} - T_{kij}\big)\,, \tag{2.85}$$

where

$$\overset{*}{\Gamma}{}_{ijk} = \frac{1}{2}\big(g_{ij,k} + g_{ik,j} - g_{jk,i}\big) \tag{2.86}$$

are the components of the Levi-Civita connection.

Proof: On a D-dimensional manifold definitions of the torsion and the nonmetricity tensors

$$T_{ijk} = 2\Gamma_{i[jk]}\,, \qquad Q_{kij} = -g_{ij,k} + \Gamma_{ijk} + \Gamma_{jik} \tag{2.87}$$

can be viewed as a linear system of D^3 linear equations for D^3 independent variables Γ_{ijk}

$$\Gamma_{i[jk]} = \frac{1}{2}T_{ijk}\,, \qquad \Gamma_{(ij)k} = \frac{1}{2}(Q_{kij} + g_{ij,k})\,. \tag{2.88}$$

For $T_{ijk} = Q_{kij} = 0$, the system has a unique solution — the Levi-Civita connection $\overset{*}{\Gamma}{}_{ijk}$. Thus the determinant of the matrix of the system (2.88) is nonsingular. Consequently also for arbitrary tensors T_{ijk} and Q_{kij}, the system has a unique solution. In order to check the specific form of the solution (2.85), it is enough to substitute the definitions (2.87). ■

Nonmetricity of the metric-coframe connection. We calculate now the nonmetricity tensor of the metric-coframe connection (2.44)

$$\begin{aligned} Q_{kij} \quad = \quad & \Big(-g_{ij,k} + \overset{\circ}{\Gamma}{}_{ijk} + \overset{\circ}{\Gamma}{}_{jik}\Big) + \\ & (\alpha_1 - \beta_2)(C_{ijk} - C_{jki}) + (\alpha_2 + \beta_1)(g_{ik}C_j + g_{jk}C_i) + 2\alpha_3 g_{ij}C_k\,. \end{aligned} \tag{2.89}$$

The first parenthesis represent the nonmetricity tensor of the Weitzenböck connection. This expression vanishes identically, i.e., the Weitzenböck connection is metric-compatible. Indeed, we have

$$g_{ij,k} = \eta_{\alpha\beta}(\vartheta^\alpha{}_{i,k}\vartheta^\beta_j + \vartheta^\alpha{}_i\vartheta^\beta_{j,k}) = \overset{\circ}{\Gamma}{}_{ijk} + \overset{\circ}{\Gamma}{}_{jik}\,. \tag{2.90}$$

Consequently, (2.89) is simplified to

$$Q_{kij} = (\alpha_1 - \beta_2)(C_{ijk} + C_{jki}) + (\alpha_2 + \beta_1)(g_{ik}C_j + g_{jk}C_i) + 2\alpha_3 g_{ij}C_k\,. \tag{2.91}$$

Relative to the coframe field, we have, using (2.52,2.82), the 1-form of nonmetricity

$$\begin{aligned} Q_{\alpha\beta} &= -\frac{1}{4}\,(\alpha_1-\beta_2)\Big(e_\alpha\rfloor d\vartheta_\beta + e_\beta\rfloor d\vartheta_\alpha\Big) + \frac{1}{2}\,\alpha_3\eta_{\alpha\beta}A + \\ &\quad \frac{1}{4}\,(\alpha_2+\beta_1)\Big[(e_\alpha\rfloor A)\vartheta_\beta + (e_\beta\rfloor A)\vartheta_\alpha\Big]\,. \end{aligned} \tag{2.92}$$

Irreducible decomposition of the nonmetricity. We are looking now for an irreducible decomposition of the nonmetricity 1-form Q_{ab} under the pseudo-orthogonal group. Since Q_{ab} is a tensor-valued 1-form it can be calculated in an arbitrary basis. Certainly, the basis of the coframe field is the best for these purposes. We have only remember that for a transformation to an arbitrary basis we have simple multiply the corresponding quantity $Q_{\alpha\beta}$ by the matrix of the transformation. We cannot, however, transform the coframe basis to an arbitrary basis. This is because the coframe field is a fixed building block of our construction.

The irreducible decomposition of the nonmetricity 1-form under the pseudo-orthogonal group $SO(1,n)$ is constructed by the in correspondence to the Young diagrams. For actual calculations we use the algorithm given in [5]. The resulting decomposition is given as a sum of four independent pieces

$$Q_{\alpha\beta} = {}^{(1)}Q_{\alpha\beta} + {}^{(2)}Q_{\alpha\beta} + {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}\,. \tag{2.93}$$

For the nonmetricity 1-form (2.82),the irreducible parts are

$${}^{(1)}Q_{\alpha\beta} = \mu_1\Big[(n-1)e_{(\alpha}\rfloor d\vartheta_{\beta)} + (e_{(\alpha}\rfloor A)\vartheta_{\beta)} - 4\eta_{\alpha\beta}A\Big]\,, \tag{2.94}$$

$${}^{(2)}Q_{\alpha\beta} = \mu_2\Big[(n-1)e_{(\alpha}\rfloor d\vartheta_{\beta)} + (e_{(\alpha}\rfloor A)\vartheta_{\beta)} + 2\eta_{\alpha\beta}A\Big]\,, \tag{2.95}$$

$${}^{(3)}Q_{\alpha\beta} = \mu_3\Big[(e_{(\alpha}\rfloor A)\vartheta_{\beta)} + \frac{2}{n}\,\eta_{\alpha\beta}A\Big]\,, \tag{2.96}$$

$${}^{(4)}Q_{\alpha\beta} = \mu_4\Big[\frac{1}{n}\,\eta_{\alpha\beta}A\Big]\,. \tag{2.97}$$

The coefficients of these quantities depend on the parameters of the general connection as

$$\mu_1 = -\frac{1}{6(n-1)}\,(\alpha_1-\beta_2)\,, \qquad \mu_2 = \frac{1}{2}\,\mu_1\,, \tag{2.98}$$

$$\mu_3 = \frac{1}{4}\Big[\frac{1}{n-1}\,(\alpha_1-\beta_2) + (\alpha_2+\beta_1)\Big]\,, \tag{2.99}$$

$$\mu_4 = \frac{1}{2}\Big[-(\alpha_1-\beta_2) + n\alpha_3 + (\alpha_2+\beta_1)\Big]\,. \tag{2.100}$$

Metric compatible metric-coframe connection. Let us look for which values of the coefficients the connection is *metric-compatible*, i.e., has an identically zero non-metricity tensor. Recall that both quantities, the metric tensor and the connection, are constructed from the same building block — the coframe field ϑ^α. It is clear from (2.92) that the metric-coframe connection is metric-compatible if

$$\alpha_1 = \beta_2\,, \qquad \alpha_2 = -\beta_1\,, \qquad \alpha_3 = 0\,. \tag{2.101}$$

The necessity of this condition can be derived from the irreducible decomposition of the nonmetricity tensor. Four irreducible pieces of the non-metricity tensor are mutually independent, so they have to vanish simultaneously. Hence we have a condition $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$ which turns out to be equivalent to (2.101). Note that this requirement is necessary only for a manifold of the dimension $D \geq 3$, where the irreducible decomposition (2.93) is valid. On a two-dimensional manifold, the metric-coframe connection is metric-compatible if and only if

$$\alpha_1 - \beta_2 = \alpha_2 + \beta_1 = \alpha_3 \,. \tag{2.102}$$

On a one-dimensional manifold, every connection is metric-compatible.

Metric compatible and torsion-free metric-coframe connection. Let us look now for a general coframe connection of a zero torsion and zero non-metricity, i.e., for a symmetric metric compatible connection constructed from the coframe field. The system of conditions (2.75) and (2.101) has a unique solution

$$\alpha_1 = \beta_2 = -\beta_3 = -1 \,, \qquad \beta_1 = \alpha_2 = \alpha_3 = 0 \,. \tag{2.103}$$

Consequently, a metric-compatible symmetric connection is unique. This is in a correspondence to the original Levi-Civita theorem, and the unique connection is of Levi-Civita. Moreover, substituting (2.103) into (2.44) we can express now the standard Levi-Civita connection $\overset{*}{\Gamma}{}^i{}_{jk}$ via the flat connection of Weitzenböck $\overset{\text{o}}{\Gamma}{}^i{}_{jk}$ —

$$\overset{*}{\Gamma} ijk \quad = \quad \overset{\text{o}}{\Gamma}{}_{i(jk)} + C_{kij} - C_{jki} \,. \tag{2.104}$$

In the basis constructed from the coframe field itself, the nonmetricity 1-form for the Levi-Civita connection reads

$$\overset{*}{\Gamma}{}_{\alpha\beta} \quad = \quad e_\alpha \rfloor d\vartheta^\beta - e_\beta \rfloor d\vartheta^\alpha - \frac{1}{2}\, e_\alpha \rfloor e_\beta \rfloor B \,. \tag{2.105}$$

It is in a correspondence with a formula given in [5].

2.5 Gauge transformations

Local transformations of the coframe field. The geometrical structure considered above is well defined for a fixed coframe field e_α. Moreover, it is invariant under rigid coframe transformations. The gauge paradigm suggests now to look for a localization of such transformations:

$$\vartheta^\alpha \mapsto L^\alpha{}_\beta\, \vartheta^\beta \,, \qquad e_\alpha \mapsto L_\alpha{}^\beta\, e_\beta \,, \tag{2.106}$$

or, in the components,

$$\vartheta^\alpha{}_i \mapsto L^\alpha{}_\beta\, \vartheta^\beta{}_i \,, \qquad e_\alpha{}^i \mapsto L_\alpha{}^\beta\, e_\beta{}^i \,. \tag{2.107}$$

Here the matrix $L^\alpha{}_\beta$ and its inverse $L_\alpha{}^\beta$ are functions of a point $x \in M$. We require the volume element (2.8) and the metric tensor (2.12) both to be invariant under the pointwise transformations (2.106). Consequently, $L^\alpha{}_\beta$ is assumed to be a pseudo-orthonormal matrix whit enters are smooth functions of a point. We will also use an infinitesimal version of the transformation (2.107) with $L^\alpha{}_\beta = \delta^\alpha_\beta + X^\alpha{}_\beta$. In the components, it takes the form

$$\vartheta^\alpha{}_i \mapsto \vartheta^\alpha{}_i + X^\alpha{}_\beta\, \vartheta^\beta{}_i \,, \qquad e_\alpha{}^i \mapsto e_\alpha{}^i - X^\beta{}_\alpha\, e_\beta{}^i \,. \tag{2.108}$$

As the elements of the algebra $so(1,n)$, the matrix $X_{\alpha\beta} = \eta_{\alpha\mu} X^\mu{}_\beta$ is antisymmetric. We define a corresponded antisymmetric tensor

$$F_{ij} = \vartheta^\alpha{}_i \vartheta^\beta{}_j X_{\alpha\beta} \,. \tag{2.109}$$

Connection invariance postulate. Recall that we are looking for a most general geometric structure that can be explicitly constructed from the coframe field. Moreover, we are interested not in a one fixed coframe field, but rather in a family of fields related by the left action of the elements of some continuous group G.

In a general setting, the different geometrical structures such as the volume element, the metric tensor, and the field of affine connections, are completely independent. We have already postulated the invariance of the volume element and of the metric tensor under the coframe transformations. It is natural to involve now an additional invariance requirement concerning the affine connection.

> Connection invariance postulate: Affine coframe connection is assumed to be invariant under pointwise transformations of the coframe field
> $$\Gamma^i{}_{jk}\left(\vartheta^\alpha\right) = \Gamma^i{}_{jk}\left(L^\alpha{}_\beta \vartheta^\beta\right) . \tag{2.110}$$

Since the coframe connection is constructed from the first order derivatives of the coframe field, (2.110) is a first order PDE for the elements of the group G and for the components of the coframe field.

Weitzenböck connection transformation. Since the Weitzenböck connection is a basis tool of our construction, it is useful to calculate the change of this quantity under the coframe transformations (2.106). We have

$$\Delta \overset{\circ}{\Gamma}{}^i{}_{jk} = e_\alpha{}^i \vartheta^\beta{}_k Y^\alpha{}_{\beta j} , \qquad \text{where} \qquad Y^\alpha{}_{\beta j} = L^\alpha{}_\gamma L^\gamma{}_{\beta,j} . \tag{2.111}$$

All matrices involved here are nonsingular, consequently the Weitzenböck connection is preserved only under the rigid transformations of the coframe field with $L^\gamma{}_{\beta,j} = 0$.

Let us rewrite (2.111) in alternative forms. Since the metric tensor is invariant under the transformations (2.106) we have

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = \Delta\left(g_{im} \overset{\circ}{\Gamma}{}^m{}_{jk}\right) = g_{im} \Delta \overset{\circ}{\Gamma}{}^m{}_{jk} . \tag{2.112}$$

Consequently

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = \vartheta^\alpha{}_i \vartheta^\beta{}_k Y_{\alpha\beta j} , \qquad \text{where} \qquad Y_{\alpha\beta j} = \eta_{\alpha\mu} Y^\mu{}_{\beta j} . \tag{2.113}$$

In the infinitesimal approximation, (2.111) takes the form

$$\Delta \overset{\circ}{\Gamma}{}^i{}_{jk} = e_\alpha{}^i \vartheta^\beta{}_k X^\alpha{}_{\beta,j} . \tag{2.114}$$

while (2.112) with $X_{\alpha\beta} = \eta_{\alpha\mu} X^\mu{}_\beta$ reads

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = \vartheta^\alpha{}_i \vartheta^\beta{}_k X_{\alpha\beta,j} . \tag{2.115}$$

Note that since $X_{\alpha\beta}$ is antisymmetric, we have in this approximation

$$\Delta \overset{\circ}{\Gamma}{}_{ijk} = -\Delta \overset{\circ}{\Gamma}{}_{kji} . \tag{2.116}$$

We will also consider an additional physical meaningful approximation when the derivatives of the coframe is considered to be small relative to the derivatives of the transformation matrix. In this case, (2.114) and (2.115) read

$$\Delta \overset{\circ}{\Gamma}{}^i{}_{jk} = F^i{}_{k,j} , \qquad \text{where} \qquad F^i{}_k = e_\alpha{}^i \vartheta^\beta{}_k X^\alpha{}_\beta , \tag{2.117}$$

and

$$\Delta \overset{\circ}{\Gamma}_{ijk} = F_{ik,j}\,, \qquad \text{where} \qquad F_{ij} = \vartheta^\alpha{}_i \vartheta^\beta{}_j X_{\alpha\beta}\,. \tag{2.118}$$

Transformations preserved the geometric structure. Since the coframe field appears in the coframe geometrical structure only implicitly, (2.106) is a type of a gauge transformation. Invariance of the metric tensor and of the volume element restricts $L^\alpha{}_\beta$ to a pseudo-orthonormal matrix $G = SO(1,n)$. Let us ask now, under what conditions the general coframe connection (2.44) is invariant under the coframe transformations (2.106). First we rewrite (2.44) via the Levi-Civita connection. Using (2.103) we have

$$\overset{\circ}{\Gamma}_{ijk} = \overset{*}{\Gamma}_{ijk} + C_{ijk} - C_{kij} + C_{jki}\,. \tag{2.119}$$

Thus (2.44) takes the form

$$\Gamma_{ijk} = \overset{*}{\Gamma}_{ijk} + (\alpha_1+1)C_{ijk} + \alpha_2 g_{ik}C_j + \alpha_3 g_{ij}C_k + \beta_1 g_{jk}C_i + (\beta_2+1)C_{jki} + (\beta_3-1)C_{kij}\,. \tag{2.120}$$

Since the Levi-Civita connection $\overset{*}{\Gamma}_{ijk}$ is invariant under the transformations (2.106), the equation $\Delta\Gamma_{ijk} = 0$ takes the form

$$(\alpha_1+1)\Delta C_{ijk} + \alpha_2 g_{ik}\Delta C_j + \alpha_3 g_{ij}\Delta C_k + \beta_1 g_{jk}\Delta C_i + (\beta_2+1)\Delta C_{jki} + (\beta_3-1)\Delta C_{kij} = 0\,. \tag{2.121}$$

Hence in order to have an invariant coframe connection, we have to look for possible solutions of equation (2.121).

Trivial solutions of the invariance equation. Consider first two trivial solutions of (2.121) which turn out to be non-dynamical.

(i) *Arbitrary transformations — Levi-Civita connection.*
The equation (2.121) is evidently satisfied when all the numerical coefficients mutually equal to zero. It is easy to check that these six relations are equivalent to (2.103). Thus the corresponded connection is of Levi-Civita. In this case, the elements of the matrix $L^\alpha{}_\beta$ are arbitrary functions of a point. Thus we come to a trivial fact that the Levi-Civita connection is a unique coframe connection which is invariant under arbitrary local $SO(1,n)$ transformations of the coframe field.

(ii) *Rigid transformations.*
Another trivial solution of the system (2.121) emerges when we require $\Delta C_{ijk} = 0$. All permutations and traces of this tensor are also equal to zero so (2.121) is trivially valid. Due to (2.113), it means that the matrix of transformations is independent on a point. In this case, an arbitrary coframe connection, in particular the Weitzenböck connection, remains unchanged. Thus we come to another trivial fact that the coframe connection is invariant under rigid transformations of the coframe field.

Dynamical solution. We will look now for nontrivial solutions of the system (2.121). Three traces of this system yield the equations of the type $\lambda\Delta C_i = 0$, where λ is a linear combination of the coefficients α_i, β_i. Thus we have to apply the first condition

$$\Delta C_i = 0\,. \tag{2.122}$$

The system (2.121) remains now in the form

$$(\alpha_1+1)\Delta C_{ijk} + (\beta_2+1)\Delta C_{jki} + (\beta_3-1)\Delta C_{kij} = 0\,. \tag{2.123}$$

Applying the complete antisymetrization in three indices we derive the second equation

$$\Delta C_{[ijk]} = 0\,. \tag{2.124}$$

The equation (2.123) remains now in the form

$$(\beta_2 - \alpha_1)\Delta C_{jki} + (\beta_3 - \alpha_1 - 2)\Delta C_{jki} = 0\,. \tag{2.125}$$

We have to restrict now the coefficients, otherwise we obtain $\Delta C_{ijk} = 0$, i.e., only the rigid transformations. Consequently we require

$$\beta_2 = \alpha_1\,, \qquad \beta_3 = \alpha_1 + 2\,. \tag{2.126}$$

Thus we have proved

Proposition 6: *The coframe connection*

$$\Gamma_{ijk} = \overset{*}{\Gamma}_{ijk} + (\alpha_1 + 1)C_{[ijk]} + \alpha_2 g_{ik}C_j + \alpha_3 g_{ij}C_k + \beta_1 g_{jk}C_i\,. \tag{2.127}$$

is invariant under the coframe transformations satisfied the equations

$$\Delta C_i = 0\,. \qquad \Delta C_{[ijk]} = 0\,. \tag{2.128}$$

Observe that this family includes the Levi-Civita connection, which is invariant under arbitrary transformations of the coframe field. The torsion tensor of the connection (2.127) is expressed as

$$T_{ijk} = (\alpha_1 + 1)C_{[ijk]} + (\alpha_2 - \alpha_3)(g_{ik}C_j - g_{ij}C_k)\,. \tag{2.129}$$

Thus a torsion-free subfamily of (2.127) is given by

$$\Gamma_{ijk} = \overset{*}{\Gamma}_{i(jk)} + \alpha_2(g_{ik}C_j + g_{ij}C_k) + \beta_1 g_{jk}C_i\,. \tag{2.130}$$

The nonmetricity tensor of the connection (2.127) reads

$$\mathcal{Q}_{kij} = (\alpha_2 + \beta_1)(g_{ik}C_j + g_{jk}C_i) + 2\alpha_3 g_{jj}C_k\,. \tag{2.131}$$

Thus a metric compatible subfamily of (2.127) is given by

$$\Gamma_{ijk} = \overset{*}{\Gamma}_{i(jk)} + (\alpha_1 + 1)C_{[ijk]} + \alpha_2(g_{ik}C_j - g_{jk}C_i)\,. \tag{2.132}$$

From (2.129) and (2.131) we derive an interesting conclusions:

$$\Delta Q_{kij} = 0 \quad \Longleftrightarrow \quad \Delta C_i = 0\,. \tag{2.133}$$

and, together with this relation,

$$\Delta T_{ijk} = 0 \quad \Longleftrightarrow \quad \Delta C_{[ijk]} = 0\,. \tag{2.134}$$

Thus the relations (2.128) obtain a geometric meaning, they correspond to invariance of the torsion and nonmetricity tensors under coframe transformations.

2.6 Maxwell-type system

Let us examine now what physical meaning can be given to the invariance conditions [41]

$$\Delta C_{[ijk]} = 0\,, \qquad \Delta C_i = 0\,. \tag{2.135}$$

Denote $K_{ijk} = \Delta C_{ijk}$. Thus (2.135) takes the form

$$K_{[ijk]} = 0\,, \qquad K^m{}_{im} = 0\,. \tag{2.136}$$

The tensor K_{ijk} depends on the derivatives of the Lorentz parameters $X_{\alpha\beta}$ and on the components of the coframe field

$$K_{ijk} = \frac{1}{2}\,\vartheta^\alpha{}_k\Big(X_{\alpha\beta,j}\vartheta^\beta{}_i - X_{\alpha\beta,i}\vartheta^\beta{}_j\Big)\,. \tag{2.137}$$

Thus, in fact, we have in (2.136), two first order partial differential equations for the entries of an antisymmetric matrix $X_{\alpha\beta}$. Let us construct from this matrix an antisymmetric tensor F_{ij}

$$F_{ij} = X_{\mu\nu}\vartheta^\mu{}_i\vartheta^\nu{}_j\,, \qquad X_{\mu\nu} = F_{ij}e_\mu{}^ie_\nu{}^j\,. \tag{2.138}$$

Substituting into (2.137), we derive

$$\begin{aligned} K_{ijk} &= F_{k[i,j]} - \frac{1}{2}\,X_{\alpha\beta}\Big[(\vartheta^\alpha{}_k\vartheta^\beta{}_i)_{,j} - (\vartheta^\alpha{}_k\vartheta^\beta{}_j)_{,i}\Big] \\ &= F_{k[i,j]} - F_{km}C^m{}_{ij} - \frac{1}{2}\Big(F_{mi}\,\overset{\circ}{\Gamma}{}^m{}_{kj} - F_{mj}\,\overset{\circ}{\Gamma}{}^m{}_{ki}\Big)\,. \end{aligned} \tag{2.139}$$

Consequently, the first equation from (2.136) takes the form

$$F_{[ij,k]} = \frac{2}{3}(C^m{}_{ij}F_{km} + C^m{}_{jk}F_{im} + C^m{}_{ki}F_{jm})\,, \tag{2.140}$$

while the second equation from (2.136) is rewritten as

$$F^i{}_{j,i} = -2F^i{}_mC^m{}_{ij} + F_{kj}g^{ki}{}_{,i} + F_{mj}g^{ki}\,\overset{\circ}{\Gamma}{}^m{}_{ki} - F_{mi}g^{ki}\,\overset{\circ}{\Gamma}{}^m{}_{kj}\,. \tag{2.141}$$

Observe first a significant approximation to (2.140—2.141). If the right hand sides in both equations are neglected, the equations take the form of the ordinary Maxwell equations for the electromagnetic field in vacuum —

$$F_{[ij,k]} = 0\,, \qquad F^i{}_{j,i} = 0\,. \tag{2.142}$$

In the coframe models, the gravity is modeled by a variable coframe field, i.e., by nonzero values of the quantities $\overset{\circ}{\Gamma}{}_{ij}{}^k$. Consequently, the right hand sides of (2.140—2.141) can be viewed as curved space additions, i.e., as the gravitational corrections to the electromagnetic field equations. In the flat spacetime, when a suitable coordinate system is chosen, these corrections are identically equal to zero. Consequently, in the flat spacetime, the invariance conditions (2.136) take the form of the vacuum Maxwell system.

On a curved manifold, the standard Maxwell equations are formulated in a covariant form. Let us show that our system (2.140—2.141) is already covariant. We rewrite (2.139) as

$$K_{ijk} = \frac{1}{2}(F_{ki,j} - F_{km}\,\overset{\circ}{\Gamma}{}^m{}_{ij} - F_{mi}\,\overset{\circ}{\Gamma}{}^m{}_{kj}) - \frac{1}{2}(\ i \longleftrightarrow j\ )\,. \tag{2.143}$$

Consequently,

$$K_{ijk} = F_{k[i;j]}\,, \tag{2.144}$$

where the covariant derivative (denoted by the semicolon) is taken relative to the Weitzenböck connection. Consequently, the system (2.140—2.141) takes the covariant form

$$F_{[ij;k]} = 0\,, \qquad F^i{}_{j;i} = 0\,. \tag{2.145}$$

These equations are literally the same as the electromagnetic sector field equations of the Maxwell-Einstein system. The crucial difference is encoded in the type of the covariant derivative. In the Maxwell-Einstein system, the covariant derivative is taken relative to the Levi-Civita connection, while, in our case, the corresponding connection is of Weitzenböck. Observe that, due to our approach, the Weitzenböck connection is rather natural in (2.145). Indeed, since the electromagnetic-type field describes the local change of the coframe field, it should itself be referred only to the global changes of the coframe. As we have shown, such global transformations correspond precisely to the teleparallel geometry with the Weitzenböck connections.

3 Geometrized coframe field model

3.1 Generalized Einstein-Hilbert Lagrangian

One of the most important feature of the Einstein gravity theory is its pure geometrical content. The basic field variable of this theory is the metric tensor field g_{ij}. The action integral is given by the Einstein-Hilbert Lagrangian

$${}^{(\texttt{GR})}\mathcal{A} = \int_M R\left(\overset{*}{\Gamma}{}^i_{jk}(g), g\right) * 1\,, \tag{3.1}$$

where R is the curvature scalar constructed from the metric tensor and its partial derivatives while $*1$ is the invariant volume element constructed from the metric tensor. When we restrict to the quasilinear second order field equations the Lagrangian (3.1) is a unique possible.

The coframe field model also constructed from the geometrical field variable — coframe. Its Lagrangian however is taken as an arbitrary linear combination of the global $SO(1,3)$ invariants. The geometrical sense of this expression is not clear. Although the coframe Lagrangian can be written in term of the torsion of the flat connection it does not mean that it corresponds to the Weitzenböck geometry with a flat curvature and a non-zero connection. Indeed also the standard Einstein-Hilbert Lagrangian (3.1) can be rewritten in such a form. Moreover, as we have seen in the previous section, there is a wide class of connections all constructed from Weitzenböck connection and its torsion. In particular, using the coframe Lagrangian in the form (1.4) we cannot answer the question: *What special geometry corresponds to the set of viable coframe models?*

Our proposal is to consider for the coframe Lagrangian an expression similar to (3.1)

$${}^{(\texttt{cof})}\mathcal{A} = \int_M R\Big(\Gamma^i_{jk}(\vartheta^\alpha), g(\vartheta^\alpha)\Big) * 1\,, \tag{3.2}$$

which is constructed from the general free parametric coframe connection. Also the invariant volume element $*1$ is constructed here from the coframe field. Since the Levi-Civita connection is included as a special case of general coframe connection we have in (3.2) a generalization of the standard GR.

3.2 Curvature of the coframe connection

Riemannian curvature 2-form. We start with the definitions of the Riemannian curvature machinery. Although it is a classical subject of differential geometry [50], in the case of a general connection of non-zero torsion and nonmetricity, slightly different notations are in use. Moreover, in this case, it is useful to apply the formalism of differential forms. We accept the agreements used in metric-affine gravity [5].

Let a connection 1-form $\mathit{\Gamma}_a{}^b$ referred to a general nonholonomic basis (θ^a, f_a) be given. The *curvature 2-form* is defined as

$$\mathcal{R}_a{}^b = d\mathit{\Gamma}_a{}^b - \mathit{\Gamma}_a{}^c \wedge \mathit{\Gamma}_c{}^b\,. \tag{3.3}$$

It satisfies two fundamental identities:

The first Bianchy identity involves the first order derivatives of the connection

$$D\mathcal{T}^a - \mathcal{R}_b{}^a \wedge \theta^b = 0\,, \qquad \text{or} \qquad d\mathcal{T}^a + \mathit{\Gamma}_b{}^a \wedge \mathcal{T}^b - \mathcal{R}_b{}^a \wedge \theta^b = 0\,. \tag{3.4}$$

The second Bianchy identity involves the second order derivatives of the connection

$$D\mathcal{R}_b{}^a = 0\,, \qquad \text{or} \qquad d\mathcal{R}_a{}^b + \mathit{\Gamma}_a{}^c \wedge \mathcal{R}_c{}^b - \mathit{\Gamma}_c{}^b \wedge \mathcal{R}_a{}^c = 0\,. \tag{3.5}$$

It is useful to consider the Riemannian curvature of the coframe connection to be referred to a basis composed from the elements of the coframe field itself. The corresponded quantity

$$\mathcal{R}_\alpha{}^\beta = d\mathit{\Gamma}_\alpha{}^\beta - \mathit{\Gamma}_\alpha{}^\gamma \wedge \mathit{\Gamma}_\gamma{}^\beta\,. \tag{3.6}$$

is related to the generic basis expression by the standard tensorial rule with the matrices of transformation $\vartheta^\alpha{}_i f_a{}^i$

$$\mathcal{R}_a{}^b = \mathcal{R}_\alpha{}^\beta(\vartheta^\alpha{}_i f_a{}^i)(e_\beta{}^j\theta^b{}_j)\,. \tag{3.7}$$

From (3.6), we see that the Riemannian curvature of the Weitzenböck connection is zero being referred to a basis of the coframe field. Due to (3.6), it is zero in an arbitrary basis.

Being referred to a coordinate basis, the Riemannian curvature 2-form reads

$$\mathcal{R}_i{}^j = d\mathit{\Gamma}_i{}^j - \mathit{\Gamma}_i^k \wedge \mathit{\Gamma}_k{}^j \tag{3.8}$$
$$= d\Gamma^j{}_{in} \wedge dx^n - \Gamma^k{}_{im}\Gamma^j{}_{kn}dx^m \wedge dx^n \tag{3.9}$$
$$= \Big(\Gamma^j{}_{in,m} - \Gamma^k{}_{im}\Gamma^j{}_{kn}\Big)dx^m \wedge dx^n\,. \tag{3.10}$$

The components of the Riemannian curvature 2-form

$$\mathcal{R}_i{}^j = \frac{1}{2}\,R^j{}_{imn}dx^m \wedge dx^n \tag{3.11}$$

are arranged in the familiar expression of the *Riemannian curvature tensor*

$$R^j{}_{imn} = \Gamma^j{}_{in,m} - \Gamma^j{}_{im,n} + \Gamma^k{}_{in}\Gamma^j{}_{km} - \Gamma^k{}_{im}\Gamma^j{}_{kn}\,. \tag{3.12}$$

Curvature scalar density. Curvature scalar plays an important role in physical applications. In fact, it is used as an integrand in action of geometrical field models — Hilbert-Einstein Lagrangian density

$$\mathcal{L} = R\,\mathrm{vol} = R * 1\,, \tag{3.13}$$

where star denotes the Hodge dual. In term of the curvature 2-form, this expression is rewritten as

$$\mathcal{L} = \mathcal{R}_{ij} \wedge *\,(dx^i \wedge dx^j) = \mathcal{R}_{\alpha\beta} \wedge *\,\vartheta^{\alpha\beta}\,. \tag{3.14}$$

where the abbreviation $\vartheta^{\alpha\beta}=\vartheta^\alpha\wedge\vartheta^\beta$ is used. Extracting in (3.14) the total derivative term we obtain

$$\begin{aligned}\mathcal{L} &= (d\Gamma_{\alpha\beta}-\Gamma_\alpha{}^\gamma\wedge\Gamma_{\gamma\beta})\wedge*\vartheta^{\alpha\beta}\\ &= d\left(\Gamma_{\alpha\beta}\wedge*\vartheta^{\alpha\beta}\right)+\Gamma_{\alpha\beta}\wedge d*\vartheta^{\alpha\beta}-\Gamma_\alpha{}^\gamma\wedge\Gamma_{\gamma\beta}\wedge*\vartheta^{\alpha\beta}\,. \qquad (3.15)\end{aligned}$$

For actual calculation of this quantity, it is useful to express the connection 1-form in the basis of the coframe field. We denote

$$\Gamma_{\alpha\beta}=K_{\alpha\gamma\beta}\vartheta^\gamma\,. \qquad (3.16)$$

Substituting it in the total derivative term of (3.15) we have

$$\begin{aligned}d\left(\Gamma_{\alpha\beta}\wedge*\vartheta^{\alpha\beta}\right) &= d\left(K_{\alpha\gamma\beta}\vartheta^\gamma\wedge*\vartheta^{\alpha\beta}\right)=(-1)^n d\left[K_{\alpha\gamma\beta}*\left(e^\gamma\rfloor\vartheta^{\alpha\beta}\right)\right]\\ &= (-1)^n d\left[(K^\alpha{}_{\alpha\beta}-K_{\beta\alpha}{}^\alpha)*\vartheta^\beta\right]\,. \qquad (3.17)\end{aligned}$$

The second term of (3.15) reads

$$\Gamma_{\alpha\beta}\wedge d*\vartheta^{\alpha\beta}=K_{\alpha\gamma\beta}\vartheta^\gamma\wedge d*\vartheta^{\alpha\beta}=K_{\alpha\gamma\beta}\left[d\vartheta^\gamma\wedge*\vartheta^{\alpha\beta}-d\left(\vartheta^\gamma\wedge*\vartheta^{\alpha\beta}\right)\right]\,. \qquad (3.18)$$

Calculate:

$$d\vartheta^\gamma\wedge*\vartheta^{\alpha\beta}=\frac{1}{2}C^\gamma{}_{\mu\nu}\vartheta^{\mu\nu}\wedge*\vartheta^{\alpha\beta}=(-1)^{n+1}C^{\gamma\alpha\beta}*1\,. \qquad (3.19)$$

and

$$d\left(\vartheta^\gamma\wedge*\vartheta^{\alpha\beta}\right)=(-1)^n d*(\eta^{\alpha\gamma}\vartheta^\beta-\eta^{\beta\gamma}\vartheta^\alpha)=(-1)^n\left(\eta^{\beta\gamma}C^\alpha-\eta^{\alpha\gamma}C^\beta\right)*1\,. \qquad (3.20)$$

Consequently the second term of (3.15) takes the form

$$\Gamma_{\alpha\beta}\wedge d*(\vartheta^\alpha\wedge\vartheta^\beta) = (-1)^n\left[K_{\alpha\gamma\beta}C^{\gamma\alpha\beta}-(K^\alpha{}_{\alpha\beta}-K_{\beta\alpha}{}^\alpha)\,C^\beta\right]*1 \qquad (3.21)$$

The third term of (3.15) reads

$$\begin{aligned}\Gamma_\alpha{}^\gamma\wedge\Gamma_{\gamma\beta}\wedge*(\vartheta^\alpha\wedge\vartheta^\beta) &= K_{\alpha\mu}{}^\gamma K_{\gamma\nu\beta}\vartheta^{\mu\nu}\wedge*\vartheta^{\alpha\beta}\\ &= (-1)^n\left(K^{\alpha\beta\gamma}K_{\gamma\alpha\beta}-K^\alpha{}_{\alpha\gamma}K^{\gamma\beta}{}_\beta\right)*1\,. \qquad (3.22)\end{aligned}$$

Consequently the Lagrangian density takes the form

$$\begin{aligned}\mathcal{L}(-1)^n &= d\left[(K^\alpha{}_{\alpha\beta}-K_{\beta\alpha}{}^\alpha)*\vartheta^\beta\right]+\left[K_{\alpha\gamma\beta}C^{\gamma\alpha\beta}-(K^\alpha{}_{\alpha\beta}-K_{\beta\alpha}{}^\alpha)\,C^\beta\right]*1\\ &\quad-\left(K^{\alpha\beta\gamma}K_{\gamma\alpha\beta}-K^\alpha{}_{\alpha\gamma}K^{\gamma\beta}{}_\beta\right)*1\,. \qquad (3.23)\end{aligned}$$

Due to (2.52), the tensor $K_{\alpha\gamma\beta}$ is of the form

$$K_{\alpha\gamma\beta}=\alpha_1C_{\beta\gamma\alpha}+\alpha_2C_\gamma\eta_{\alpha\beta}+\alpha_3C_\alpha\eta_{\beta\gamma}+\beta_1C_\beta\eta_{\alpha\gamma}+\beta_2C_{\gamma\alpha\beta}+\beta_3C_{\alpha\beta\gamma}\,, \qquad (3.24)$$

Substituting this expression in (3.23) we obtain a total derivative term plus a sum of terms which are quadratic in $C_{\alpha\beta\gamma}$. Since (1.4) is the most general expression quadratic in $C_{\alpha\beta\gamma}$, the following statement is clear.

Proposition 7 *The Hilbert-Einstein Lagrangian of the general metric-coframe connection (2.52) is equivalent up to a total derivative term to the general coframe Lagrangian*

$$R(\Gamma_{\alpha\beta})*1=\zeta_0 d(C_\alpha*\vartheta^\alpha)+\left(\zeta_1C_{\alpha\beta\gamma}C^{\alpha\beta\gamma}+\zeta_2C_{\alpha\beta\gamma}C^{\beta\gamma\alpha}+\zeta_3C_\alpha C^\alpha\right)*1\,, \qquad (3.25)$$

where the parameters ζ_i are expressed by second order polynomials of the coefficients α_i,β_i.

The actual expressions for the coefficients ζ_i are rather involved. We discuss the parameter ζ_0 in sequel.

3.3 Einstein-Hilbert Lagrangian without second order derivatives

It is well known that in GR the Einstein-Hilbert Lagrangian involves the second order derivatives of the metric tensor. These terms joint in a total derivative term which is not relevant for the field equation. Although, the total derivative terms cannot consistently dropped out. In particular, the quantization procedure requires an addition of a boundary term in order to compensate the total derivative [56], [57]. Let us calculate the total derivative term in our model. Withe (3.24) we have

$$K^{\alpha}{}_{\alpha\beta} = \eta^{\alpha\gamma} K_{\alpha\gamma\beta} = [\alpha_2 + \alpha_3 + (n+1)\beta_1 + \beta_2 - \beta_3]\, C_\beta \,, \tag{3.26}$$

and

$$K_{\beta\alpha}{}^{\alpha} = \eta^{\beta\gamma} K_{\alpha\gamma\beta} = [\alpha_1 + \alpha_2 + (n+1)\beta_1 + \beta_2 - \beta_3]\, C_\beta \,. \tag{3.27}$$

Thus

$$d\left[(K^{\alpha}{}_{\alpha\beta} - K_{\beta\alpha}{}^{\alpha}) * \vartheta^\beta\right] = -\left[\alpha_1 + n(\alpha_3 - \beta_1) + 2\beta_2 - \beta_3\right] d\left(C_\beta * \vartheta^\beta\right) \,. \tag{3.28}$$

Consequently, the coefficient ζ_0 in (3.25) takes the form

$$\zeta_0 = \alpha_1 + n(\alpha_3 - \beta_1) + 2\beta_2 - \beta_3 \,. \tag{3.29}$$

For the Weitzenböck connection, this coefficient is zero together with all other terms of the Lagrangian. For the Levi-Civita connection, $\zeta_0 = -2$ on a manifold of an arbitrary dimension.

We can identify now a family of coframe connections without a total derivative term at all. It is enough to require

$$\alpha_1 + n(\alpha_3 - \beta_1) + 2\beta_2 - \beta_3 = 0 \,. \tag{3.30}$$

The corresponding connection is given by

$$\begin{aligned} \Gamma_{ijk} \;=\; & \overset{o}{\Gamma}_{ijk} + \alpha_1(C_{ijk} + C_{kij}) + \alpha_2 g_{ik} C_j + \alpha_3(g_{ij}C_k + nC_{kij}) + \\ & \beta_1(g_{jk}C_i - nC_{kij}) + \beta_2(C_{jki} + 2C_{kij}) \,. \end{aligned} \tag{3.31}$$

This family includes the metric-compatible connections

$$\Gamma_{ijk} = \overset{o}{\Gamma}_{ijk} + \alpha_1(C_{ijk} + C_{jki} + 3C_{kij}) + \alpha_2(g_{ik}C_j - g_{jk}C_i + nC_{kij}) \,, \tag{3.32}$$

and the symmetric (torsion-free) connections

$$\Gamma_{ijk} = \overset{o}{\Gamma}_{i(jk)} + \alpha_2\left(g_{ik}C_j + g_{ij}C_k\right) + \beta_1 g_{jk} C_i + \left[1 - n(\alpha_2 - \beta_1)\right]\left(C_{jki} + C_{kji}\right) \,. \tag{3.33}$$

Also the gauge invariant connections (2.127) can be found into the family (3.31).

Consequently we identified a remarkable property of the coframe geometry. There is a family of coframe connections which standard Einstein-Hilbert Lagrangian does not involve second order derivatives terms at all. It means that there is a family of coframe models with a geometrical Lagrangian which is completely equivalent to the Yang-Mills Lagrangians of particle physics.

4 Conclusion

GR is a well-posed classical field theory for 10 independent variables — the components of the metric tensor. Although, this theory is completely satisfactory in the pure gravity sector, its possible extensions to other physics phenomena is rather problematic. In particular, the description of fermions on a curved space and the supergravity constructions require a richer set of 16 independent variables. These variables can be assembled in a coframe field, i.e., a local set of four linearly independent 1-forms. Moreover, in supergravity, it is necessary to involve a special flat connection constructed from the derivatives of the coframe field. These facts justify the study of the field models based on a coframe variable alone.

The classical field construction of the coframe gravity is based on a Yang-Mills-type Lagrangian which is a linear combination of quadratic terms with dimensionless coefficients. Such model turns to be satisfactory in the gravity sector and has the viable Schwarzschild solutions even being alternative to the standard GR. Moreover, the coframe model treating of the gravity energy makes it even preferable than the ordinary GR where the gravity energy cannot be defined at all. A principle problem that the coframe gravity construction does not have any connection to a specific geometry even being constructed from the geometrical meaningful objects. A geometrization of the coframe gravity is an aim of this chapter.

We construct a general family of coframe connections which involves as the special cases the Levi-Civita connection of GR and the flat Weitzenböck connection. Every specific connection generates a geometry of a specific type. We identify the subclasses of metric-compatible and torsion-free connections. Moreover we study the local linear transformations of the coframe fields and identify a class of connections which are invariant under restricted coframe transformations. Quite remarkable that the restriction conditions are necessary approximated by a Maxwell-type system of equations.

On a basis of the coframe geometry, we propose a geometric action for the coframe gravity. It has the same form as the Einstein-Hilbert action of GR, but the scalar curvature is constructed from the general coframe connection. We show that this geometric Lagrangian is equivalent to the coframe Lagrangian up to a total derivative term. Moreover there is a family of coframe connections which Lagrangian does not include the higher order terms. In this case, the equivalence is complete.

However, the Hilbert-Einstein-type action itself is not enough to predict a unique coframe connection. Indeed, the coframe connection has six free parameters, while the action involves only four of their combinations. Moreover, one combination represents a total derivative term in Lagrangian which does not influence the field equations. So the gravity action itself is not defined uniquely the geometry on the base manifold. It should not be, however, a problem. Indeed, the gravitational field is not a unique physical field. Moreover, gravity does not even exist without matter fields as its origin. An action for an arbitrary (non-scalar) field necessary involves the connection. So the problem can be formulated as following: To find out which matter field has to be added to the coframe Lagrangian in order to predict uniquely the type of the coframe connection and consequently the geometry of the underlying manifold. This problem can serve as a basis for future investigation.

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5 Appendix — differential form notations

We collect here some algebraic rules which are useful for calculations with the differential forms. Recall that we are working on an $n+1$ dimensional manifold.

1. Interior product
In a basis of 1-forms ϑ^α, a p-form Ψ is expressed as

$$\Psi = \frac{1}{p!}\Psi_{\alpha_1\cdots\alpha_p}\vartheta^{\alpha_1}\wedge\cdots\wedge\vartheta^{\alpha_p}\,. \qquad \text{(A.1)}$$

Interior product couples the basis vectors and basis 1-forms as

$$e_\alpha\rfloor\vartheta^\beta = \delta^b_\alpha\,. \qquad \text{(A.2)}$$

By bilinearity and the Leibniz-type rule,

$$e_\alpha\rfloor(w_1\wedge w_2) = (e_\alpha\rfloor w_1)\wedge w_2 + (-1)^{\deg w_1}w_1\wedge(e_\alpha\rfloor w_2)\,, \qquad \text{(A.3)}$$

the definition of the interior product is extended to forms of arbitrary degree. Mixed applications of the exterior and interior products to a p-form w satisfy the relations

$$\vartheta^\alpha\wedge(e_a\rfloor w) = pw\,, \qquad \text{(A.4)}$$

and

$$e_a\rfloor(\vartheta^\alpha\wedge w) = (n-p)w\,. \qquad \text{(A.5)}$$

2. Hodge star operator
The Hodge star operator maps p-forms into $(n+1-p)$-forms. In a pseudo-orthonormal basis ϑ^α, the metric tensor is represented by the constant components $\eta_{\alpha\beta} = \text{diag}(-1,1,\cdots,1)$. In this case, the Hodge star operator is defined as

$$*\Psi = \frac{1}{p!(n+1-p)!}\Psi_{\alpha_0\cdots\alpha_p}\eta^{\alpha_0\beta_0}\cdots\eta^{\alpha_p\beta_p}\varepsilon_{\beta_0\cdots\beta_n}\vartheta^{\beta_{p+1}\wedge\cdots\wedge\vartheta^{\beta_n}}\,, \qquad \text{(A.6)}$$

where the permutation symbol is normalized as

$$\varepsilon_{0\cdots n} = 1\,, \qquad \varepsilon^{0\cdots n} = -1\,. \qquad \text{(A.7)}$$

For the basis forms themselves, this formula can be rewritten as

$$*(\vartheta_{\alpha_0}\wedge\cdots\wedge\vartheta_{\alpha_p}) = \frac{1}{(n+1-p)!}\varepsilon_{\alpha_0\cdots\alpha_p\beta_1\cdots\beta_{n-p}}\vartheta^{\beta_1}\wedge\cdots\wedge\vartheta^{\beta_{n-p}}\,. \qquad \text{(A.8)}$$

In particular,

$$*(\vartheta_{\alpha_0}\wedge\cdots\wedge\vartheta_{\alpha_n}) = \varepsilon_{\alpha_0\cdots\alpha_n}\,, \qquad *1 = \frac{1}{n!}\varepsilon_{\alpha_0\cdots\alpha_n}\vartheta^{\alpha_1}\wedge\cdots\wedge\vartheta^{\alpha_n}\,. \qquad \text{(A.9)}$$

When the Hodge map defined by a Lorentzian-type metric $\eta_{\alpha\beta}$ it acts on a p-form w

$$**w = (-1)^{p(n+1-p)+1}w = (-1)^{pn+1}w\,. \qquad \text{(A.10)}$$

For the forms w_1, w_2 of the same degree,

$$w_1\wedge *w_2 = w_2\wedge *w_1\,. \qquad \text{(A.11)}$$

With the Hodge map, the wedge product can be transformed into the interior product and vice versa by the relations

$$*(w\wedge\vartheta_\alpha) = e_\alpha\rfloor *w\,, \qquad \text{(A.12)}$$

and

$$\vartheta_\alpha \wedge *w = (-1)^{n(n-p)} * (e_\alpha \rfloor w)\,. \tag{A.13}$$

3. Exterior derivative and coderivative of the coframe field

We express the exterior derivative of the coframe field as

$$d\vartheta^\alpha = \frac{1}{2} C^\alpha{}_{\beta\gamma} \vartheta^\beta \wedge \vartheta^\gamma \qquad C_\alpha = C^\mu{}_{\mu\alpha}\,. \tag{A.14}$$

The divergence of the coframe 1-form is

$$d * \vartheta_\alpha = -C_\alpha * 1\,. \tag{A.15}$$

Indeed, using (A.8) we calculate

$$\begin{aligned} d * \vartheta_\alpha &= \frac{1}{n!} \varepsilon_{\alpha\beta_1\cdots\beta_n} d(\vartheta^{\beta_1} \wedge \cdots \wedge \vartheta^{\beta_n}) \\ &= \frac{1}{2(n-1)!} \varepsilon_{\alpha\beta_1\cdots\beta_n} C^{\beta_1}{}_{\mu\nu} \vartheta^\mu \wedge \vartheta^\nu \wedge \vartheta^{\beta_2} \wedge \cdots \wedge \vartheta^{\beta_n}\,. \end{aligned} \tag{A.16}$$

Using (A.9) and (A.10) we have

$$\vartheta^\mu \wedge \vartheta^\nu \wedge \vartheta^{\beta_2} \wedge \cdots \wedge \vartheta^{\beta_n} = -\varepsilon^{\mu\nu\beta_2\cdots\beta_n} * 1\,. \tag{A.17}$$

Consequently,

$$\begin{aligned} d * \vartheta_\alpha &= -\frac{1}{2(n-2)!} \varepsilon_{\alpha\beta_1\cdots\beta_{n-1}} \varepsilon^{\mu\nu\beta_2\cdots\beta_{n-1}} C^{\beta_1}{}_{\mu\nu} * 1 = \\ &= \frac{1}{2} (\delta^\mu_\alpha \delta^\nu_{\beta_1} - \delta^\nu_\alpha \delta^\mu_{\beta_1}) C^{\beta_1}{}_{\mu\nu} * 1 = C^\mu{}_{\alpha\mu} * 1 = -C_\alpha * 1\,. \end{aligned} \tag{A.18}$$

In a coordinate basis we consider the tensors

$$C^i{}_{jk} = \frac{1}{2} \left(\overset{\circ}{\Gamma}{}^i{}_{jk} - \overset{\circ}{\Gamma}{}^i{}_{kj} \right), \qquad C_i = C^m{}_{mi}\,. \tag{A.19}$$

It is easy to check the relations

$$C^i{}_{jk} = C^\alpha{}_{\beta\gamma} e_\alpha{}^i \vartheta^\beta{}_j \vartheta^\gamma{}_k\,, \qquad C_i = C_\alpha \vartheta^\alpha{}_i\,. \tag{A.20}$$

Define a non-indexed (scalar-valued) 1-form

$$\mathcal{A} = e_\mu \rfloor d\vartheta^\mu = 2\vartheta^\mu{}_{[i,j]} e_\mu{}^i \, dx^j = 2C_i dx^i = 3C_\alpha \vartheta^\alpha\,. \tag{A.21}$$

On a manifold with a metric $g = \eta_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu$ (Section 3), we define, in addition, a scalar-valued 3-form

$$\begin{aligned} \mathcal{B} &= \eta_{\mu\nu} \, d\vartheta^\mu \wedge \vartheta^\nu = -\eta_{\mu\nu} \vartheta^\mu{}_{i,j} \vartheta^\nu{}_k \, dx^i \wedge dx^j \wedge dx^k \\ &= C_{ijk} dx^i \wedge dx^j \wedge dx^k = C_{\alpha\beta\gamma} \vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma\,. \end{aligned} \tag{A.22}$$

The operations of symmetrization and antisymmetrization of tensors are used here in the normalized form:

$$(a_1 \cdots a_p) = \frac{1}{p!} \mathrm{Sym}(a_1 \cdots a_p)\,, \qquad [a_1 \cdots a_p] = \frac{1}{p!} \mathrm{Ant}(a_1 \cdots a_p)\,. \tag{A.23}$$

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