Hawking-Moss Tunneling in Noncommutative Eternal Inflation

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Abstract

The quantum behavior of noncommutative eternal inflation is quite different from the usual knowledge. Unlike the usual eternal inflation, the quantum fluctuation of noncommutative eternal inflation is suppressed by the Hubble parameter. Due to this, we need to reconsider many conceptions of eternal inflation. In this paper we study the Hawking-Moss tunneling in noncommutative eternal inflation using the stochastic approach. We obtain a brand-new form of the tunneling probability for this process and find that the Hawking-Moss tunneling is more unlikely to take place in the noncommutative case than in the usual one. We also conclude that the lifetime of a metastable de-Sitter (dS) vacuum in the noncommutative spacetime is longer than that in the commutative case.

1 Introduction

Inflation has been widely considered as a remarkably successful theory in explaining many problems in the very early universe, such as the flatness, horizon and monopole

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problems [1, 2, 3, 4, 5]. During inflation, quantum effects play a crucial role and may bring the universe into a self-reproducing process which is dubbed "eternal inflation" [6, 7, 8]. The string theory landscape indicates that there are a huge number of meta-stable vacua surrounded by various kinds of effective potentials [9, 10, 11, 12]. The realization of string landscape provides an important arena for eternal inflation.

In eternal inflation driven by the false vacuum, the false vacuum is a meta-stable state and would decay through a mix of semiclassical tunneling and stochastic evolution. The probability of finding the inflaton at the top of the plateau in its potential decreases exponentially with time [13]. However, the false vacuum is also expanding exponentially while decaying. When the rate of exponential expansion is larger than the decay rate during this process, the total volume of the false vacuum will grow eternally although the false vacuum is decaying. In this case, bubbles form by random nucleation and then start to expand. Every growing bubble can be viewed as an open FRW universe [14], and we are living in one of such "pocket universes" [15]. Another approach to eternal inflation is achieved in chaotic inflation when the quantum fluctuation of the inflaton dominates over its classical motion. As the inflaton is rolling down the potential classically, its change during one Hubble time ($\delta t = \frac{1}{H}$) can be divided into $\delta\varphi = \Delta\varphi + \delta_q\varphi$, where $\Delta\varphi$ denotes the classical value and $\delta_q\varphi$ represents the quantum one. For a Gaussian probability distribution, when $\delta_q\varphi > 0.61\Delta\varphi$, the quantum behavior overwhelms the classical evolution and inflation becomes eternal.

When dealing with the decaying process of false vacua, one have to make the inflaton tunnel from one false vacuum to another. One method was provided by Coleman and De Luccia (CDL) [14]; another method was investigated by Hawking and Moss [16]. During the Hawking-Moss tunneling, the potential between two vacua is so flat that CDL instanton can not exist. It was shown that the probability of tunneling from one false vacuum φ_0 to another is given by

$$\begin{aligned} P_C &\sim \exp\left(-\frac{24\pi^2}{V(\varphi_0)} + \frac{24\pi^2}{V(\varphi_{\rm top})}\right) \\ &\sim \exp\left(-8\pi^2 \cdot \frac{H(\varphi_{\rm top})^2 - H(\varphi_0)^2}{H(\varphi_0)^2 H(\varphi_{\rm top})^2}\right) \ , \end{aligned} \tag{1}$$

which is related to the value of the top barrier. A proper scenario of this tunneling can be realized in the stochastic approach to inflation [17, 18, 19, 20, 21, 22]. Here the potential of the scalar field is flat enough for slow rolling which requires $V'' <$

V. In the stochastic description, the quantum fluctuations can be simulated by the stochastic noise and the scalar field walks in random.

As is known, eternal inflation happens when the energy scale of the universe is extremely high. Thus we would like to take into consideration more fundamental theories in logic, namely, the string theory. Recently, we have considered the effects of spacetime noncommutativity on slow-roll eternal inflation in Ref. [23]. In noncommutative inflation, it is generally assumed that the background evolution of inflaton is not modified but the fluctuations are affected by noncommutativity (see Ref. [24, 25, 26, 27], for a review in [28] and references therein). In order to introduce the noncommutativity [29, 30] into the 4-dimensional flat Friedmann-Robertson-Walker universe, we would like to define another time coordinate τ,

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 = a^{-2}(\tau)d\tau^2 - a^2(\tau)d\vec{x}^2 \ , \tag{2}$$

where a is the scale factor. Then the spacetime uncertainty relation can be realized by the commutation relation:

$$[\tau, x]_* = iM_{\rm N}^{-2} \ , \tag{3}$$

where $M_{\rm N}$ is the energy scale of noncommutativity and the $*$-product is defined as

$$\begin{aligned}(f*g)(x,\tau) &= \exp\left(-\frac{i}{2}M_{\rm N}^{-2}\left(\partial_x\partial_{\tau'} - \partial_\tau\partial_y\right)\right) \\ &\times \ f(x,\tau)g(y,\tau')\mid_{y=x,\tau'=\tau} \ . \end{aligned} \tag{4}$$

From the result of the paper [23], we can see that the quantum fluctuation still satisfies the Gaussian distribution, but the form of its amplitude in IR region is changed to $\delta_q\varphi \simeq \frac{1}{2\pi}\frac{M_{\rm N}^2}{H}$. Therefore, when the Hubble parameter is lifted highly enough, eternal inflation would cease. This is strongly different from the normal scenario of eternal inflation. In this paper, we shall use the analysis of noncommutativity mentioned above (especially the IR region) and the stochastic approach to study Hawking-Moss tunneling of eternal inflation.

This paper is organized as follows. In Section 2, we study the Hawking-Moss tunneling in noncommutative eternal inflation using the stochastic approach. In Section 3, we make conclusions that Hawking-Moss tunneling is more unlikely to happen in the noncommutative case than in the usual one and the lifetime of a metastable de-Sitter vacuum in the noncommutative spacetime is longer than that in the commutative case.

2 Stochastic Approach to Noncommutative Inflation

One can describe inflation by analyzing the stochastic probability distribution $P(\varphi, t)$, which represents the probability to find the inflaton field φ at the time t. In our note we consider the probability distribution averaged in a Hubble volume observed by a comoving observer. The inflaton evolves as a Brownian particle. Consequently, the probability distribution $P(\varphi, t)$ satisfies the Fokker-Planck (FP) equation(see detailed introduction in Ref. [20, 31]):

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \varphi}\left(\frac{\partial(DP)}{\partial \varphi} + \gamma \frac{dV}{d\varphi} P\right) , \tag{5}$$

where D is the diffusion coefficient and γ is the mobility coefficient[1]. Using slow roll approximation $\dot{\varphi} \simeq -V'/(3H)$, we can establish that $\gamma = \frac{1}{3H}$. In the following we need to derive the form of the coefficient D.

Following the usual knowledge of inflation, the background evolution of noncommutative inflation can be described by

$$3H^2 \simeq V(\varphi) , \tag{6}$$

where we take the normalization $M_p^2 = 1/8\pi G = 1$. According to the calculation in Ref. [23], the IR quantum fluctuation in the momentum space $\delta_q \varphi_k$ is linked to the canonical perturbation u_k by $u_k \simeq a \delta_q \varphi_k$, and when the perturbation begins to be generated the initial conditions require u_k to be canonically normalized as $u_k \simeq \frac{1}{\sqrt{2k}}$ with $a \simeq Hk/M_{\rm N}^2$. Therefore, the IR quantum fluctuation in momentum space can be generally given by,

$$\delta_q \varphi_k \simeq \frac{1}{\sqrt{2k}} \frac{M_{\rm N}^2}{Hk} . \tag{7}$$

After that, the fluctuations outside the horizon are nearly frozen. It can be shown that the initial wave length for the k mode is $\lambda_k = H/M_{\rm N}^2$ [23], so it is appropriate

[1] A general form of FP equation is given by

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial \varphi}\left(D^{1-d}\frac{\partial(D^d P)}{\partial \varphi} + \gamma \frac{dV}{d\varphi} P\right) .$$

However, the choice of the parameter d does not affect the calculation of probability distribution a lot. Therefore we do not plan to discuss it in our note but only focus on the case $d = 1$.

to do the spacial average at a length scale $H/M_{\rm N}^2$. During one Hubble time, we can calculate the IR quantum fluctuation in coordinate space $\delta_q\varphi$ as follows,

$$\begin{aligned}\delta_q\varphi|_H &\equiv \sqrt{\langle\delta_q\varphi^2\rangle}|_H = \left(\int_{k=aM_{\rm N}^2/H}^{k=e\times aM_{\rm N}^2/H}\frac{dk}{k}\frac{k^3}{2\pi^2}\delta_q\varphi_k\delta_q\varphi_{-k}\right)^{\frac{1}{2}} \\ &\simeq \frac{1}{2\pi}\frac{M_{\rm N}^2}{H}\ . \end{aligned} \tag{8}$$

Note that due to the nearly scale invariance of the spectrum, the result of (8) is not sensitive to the length scale $H/M_{\rm N}^2$ where we do the spacial average. We might have chosen the Hubble length H^{-1} or the noncommutativity scale $M_{\rm N}^{-1}$, and the result of the integration in leading order does not change.

Besides, since $M_{\rm N}^{-1}$ arises as another important time scale rather than H^{-1}, it is worthy to calculate the quantum fluctuations during the time $M_{\rm N}^{-1}$. We have

$$\begin{aligned}\delta_q\varphi|_{M_{\rm N}} &\equiv \sqrt{\langle\delta_q\varphi^2\rangle}|_{M_{\rm N}} = \left(\int_{k=aM_{\rm N}^2/H}^{k=e^{H/M_{\rm N}}\times aM_{\rm N}^2/H}\frac{dk}{k}\frac{k^3}{2\pi^2}\delta_q\varphi_k\delta_q\varphi_{-k}\right)^{\frac{1}{2}} \\ &\simeq \frac{1}{2\pi}\frac{M_{\rm N}^{\frac{3}{2}}}{H^{\frac{1}{2}}}\ , \end{aligned} \tag{9}$$

which is a bit different from Eq. (8). We will show later, however, this is also insensitive to the simulation of Langevin equation.

We can simulate the quantum fluctuation in IR region by the Langevin equation which is expressed as

$$\dot{\varphi} \simeq -\frac{V'}{3H} - \frac{M_{\rm N}^2}{3H^{\frac{1}{2}}}\eta\ . \tag{10}$$

Here η is a stochastic noise term added to simulate the quantum fluctuation of the inflaton and we make its form to be Gaussian satisfying

$$<\eta(t)>=0\ , \quad <\eta(t)\eta(t')>=\frac{9}{4\pi^2}\delta(t-t')\ . \tag{11}$$

This simulation is quite general and very efficient in the IR region of noncommutative inflation no matter what time scale we use. When we consider the quantum fluctuations in one Hubble scale, we can recover $<\delta_q\varphi^2>\simeq M_{\rm N}^4/(4\pi^2H^2)$ with the fluctuation of φ integrated in one Hubble time,

$$\delta_q\varphi = -H\int^{\frac{1}{H}}\frac{M_{\rm N}^2\eta}{3H^{\frac{3}{2}}}dt\ ; \tag{12}$$

moreover, when we consider the quantum fluctuations in one noncommutative scale, then we can recover $<\delta_q\varphi^2>\simeq M_{\rm N}^3/(4\pi^2 H)$ with the fluctuation of φ integrated during the time scale $1/M_{\rm N}$,

$$\delta_q\varphi = -M_{\rm N}\int^{\frac{1}{M_{\rm N}}}\frac{M_{\rm N}\eta}{3H^{\frac{1}{2}}}dt\ . \tag{13}$$

Therefore, we have the expression $\frac{d}{dt}<\varphi^2>\simeq M_{\rm N}^4/(4\pi^2 H)$ both in the Hubble scale and the noncommutative scale. According to the property of Brownian motion, the diffusion coefficient D is approximately equal to half of $\frac{d}{dt}<\varphi^2>$, and thus we have $D = M_{\rm N}^4/(8\pi^2 H)$.

For simplicity, we study the stationary ansatz of Eq. (5): $\partial_t P_{\rm N} = 0$. Consequently the FP equation (5) in IR region of noncommutative case can be solved as

$$\begin{aligned} P_{\rm N}(\varphi,t) &\sim \exp\left\{-\frac{8\pi^2}{3M_{\rm N}^4}(V-V_0)\right\} \\ &\sim \exp\left\{-8\pi^2\cdot\frac{H^2-H_0^2}{M_{\rm N}^4}\right\}\ , \end{aligned} \tag{14}$$

where V_0 (and H_0) appears from a proper normalization.

Keeping in mind that $H > M_{\rm N}$ in the IR region of noncommutative eternal inflation, and comparing the denominator on the exponential of the noncommutative result (14) with the commutative result (1), we conclude that the tunneling probability is more suppressed by the spacetime noncommutativity. This suppression has clear physical interpretation. Since the spacetime noncommutativity generally suppresses the quantum fluctuation of the inflaton, it should make quantum behaviors of the inflaton, such as the Hawking-Moss tunneling, more unlikely to happen.

We also note that the equation (1) and (14) can be linked smoothly describing the energy density crossing the noncommutative UV/IR boundary. It is known that when $H < M_{\rm N}$, the noncommutative inflation is in the UV region, and it is in the IR region when $H > M_{\rm N}$. The probability distribution of inflaton in the UV region is described by Eq. (1) in which the maximal value of the potential is $V \simeq 3H^2 = 3M_{\rm N}^2$. This is just the minimal value of the potential in the IR region. Consequently the distribution function of noncommutative eternal inflation in the whole parameter space is continuous, and hence, there is no pathology when the field φ tunnels through the UV/IR boundary(See Fig. 1).

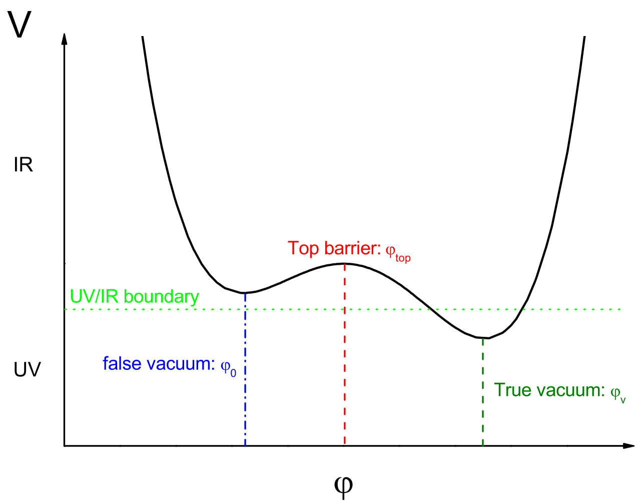

Figure 1: A sketch map of Hawking-Moss tunneling from a false vacuum φ_0 to the true one φ_v. When the inflaton φ lies above the green dot line, its distribution function satisfies Eq. (14) and the probability of tunneling is suppressed exponentially with respect to V; meanwhile, if φ is placed below the green line, the form of distribution function returns the usual one (1).

In order to make this result more explicitly and to investigate the details of physics, we consider the example $V(\varphi) = \lambda\varphi^4$ when φ is near one minimal of the potential. To solve the equations (6) and (10), we define for simplicity $\sigma \equiv \varphi^2$, then we have

$$\dot{\sigma} + \alpha\sigma + \beta\eta = 0 \ , \quad \alpha \equiv 8\sqrt{\frac{\lambda}{3}} \ , \quad \beta \equiv \frac{2}{3}M_{\rm N}^2\sqrt[4]{\frac{3}{\lambda}} \ . \tag{15}$$

The solution of (15) can be written as

$$\sigma(t) = \sigma_0 e^{-\alpha t} + \beta e^{-\alpha t}\int_0^t e^{\alpha t_1}\eta(t_1)dt_1 \ . \tag{16}$$

where $\sigma_0 \equiv \sigma(0)$ sets the initial condition at $t = 0$.

Following [32], the distribution function for σ can also be given by

$$\begin{aligned} P_{\rm N}(\sigma_m) \ \sim & \ \int [d\eta]dt \exp\left(-\frac{2}{9}\pi^2\int_0^\infty dt_1\eta^2(t_1)\right) \\ \times & \ \delta\left(\sigma(t) - \sigma_m\right) \ , \end{aligned} \tag{17}$$

which denotes the number of times the universe arrives at the $\sigma(t) = \sigma_m$ surface during infinite time. By using $\delta(y) = \int \frac{dx}{2\pi}e^{ixy}$, and doing Gaussian integration twice, we obtain the integral

$$\begin{aligned} P_{\rm N}(\sigma_m) \ \sim & \ \int dt \sqrt{\frac{8\pi\lambda}{3M_{\rm N}^4(1 - e^{-2\alpha t})}} \\ \times & \ \exp\left(-\frac{8\pi^2\lambda}{3M_{\rm N}^4}\sigma_m^2\frac{\left(e^{\alpha t} - \frac{\sigma_0}{\sigma_m}\right)^2}{e^{2\alpha t} - 1}\right) . \end{aligned} \tag{18}$$

This integral seems problematic because there is a divergence when $t \to \infty$. However, note that the energy scale of eternal inflation along a comoving world line will eventually drop. As t becomes larger, inflation enters the UV region, and the behavior of evolution returns to the commutative case. Consequently, the full integral does not suffer from the divergence, so this measure is well defined.

Further, what we care about is the tunneling probability which corresponds to the case: $\sigma_m > \sigma_0$. By using the saddle point approximation on the exponential as in [32], we obtain

$$\begin{aligned} P_{\rm N}(\sigma_0) \ \sim & \ \exp\left(-8\pi^2 M_{\rm N}^{-4}(H_m^2 - H_0^2)\right) \\ \simeq & \ \exp\left\{-\frac{8\pi^2}{3M_{\rm N}^4}(V_m - V_0)\right\} \ , \end{aligned} \tag{19}$$

which is consistent with Eq. (14).

3 Conclusion and Discussions

To take a further discussion, we would like to compare the difference of dS decaying processes whether or not the spacetime noncommutativity is present. According to the work of Coleman and De Luccia[14], the decay time of a metastable dS vacuum has an approximate expression $T \sim P^{-1}$. By neglecting all the sub-exponential factors, we have

$$T_C = \exp\left\{24\pi^2 M_p^4 (\frac{1}{V_0} - \frac{1}{V_{top}})\right\} ; \quad (20)$$

$$T_{\rm N} = \exp\left\{\frac{8\pi^2}{3M_{\rm N}^4}(V_{top} - V_0)\right\} , \quad (21)$$

which represent the dS decay times without the spacetime noncommutativity, and the IR region with noncommutativity respectively. To be clear, we have written M_p explicitly here. It is clear that $T_N > T_C$, which is a very general result indicating that the lifetime of a metastable dS vacuum with noncommutativity is longer than that without noncommutativity.

To summarize, from the results obtained in this note we learn that the Hawking-Moss tunneling effect of noncommutative eternal inflation in the IR region is greatly different from the usual one. Its probability distribution is exponentially suppressed by the top barrier value of the potential and make Hawking-Moss tunneling more difficult to happen than in the usual case. This is because the quantum fluctuation is suppressed by spacetime noncommutativity. Consequently, we may expect the application of noncommutativity would bring a closer sight into high energy physics of eternal inflation. Based on the new form of the probability distribution, we find that the lifetime of a metastable dS vacuum in the noncommutative case is longer than in the usual one. This may leave more clues for investigating the new physics of noncommutativity which is worthy for further studies.

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References

[1] A. H. Guth, Phys. Rev. **D23**, 347 (1981).

[2] K. Sato, Mon. Not. Roy. Astron. Soc. **195**, 467 (1981).

[3] A. D. Linde, Phys. Lett. **B108**, 389 (1982).

[4] A. Albrecht, and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[5] For earlier attemps on an inflationary model, see A. A. Starobinsky, JETP Lett. **30**, 682 (1979) [Pisma Zh. Eksp. Teor. Fiz. 30 (1979) 719] ; A. A. Starobinsky, Phys. Lett. **B91**, 99 (1980).

[6] P. J. Steinhardt, **The Very Early Universe** Proceedings, pp.251, (1982).

[7] A. Vilenkin, Phys. Rev. **D27**, 2848 (1983).

[8] A. D. Linde, Mod. Phys. Lett. **A1**, 81 (1986); A. D. Linde, Phys. Lett. **B175**, 395 (1986); A. S. Goncharov, A. D. Linde, and V. F. Mukhanov, Int. J. Mod. Phys. **A2**, 561 (1987); A. D. Linde, D. A. Linde and A. Mezhlumian, Phys. Lett. **B345**, 203 (1995).

[9] R. Bousso and J. Polchinski, JHEP **0006**, 006 (2000) [arXiv:hep-th/0004134].

[10] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, Phys. Rev. **D68**, 046005 (2003) [arXiv:hep-th/0301240].

[11] L. Susskind, arXiv:hep-th/0302219.

[12] M. R. Douglas, JHEP **0305**, 046 (2003) [arXiv:hep-th/0303194]; F. Denef and M. R. Douglas, JHEP **0405**, 072 (2004) [arXiv:hep-th/0404116]; F. Denef, M. R. Douglas and S. Kachru, arXiv:hep-th/0701050; for a review see: M. R. Douglas and S. Kachru, arXiv:hep-th/0610102.

[13] A. H. Guth and S. Y. Pi, Phys. Rev. D **32**, 1899 (1985).

[14] S. R. Coleman and F. De Luccia, Phys. Rev. D **21**, 3305 (1980).

[15] A. H. Guth, Phys. Rept. **333**, 555 (2000) [arXiv:astro-ph/0002156].

[16] S. W. Hawking and I. G. Moss, Phys. Lett. B **110**, 35 (1982).

[17] A. A. Starobinsky, in Field Theory, Quantum Gravity and Strings, edited by H. de Vega and N. Sanchez (Springer-Verlag, Berlin, 1986), p. 107; A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994) [arXiv:astro-ph/9407016].

[18] S. J. Rey, Nucl. Phys. B **284**, 706 (1987).

[19] K. i. Nakao, Y. Nambu and M. Sasaki, Prog. Theor. Phys. **80**, 1041 (1988).

[20] A. D. Linde, Nucl. Phys. B **372**, 421 (1992) [arXiv:hep-th/9110037]; for a review see: A. D. Linde, D. A. Linde and A. Mezhlumian, Phys. Rev. D **49**, 1783 (1994) [arXiv:gr-qc/9306035].

[21] Recent progresses see: S. Gratton and N. Turok, Phys. Rev. D **72**, 043507 (2005) [arXiv:hep-th/0503063]; J. Martin and M. Musso, Phys. Rev. D **73**, 043516 (2006) [arXiv:hep-th/0511214].

[22] A. Linde, JCAP **0701**, 022 (2007) [arXiv:hep-th/0611043].

[23] Y. F. Cai and Y. Wang, JCAP **0706**, 022 (2007) arXiv:0706.0572 [hep-th].

[24] R. Brandenberger and P. M. Ho, Phys. Rev. D **66**, 023517 (2002) [AAPPS Bull. **12N1**, 10 (2002)] [arXiv:hep-th/0203119].

[25] S. Tsujikawa, R. Maartens and R. Brandenberger, Phys. Lett. **B574**, 141 (2003) [arXiv:astro-ph/0308169]; S. Koh and R. H. Brandenberger, JCAP **0706**, 021 (2007) [arXiv:hep-th/0702217].

[26] Q. G. Huang and M. Li, JHEP **0306**, 014 (2003) [arXiv:hep-th/0304203]; Q. G. Huang and M. Li, JCAP **0311**, 001 (2003) [arXiv:astro-ph/0308458]; Q. G. Huang and M. Li, Nucl. Phys. B **713**, 219 (2005) [arXiv:astro-ph/0311378]; W. Xue, B. Chen and Y. Wang, arXiv:0706.1843 [hep-th].

[27] Y. F. Cai and Y. S. Piao, arXiv:gr-qc/0701114;

[28] R. H. Brandenberger, arXiv:hep-th/0703173.

[29] T. Yoneya, Mod. Phys. Lett. A **4**, 1587 (1989); M. Li and T. Yoneya, Phys. Rev. Lett. **78**, 1219 (1997) [arXiv:hep-th/9611072].

[30] N. Seiberg, and E. Witten, JHEP **9909**, 032 (1999), hep-th/9908142.

[31] M. C. Wang, and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).

[32] M. Li and Y. Wang, arXiv:0706.1691 [hep-th].