

Static exteriors for nonstatic braneworld stars

J. Ponce de Leon*

Laboratory of Theoretical Physics, Department of Physics
University of Puerto Rico, P.O. Box 23343, San Juan,
PR 00931, USA

Version 2, January 2008

Abstract

We study possible static non-Schwarzschild exteriors for nonstatic spherically symmetric stars in a Randall & Sundrum type II braneworld scenario. Thus, the vacuum region outside the surface of a star is assumed to be a static solution to the equation ${}^{(4)}R = 0$, where ${}^{(4)}R$ is the scalar curvature of the 4-dimensional Ricci tensor with spherical symmetry. Firstly, we show that for nonstatic spheres the standard matching conditions are much more restrictive than for static ones; they lead to a specific requirement on the vacuum region outside of a nonstatic star, that is absent in the case of static stars. Secondly, without making any assumption about the bulk, or the material medium inside the star, we prove the following theorem on the brane: for *any* nonstatic spherical star, without rotation, there are only two possible static exteriors; these are the Schwarzschild and the “Reissner-Nordström-like” exteriors. This is quite distinct from the case of stars in hydrostatic equilibrium which admit a much larger family of non-Schwarzschild static exteriors.

PACS: 04.50.+h; 04.20.Cv

Keywords: Braneworld; Kaluza-Klein Theory; General Relativity; Space-Time-Matter theory.

*E-mail: jpdel@ltp.upr.clu.edu; jpdel1@hotmail.com

1 Introduction

In recent years there has been an increased interest in theories that envision our world as embedded in a universe with more than four large dimensions [1]-[16]. The study of stellar structure and stellar evolution might constitute an important approach to predict observable effects from extra dimensions.

An indispensable ingredient for these studies is to know how to describe the region outside of an isolated star. In general relativity this constitutes no problem because there is an unique spherically symmetric vacuum solution, namely the Schwarzschild exterior metric.

However, in more than four dimensions, the effective equations for gravity in $4D$ are weaker than the Einstein equations in ordinary general relativity in the sense that they do not constitute a closed set of differential equations. From a geometrical point of view this reflects the fact that there are many ways of producing, or embedding, a $4D$ spacetime in a given higher dimensional manifold, while satisfying the field equations [17]. From a physical point of view this is a consequence of nonlocal effects transported from the bulk to the brane by the projection of the $5D$ Weyl tensor onto the brane, which are unknown without specifying the properties of the metric in the bulk.

As a consequence, the effective picture in four dimensions allows the existence of different possible non-Schwarzschild scenarios for the description of the spacetime outside of a spherical star. In a recent paper we have studied various non-Schwarzschild exteriors in the context of static spherical stars [18]. A number of interesting results emerged from that study. Among others, that the general relativistic upper bound on the gravitational potential $M/R < 4/9$, for perfect fluid stars, can significantly be increased in these exteriors. In particular, the upper bound is $M/R < 1/2$, $M/R < 2/3$ and $M/R < 1$ for the temporal Schwarzschild [6], [19], spatial Schwarzschild [19] and Reissner-Nordström-like exteriors [6], respectively.

In this work, we concentrate our attention on *nonstatic* spherical stars, without rotation, in the context of the Randall & Sundrum type II braneworld scenario [20]. Our aim is to study possible static non-Schwarzschild exteriors. This is crucial in order to identify the difference between stellar evolution and gravitational collapse in ordinary general relativity and braneworld models, which might shed some light on possible observational clues to detect effects from extra dimensions.

We will show here two specific features of nonstatic spheres in this scenario. Firstly, we show that for nonstatic spheres the standard matching conditions are much more restrictive than for static ones; they lead to a specific requirement on the vacuum region outside of a nonstatic star, that is absent in the case of static stars. Secondly, without making any assumption about the bulk, or the material medium inside the star, we prove the following theorem on the brane: for *any* nonstatic spherical star, without rotation, there are only two possible static exteriors; these are the Schwarzschild and the “Reissner-Nordström-like” exteriors. This is quite distinct from the case of stars in hydrostatic equilibrium which admit a much larger family of non-Schwarzschild static exteriors.

The paper is organized as follows. In section 2 we review the effective field equations on the brane and the general static vacuum solutions. In section 3 we present the stellar model. In section 4 we discuss the boundary conditions and demonstrate the uniqueness of the Schwarzschild and Reissner-Nordström-like exteriors for non-static spherical stars. In section 5 we provide a simple example that illustrates the results. Section 6 is a discussion and conclusions.

2 Field equations on the brane

In order to make the paper self-consistent, let us restate some concepts that are essential in our discussion. This is also necessary, because some authors work with spacetime signature $(+, -, -, -)$, while others with $(-, +, +, +)$. Besides, there are different definitions for the Riemann-Christoffel curvature tensor. As a consequence the Einstein field equations look different. For example, $G_{\mu\nu} = -(8\pi G/c^4)T_{\mu\nu}$ in [21] and $G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ in [22]. In this work the spacetime signature is $(+, -, -, -)$; we follow the definitions of Landau and Lifshitz [22]; and the speed of light c is taken to be unity.

In the Randall & Sundrum braneworld scenario [20] the effective equations for gravity in $4D$ are obtained from dimensional reduction of the five-dimensional equations ${}^{(5)}G_{AB} = k_{(5)}^2 {}^{(5)}T_{AB}$. In this scenario our universe is identified with a singular hypersurface (the *brane*) embedded in a 5-dimensional anti-de Sitter bulk (${}^{(5)}T_{AB} =$

$-\Lambda_{(5)}g_{AB})$ with \mathbf{Z}_2 symmetry with respect to the brane. The effective field equations in $4D$ are [23]

$${}^{(4)}G_{\mu\nu} = -\Lambda_{(4)}g_{\mu\nu} + 8\pi GT_{\mu\nu} + \epsilon k_{(5)}^4 \Pi_{\mu\nu} - \epsilon E_{\mu\nu}, \quad (1)$$

where ${}^{(4)}G_{\mu\nu}$ is the usual Einstein tensor in $4D$; $\Lambda_{(4)}$ is the $4D$ cosmological constant, which is expressed in terms of the $5D$ cosmological constant $\Lambda_{(5)}$ and the brane tension λ , as

$$\Lambda_{(4)} = \frac{1}{2}k_{(5)}^2 \left(\Lambda_{(5)} + \epsilon k_{(5)}^2 \frac{\lambda^2}{6} \right); \quad (2)$$

ϵ is taken to be -1 or $+1$, depending on whether the extra dimension is spacelike or timelike, respectively; G is the Newtonian gravitational constant

$$8\pi G = \epsilon k_{(5)}^4 \frac{\lambda}{6}; \quad (3)$$

$T_{\mu\nu}$ is the energy momentum tensor (EMT) of matter confined in $4D$; $\Pi_{\mu\nu}$ is a tensor quadratic in $T_{\mu\nu}$

$$\Pi_{\mu\nu} = -\frac{1}{4}T_{\mu\alpha}T_{\nu}^{\alpha} + \frac{1}{12}TT_{\mu\nu} + \frac{1}{8}g_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{24}g_{\mu\nu}T^2; \quad (4)$$

and $E_{\alpha\beta}$ is the projection onto the brane of the Weyl tensor in $5D$. Explicitly, $E_{\alpha\beta} = {}^{(5)}C_{\alpha A \beta B} n^A n^B$, where n^A is the $5D$ unit vector ($n_A n^A = \epsilon$) orthogonal to the brane. This quantity connects the physics in $4D$ with the geometry of the bulk.

Therefore, giving the EMT of matter in $4D$ is not enough to solve the above equations, because $E_{\alpha\beta}$ is unknown without specifying, *both* the metric in $5D$, and the way the $4D$ spacetime is identified [17], [24]. In other words, the set of equations (1) is not closed in $4D$. The only quantity that can be specified without resorting to the bulk metric, or the details of the embedding, is the curvature scalar ${}^{(4)}R = {}^{(4)}R_{\alpha}^{\alpha}$, because $E_{\mu\nu}$ is traceless. In particular, in empty space ($T_{\mu\nu} = 0, \Lambda_{(4)} = 0$)

$${}^{(4)}R = 0. \quad (5)$$

Thus, the braneworld theory provides only one equation for the vacuum region outside the surface of a star. In the case of static spherically symmetric exteriors there are two metric functions, say g_{TT} and g_{RR} , to be determined. As a consequence, (5) admits a non denumerable infinity of solutions parameterized by some arbitrary function of the radial coordinate R [25]. Since this is a second order differential equation for g_{TT} and first order for g_{RR} , the simplest way for generating static solutions is to provide a smooth function of R for g_{TT} . Then, the field equation ${}^{(4)}R = 0$ reduces to a first order differential equation for g_{RR} , whose static solutions and their general properties have thoroughly been discussed in the literature [19], [25], [26], [27].

3 The stellar model

An observer in $4D$, who is confined to making physical measurements in our ordinary spacetime, can interpret the effective equations (1) as the conventional Einstein equations with an effective EMT, $T_{\mu\nu}^{eff}$, defined as

$$8\pi GT_{\mu\nu}^{eff} \equiv -\Lambda_{(4)}g_{\mu\nu} + 8\pi GT_{\mu\nu} + \frac{48\pi G}{\lambda}\Pi_{\mu\nu} - \epsilon E_{\mu\nu}. \quad (6)$$

Thus, if we are dealing with a perfect fluid star with density ρ and pressure p , then the effective density and pressure are given by ($\Lambda_{(4)} = 0$)

$$\begin{aligned} \rho^{eff} &= \rho - \frac{\epsilon k_{(5)}^4}{48\pi G} \rho^2 - \frac{\epsilon E_0^0}{8\pi G}, \\ p_{rad}^{eff} &= p - \frac{\epsilon k_{(5)}^4}{48\pi G} (\rho + 2p)\rho + \frac{\epsilon E_1^1}{8\pi G}, \\ p_{\perp}^{eff} &= p - \frac{\epsilon k_{(5)}^4}{48\pi G} (\rho + 2p)\rho + \frac{\epsilon E_2^2}{8\pi G}. \end{aligned} \quad (7)$$

It should be noted that the effective matter quantities do not have to satisfy the regular energy conditions [28], because they involve terms of geometric origin.

For a nonstatic, spherically symmetric distribution of matter in $4D$, the line element can be written as

$$ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - R^2(r,t) (d\theta^2 + \sin^2 \theta d\phi^2). \quad (8)$$

In a comoving frame, the field equations relate the effective density ρ^{eff} and radial pressure p_{rad}^{eff} to the mass function (from now on we set $G = 1$)

$$m(r,t) = \frac{R}{2} \left(1 + e^{-\nu} \dot{R}^2 - e^{-\lambda} R'^2 \right), \quad (9)$$

as follows [29], [30]

$$m' = 4\pi \rho^{eff} R^2 R', \quad (10)$$

$$\dot{m} = -4\pi p_{rad}^{eff} R^2 \dot{R}, \quad (11)$$

where dots and primes denote differentiation with respect to t and r , respectively. Thus,

$$m(r,t) = 4\pi \int_0^r R^2 \rho^{eff}(\bar{r}, t) R' d\bar{r}, \quad (12)$$

can be interpreted as the “total mass-energy interior to shell r at time t ” measured by an observer riding in a given shell [31]. We note that a similar expression, but for a static interiors, is used in [6].

We assume that the source is bounded, namely that the three-dimensional hypersurface Σ , defined by the equation

$$\Sigma : r - r_b = 0, \quad (13)$$

where r_b is a constant, separates the spacetime into two regions: the stellar interior described by (8) and an exterior vacuum region, which we assume is described by a static spherically symmetric line element in curvature coordinates. Namely,

$$ds^2 = A(R)dT^2 - B(R)dR^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (14)$$

where the metric functions A and B are solutions of the field equation ${}^{(4)}R = 0$.

In this coordinates the equation of the boundary takes the form

$$R = R_b(T), \quad (15)$$

where b stands for boundary. In the vacuum region, outside of the source the projection onto the brane of the Weyl tensor in $5D$ can be interpreted as an effective energy-momentum tensor ($\Lambda_{(4)} = 0$), viz.,

$$T_{\mu\nu}^{eff} = -\frac{\epsilon}{8\pi} E_{\mu\nu} \quad (16)$$

where¹

$$8\pi T_0^0 = \frac{1}{R^2 B^2} \left[R \frac{dB}{dR} + B(B-1) \right], \quad (17)$$

$$8\pi T_1^1 = -\frac{1}{R^2 AB} \left[R \frac{dA}{dR} - A(B-1) \right], \quad (18)$$

$$8\pi T_2^2 = 8\pi T_3^3 = -\frac{1}{2RB} \left[\frac{1}{A} \frac{dA}{dR} - \frac{1}{B} \frac{dB}{dR} + \frac{R}{A} \frac{d^2 A}{dR^2} - \frac{R}{2A} \left(\frac{1}{A} \frac{dA}{dR} + \frac{1}{B} \frac{dB}{dR} \right) \frac{dA}{dR} \right]. \quad (19)$$

In addition, outside the surface

$$T_0^0 + T_1^1 + T_2^2 + T_3^3 = 0, \quad (20)$$

which is a consequence of the fact that $E_{\mu\nu}$ is traceless. We are not going to discuss here the extension of these metrics to the bulk geometry. Finding an exact solution in $5D$ that is consistent with a particular induced metric in $4D$ is not an easy task. However, the existence of such a solution is guaranteed by Campbell-Maagard’s embedding theorems [33], [34].

¹In what follows, in order to simplify the notation, we will suppress the “*eff*” over the matter quantities.

4 Boundary conditions

We recall that two regions of the spacetime are said to match across a separating non-singular surface Σ if the first and second fundamental forms are continuous across Σ . These are essentially Israel's boundary conditions in vacuum².

The continuity of the first fundamental form (the metric tensor induced on Σ) gives at once

$$R_b = R(r_b, t) \equiv R_b(t), \quad (21)$$

and relates the coordinates t and T , viz.,

$$\left(\frac{dT}{dt}\right)^2 = \frac{e^{\nu(r_b, t)} B(R_b)}{A(R_b)} \left(\frac{1}{B(R_b)} + U_b^2\right), \quad (22)$$

where $U_b = e^{-\nu(r_b, t)/2} \dot{R}_b$.

Next, the continuity of the second fundamental form across Σ (or the extrinsic curvature tensor on Σ) requires continuity of the mass function and the radial pressure. In curvature coordinates (14) $r = R$ and $e^\lambda = B(R)$, therefore the mass function (9) reduces to

$$m(R) = \frac{R}{2} \left[1 - \frac{1}{B(R)}\right]. \quad (23)$$

The same can be obtained from (12) after substituting (17) into it, and performing the integration. Thus, demanding the mass function to be continuous across the boundary we get

$$m(r_b, t) = \frac{R_b(t)}{2} \left[1 - \frac{1}{B(R_b(t))}\right]. \quad (24)$$

Let us now evaluate the radial pressure at the surface. From (11) and (24) we find

$$8\pi p_{rad}(r_b, t) = -\frac{1}{R^2 B^2} \left[R \frac{dB}{dR} + B(B-1) \right], \quad \text{evaluated at } R = R_b \quad (25)$$

Thus, from (17) we find

$$p_{rad}(r_b) = -T_0^0(R)|_{R=R_b}. \quad (26)$$

On the other hand, continuity of the second fundamental form requires $p_{rad} = -T_1^1$ at the boundary. Therefore, the exterior solution must satisfy

$$T_0^0 = T_1^1, \quad \text{at } R = R_b. \quad (27)$$

4.1 The ‘‘extra’’ requirement

We should observe that the relations (25)-(27), which are consequence of the effective field equation (11), are exclusive for nonstatic distributions of matter for the reason that in the case of static interiors there is no a similar relation between the mass function and pressure. Therefore, for nonstatic distributions the boundary conditions impose the fulfillment of the extra requirement (27), which is *absent* in the static case.

The question is how to interpret this extra requirement. Firstly, we note that the hypersurface, (say $\Sigma_{(T_0^0=T_1^1)}$) at which $T_0^0 = T_1^1$ has a *fixed* radius, instead of being a dynamical one as required by (15), which is incompatible with the notion of nonstatic distribution of matter. Secondly, it is easy to demonstrate that $\Sigma_{(T_0^0=T_1^1)}$ is either a horizon

²In the general case where the separating surface is a thin layer of matter, which is *not* the situation for stars, with surface energy-momentum tensor $S_{\alpha\beta}$ the extrinsic curvature is discontinuous across the layer. If we denote the unit spacelike vector normal to Σ by n^μ , the induced metric on Σ by $\lambda_{\alpha\beta} = (g_{\alpha\beta} - n_\alpha n_\beta)$, and the extrinsic curvature tensor on Σ by $K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n \lambda_{\mu\nu}$, then the discontinuity of $K_{\alpha\beta}$ is given by $(K_{\mu\nu}|_{\Sigma^+} - K_{\mu\nu}|_{\Sigma^-}) = 8\pi(S_{\mu\nu} - \frac{1}{2}S\lambda_{\mu\nu})$. This condition and $\lambda_{\mu\nu}|_{\Sigma^+} = \lambda_{\mu\nu}|_{\Sigma^-}$ constitute the so-called Israel's boundary conditions [32].

or a spherical surface of infinite radius. In order to show this, we use a technique employed by Bronnikov *et al* [27]. Namely, we introduce a new radial coordinate u defined by

$$du = \sqrt{A(R)B(R)} dR, \quad (28)$$

which leaves $\rho^{ext} = T_0^0$ and $p_{rad}^{ext} = -T_1^1$ invariant. The line element (14) becomes

$$ds^2 = \mathcal{A}(u)dT^2 - \frac{du^2}{\mathcal{A}(u)} - R^2(u) [d\theta^2 + \sin^2\theta d\phi^2], \quad (29)$$

where $\mathcal{A}(u) = A(R)$ and $R(u) = R$. In this coordinates we find

$$T_0^0 - T_1^1 = -\frac{2\mathcal{A}(u)}{R(u)} \left[\frac{d^2 R(u)}{du^2} \right]. \quad (30)$$

This shows that the condition $T_0^0 = T_1^1$, required by (27), is satisfied either (i) at a hypersurface $\Sigma_{(T_0^0=T_1^1)}$ where $A(R_b) = 0$, i.e., at a horizon, or (ii) at spatial infinity, $R = \infty$.

From the above discussion, it is clear that (27) is *not* a condition defining the boundary of a star. In what follows we will interpret it as an “equation of state” for a static vacuum region outside of a *nonstatic* star. Namely,

$$\rho^{ext} = -p_{rad}^{ext}, \quad \text{for } R \geq R_b(t), \quad (31)$$

where $\rho^{ext} = T_0^0$ and $p_{rad}^{ext} = -T_1^1$. We remark that for static interiors this condition is gone.

The preceding analysis is totally general. Therefore, it is useful to illustrate it with some examples. Firstly, let us consider the “temporal Schwarzschild” metric [6], [19]

$$ds^2 = \left(1 - \frac{2M}{R}\right) dT^2 - \frac{(1 - 3M/2R)}{(1 - 2M/R)[1 - (3M/2R)c]} dR^2 - R^2 d\Omega^2, \quad (32)$$

where c is an arbitrary dimensionless constant³ and M is the total gravitational mass measured by an observer at spatial infinity. For $c = 1$ it reduces to the Schwarzschild vacuum solution of general relativity.

For this metric we find

$$\rho^{ext} + p_{rad}^{ext} = \frac{3M(c-1)(2M-R)}{4\pi R^4(2-3M/R)^2}. \quad (33)$$

Thus, condition (27) is satisfied everywhere for $c = 1$. Nevertheless, for $c \neq 1$ it is satisfied at the horizon $R = 2M$ and at $R = \infty$, in agreement with the statement above.

Secondly, let us consider the “spatial Schwarzschild” metric [19]

$$ds^2 = \frac{1}{b^2} \left(b - 1 + \sqrt{1 - \frac{2bM}{R}} \right)^2 dT^2 - \left(1 - \frac{2bM}{R} \right)^{-1} dR^2 - R^2 d\Omega^2, \quad (34)$$

where M is the total gravitational mass measured at spatial infinity and b is a dimensionless constant. For $b = 1$, we recover the Schwarzschild exterior metric.

For this metric we obtain

$$\rho^{ext} + p_{rad}^{ext} = \frac{bM(1-b)}{4\pi R^3 \left(b - 1 + \sqrt{1 - 2Mb/R} \right)}. \quad (35)$$

Thus, condition (27) is satisfied everywhere for $b = 1$, but only at $R = \infty$ for $b \neq 1$, in accordance with the above discussion.

³In order to avoid misunderstanding, please note that c has nothing to do with the velocity of light in vacuum.

4.2 Uniqueness of the Schwarzschild and Reissner-Nordström-like exteriors for non-static spherical stars

We now proceed to show that, in the context of Randall-Sundrum's single brane model scenario, there are only two possible static exteriors for a non-static spherical star, namely the Schwarzschild and the Reissner-Nordström-like exteriors.

The requirement $T_0^0 = T_1^1$ is equivalent to

$$A(R) = \frac{K}{B(R)}, \quad (36)$$

where K is an arbitrary positive dimensionless constant. Now, from (20) it follows that

$$T_0^0 = -T_2^2. \quad (37)$$

Then, using (17) and (19), we get

$$\frac{1}{B} \frac{d^2 B}{dR^2} - \frac{2}{B^2} \left(\frac{dB}{dR} \right)^2 + \frac{4}{RB} \frac{dB}{dR} + \frac{2(B-1)}{R^2} = 0 \quad (38)$$

The only solution to this equation is Reissner-Nordström-like. Namely,

$$B = \left(1 + \frac{C_1}{R} + \frac{C_2}{R^2} \right)^{-1}, \quad (39)$$

where C_1 and C_2 are arbitrary constants of integration. What this means is that, according to the standard matching conditions, in the Randall & Sundrum II braneworld scenario the only possible static exteriors for any *nonstatic* spherical body, are the Schwarzschild and the Reissner-Nordström-like exteriors. We should emphasize the role of the field equation ${}^{(4)}R = 0$ (or $T = 0$) in obtaining this result.

5 Example

In this section we present a simple model that illustrates the above calculations. With this aim, let us consider the case where effective density on the brane is spatially uniform, viz.,

$$\frac{\partial}{\partial r} \rho^{eff} = 0, \quad \text{and} \quad p_{rad}^{eff} = p_{\perp}^{eff}. \quad (40)$$

The most general line element corresponding to these assumptions, in comoving coordinates, is given by [35]

$$ds^2 = \frac{[r^2 + 2\dot{h}(t)/\dot{g}(t)]^2}{[\frac{1}{2}g(t)r^2 + h(t)]^2} \left\{ C^2 dt^2 - \frac{1}{[r^2 + 2\dot{h}(t)/\dot{g}(t)]^2} (dr^2 + r^2 d\Omega^2) \right\}. \quad (41)$$

where the functions $g(t)$ and $h(t)$, as well as the constant C , are arbitrary except for the fact that they have to carry the dimensions

$$[C] = L^{-2}, \quad [g] = L^{-2}, \quad [h] = L^0, \quad (42)$$

for the metric coefficients to be dimensionless, as expected. For this metric we obtain

$$m(r, t) = R^3(r, t) \left(g(t)h(t) + \frac{\dot{g}^2(t)}{8C^2} \right), \quad \text{with} \quad R(r, t) = \frac{r}{\frac{1}{2}g(t)r^2 + h(t)}. \quad (43)$$

Thus, at the boundary

$$R_b(t) \equiv R(r_b, t) = \frac{r_b}{\frac{1}{2}g(t)r_b^2 + h(t)}, \quad (44)$$

and

$$1 - 2R_b^2 \left[g(t)h(t) + \frac{\dot{g}^2(t)}{8C^2} \right] = \frac{1}{B(R_b)}. \quad (45)$$

The function $g(t)$ can be expressed in terms of R_b ,

$$g(t) = -\frac{2[R_b(t)h(t) - r_b]}{r_b^2 R_b(t)}. \quad (46)$$

Without loss of generality, in order to simplify the equations bellow, instead of $h(t)$ it is preferable to work with the function $\beta(t)$ defined as

$$\beta(t) = \frac{\alpha}{2}g(t) - h(t), \quad (47)$$

where α is a parameter with the appropriate units. Substituting these expressions in (45) we obtain the equation that governs the evolution of the boundary, viz.,

$$\left(\frac{dR_b}{dt} \right)^2 = \frac{1}{[r_b + R_b^2(d\beta/dR_b)]^2} \left[R_b^2 C^2 (\alpha - r_b^2)^2 + 4R_b^3 C^2 r_b \beta (\alpha - r_b^2) + 4R_b^4 C^2 r_b^2 \beta^2 - \frac{R_b^2 C^2 (\alpha + r_b^2)^2}{B(R_b)} \right], \quad (48)$$

where $d\beta/dR_b = (d\beta/dt)/\dot{R}_b$.

Substituting (45) into (10) and (11) we find the energy density and pressure inside the source as follows

$$\rho = \frac{3(B-1)}{8\pi R_b^2 B}, \quad (49)$$

$$p = \frac{[(r_b^2 - r^2)\beta R_b + r_b(\alpha + r^2)] R_b (dB/dR_b) - B(B-1) [3(r_b^2 - r^2)R_b^2(d\beta/dR_b) + 2(r_b^2 - r^2)\beta R_b - r_b(\alpha + r^2)]}{8\pi [(r_b^2 - r^2)R_b^2(d\beta/dR_b) - (\alpha + r^2)r_b] R_b^2 B^2}, \quad (50)$$

where B is evaluated at the boundary, i.e., $B = B(R_b)$. For example, in general relativity where the vacuum region outside the surface is the Schwarzschild metric, we set $B(R_b) = (1 - 2M/R_b)^{-1}$, and obtain

$$\rho_{Schw} = \frac{3M}{4\pi R_b^3}, \quad (51)$$

which is a well known expression. For the pressure we get

$$p_{Schw} = \frac{3M [\beta + R_b(d\beta/dR_b)] (r_b^2 - r^2)}{4\pi [(\alpha + r^2)r_b - (r_b^2 - r^2)R_b^2(d\beta/dR_b)] R_b^2}, \quad (52)$$

where R_b is a solution of

$$\left(\frac{dR_b}{dt} \right)_{|Schw}^2 = \frac{2MC^2(\alpha + r_b^2)^2 R_b - 4\alpha r_b^2 C^2 R_b^2 + 4\beta C^2 r_b (\alpha - r_b^2) R_b^3 + 4\beta^2 r_b^2 C^2 R_b^4}{[r_b + R_b^2(d\beta/dR_b)]^2}. \quad (53)$$

Coming back to our problem, evaluating the pressure (50) at the boundary $r = r_b$ we obtain

$$8\pi p(r_b) = -\frac{R_b(dB/dR_b) + B^2 - B}{R_b^2 B^2} = -T_0^0(R)|_{R=R_b}, \quad (54)$$

which is exactly what we obtained in the general case (25), (26). Similar results can be obtained in models with non-uniform effective density.

6 Discussion and conclusions

The continuity of the second fundamental form and the field equation (11) require $T_0^0 = T_1^1$ at the surface of a nonstatic star (27). However, this equation cannot be considered as a condition defining the boundary, because it can only be satisfied either at a horizon or at spatial infinity. This was shown in section 3.1.

Consequently, in order to be able to match a nonstatic interior with a static exterior we *have* to assume that (27) is satisfied *everywhere*, not only at spatial infinity or at a horizon. This constitutes an independent equation outside the surface of a nonstatic star, namely $\rho^{ext} + p_{rad}^{ext} = 0$, which in addition to (5) provides a complete set of equations to determine the two metric functions, $A(R)$ and $B(R)$ in our notation. We have found that the only static solution to these equations is a Reissner-Nordström-like metric, which is given by (36) and (39).

It should be noted that for stars in equilibrium, the space of non-Schwarzschild static exteriors allowed by the boundary conditions is much more general than in the nonstatic case [18]. The natural question to ask here is, why?. If the boundary conditions are expressed in terms of the continuity of the first and second fundamental forms, why do we get “different” results for static and nonstatic stars?

The answer to this question is found not in the boundary conditions, but in the field equations: in the nonstatic case there is a specific relation between the time derivative of the mass function and the radial pressure, which is given by (11). In the static case there is no such relation. Therefore, in the nonstatic case there is an additional expression to be satisfied, namely (27), which is absent in the static one. As a result the space of solutions of the boundary conditions for stars in equilibrium is much greater than the one for nonstatic stars.

From a physical point of view this is a consequence of the interconnection between the brane and the bulk. Indeed, Israel’s boundary conditions and the \mathbf{Z}_2 symmetry applied to the brane relate $T_{\mu\nu}$, the EMT of the fields in $4D$, with the extrinsic curvature $K_{AB} = \frac{1}{2}\partial g_{AB}/\partial y$ of the brane, where y denotes the coordinate along the extra dimension. Thus, if $T_{\mu\nu}$ varies with time, one would expect the metric in the bulk g_{AB} as well as $E_{\mu\nu}$, which carries non-local gravitational effects from the bulk to the brane, to be, in general, nonstatic. But $E_{\mu\nu}$ is the effective ETM in the exterior region (16). Therefore, in general, the exterior spacetime around a nonstatic spherical star is, is expected to be nonstatic as well. What is amazing here is that, despite of this chain of interaction between the bulk and the brane, one can still find some static exteriors for nonstatic stars.

Our results suggest that the temporal and spatial Schwarzschild metrics, as well as other possible static exteriors, are limiting configurations (in time) of non-static exteriors. In other words, if the contraction of a star comes to a halt and it reaches hydrostatic equilibrium, one would expect that a nonstatic exterior will tend to one of the possible static exteriors.

References

- [1] I. Antoniadis, *Phys. Lett.* **B246**, 3171(1990).
- [2] R. Maartens, *Phys. Rev.* **D62**, 084023 (2000); hep-th/0004166.
- [3] Roy Maartens, *Frames and Gravitomagnetism*, ed. J Pascual-Sanchez et al. (World Sci., 2001), p93-119; gr-qc/0101059.
- [4] Naresh Dadhich and S.G. Gosh, *Phys. Lett.* **B518**, 1(2001); hep-th/0101019.
- [5] M. Govender and N. Dadhich, *Phys.Lett.* **B538**, 233(2002); hep-th/0109086.
- [6] C. Germani and Roy Maartens, *Phys. Rev.* **D64**, 124010(2001); hep-th/0107011.
- [7] M. Bruni, C. Germani and R. Maartens, *Phys. Rev. Lett.* **87**, 231302(2001); gr-qc/0108013.
- [8] G. Kofinas and E. Papantonopoulos, *J. Cosmol. Astropart. Phys.* **12**, 11(2004); gr-qc/0401047.
- [9] P.S. Wesson, *G. Rel. Gravit.* **16**, 193(1984).
- [10] J. Ponce de Leon, *Gen. Rel. Grav.* **20**, 539(1988).

- [11] P.S. Wesson and J. Ponce de Leon, *J. Math. Phys.* **33**, 3883(1992).
- [12] A.A. Coley and D.J. McManus, *J. Math. Phys.* **36**, 335(1995).
- [13] J.M. Overduin and P.S. Wesson, *Phys. Reports* **283**, 303(1997).
- [14] A.P. Billiard and A.A. Coley, *Mod. Phys. Lett.* **A12**, 2121(1997).
- [15] P.S. Wesson, *Space-Time-Matter* (World Scientific Publishing Co. Pte. Ltd. 1999).
- [16] J. Ponce de Leon, *Int.J.Mod.Phys.* D11, 1355(2002); gr-qc/0105120.
- [17] J. Ponce de Leon, *Class.Quant.Grav.* **23**, 3043(2006); gr-qc/0512067.
- [18] J. Ponce de Leon, “Stellar models with Schwarzschild and non-Schwarzschild vacuum exteriors” To be published in *Gravitation & Cosmology*; arXiv:0711.0998v1 [gr-qc].
- [19] R. Casadio, A. Fabbri and L. Mazzacurati, *Phys.Rev.* **D65**, 084040(2002); gr-qc/0111072.
- [20] L. Randall and R. Sundrum, *Phys. Rev. Lett.* **83**, 4690(1999); hep-th/9906064.
- [21] Steven Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, Inc. 1972).
- [22] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Fourth Edition (Butterworth-Heinemann, 2002).
- [23] T. Shiromizu, Kei-ichi Maeda and Misao Sasaki, *Phys. Rev.* **D62**, 02412(2000); gr-qc/9910076.
- [24] J. Ponce de Leon, *Mod. Phys. Lett.* **A21**, 947(2006); gr-qc/0511067.
- [25] M. Visser and D. L. Wiltshire, *Phys.Rev.* **D67**, 104004(2003); hep-th/0212333.
- [26] N. Dadhich, R. Maartens, P. Papadopoulos and V. Rezanian, *Phys.Lett.* **B487**, 1(2000); hep-th/0003061.
- [27] K.A. Bronnikov, H. Dehnen and V.N. Melnikov, *Phys.Rev.* **D68**, 024025(2003); gr-qc/0304068.
- [28] K.A. Bronnikov and S-W Kim, *Phys.Rev.* **D67**, 064027(2003); gr-qc/0212112.
- [29] C.W. Misner and D.H. Sharp, *Phys. Rev.* **136**, B571(1964).
- [30] M.A. Podurets, *Astron. Zh.* **41**, 28(1964); *Soviet Astron* **8**, 19(1964).
- [31] C.W Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, page 858 (W.H. Freeman and Company, 1973).
- [32] W. Israel, *Nuovo Cim.* **B44**, 1(1966);[Erratum-ibid. **B48**, 463(1967)].
- [33] S.S. Seahra and P.S. Wesson, *Class.Quant.Grav.* **20** 1321(2003); gr-qc/0302015.
- [34] P.S. Wesson, “In Defense of Campbell’s Theorem as a Frame for New Physics”; gr-qc/0507107.
- [35] J. Ponce de Leon, *J. Math. Phys.* **27**, 271(1986).