HOLST ACTIONS FOR SUPERGRAVITY THEORIES

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Abstract

Holst action containing Immirzi parameter for pure gravity is generalised to the supergravity theories. Supergravity equations of motion are not modified by such generalisations, thus preserving supersymmetry. Dependence on the Immirzi parameter does not emerge in the classical equations of motion. This is in contrast with the recent observation of Perez and Rovelli for gravity action containing original Holst term and a minimally coupled Dirac fermion where the classical equations of motion do develop a dependence on Immirzi parameter.

1 Introduction

In the first order formalism, pure gravity is described through three coupling constants; while two of them, the Newton's gravitational and the cosmological constants, are dimensionful; the third known as Immirzi parameter is dimensionless. In the action these are associated with the Hilbert-Palatini, cosmological and Holst terms respectively. Ignoring the cosmological term, we present the Holst's generalisation [1] of the Hilbert-Palatini action in the natural system of fundamental units where Newton's constant $G = 1/(8\pi)$ as[1]:

$$S \;=\; \frac{1}{2}\int d^4x \; e \; \Sigma^{\mu\nu}_{ab} \left[R_{\mu\nu}{}^{ab}(\omega) \;+\; i\eta \; \tilde{R}_{\mu\nu}{}^{ab}(\omega) \right] \tag{1}$$

[1]Our conventions are: Latin indices in the beginning of alphabet, $a, b, c, ...$, run over $1, 2, 3, 4$ and $e^a_\mu e^{b\mu} = \delta^{ab}$, $e^a_\mu e^a_\nu = g_{\mu\nu}$. The tetrad component e^4_μ is imaginary so are the connection components ω^{4i}_μ $(i = 1, 2, 3)$ and the determinant e of tetrad e^a_μ, $e^* = -e$ $= -\frac{1}{4!}\ \epsilon^{\mu\nu\alpha\beta}\epsilon_{abcd}e^a_\mu e^b_\nu e^c_\alpha e^d_\beta$. The usual antisymmetric Levi-Civita density of weight one $\epsilon^{\mu\nu\alpha\beta}$ has values $\pm$ 1 or 0 and $\epsilon_{\mu\nu\alpha\beta}$ takes values $\pm\, e^2$ or 0; and completely antisymmetric $\epsilon^{abcd} = \epsilon_{abcd}$ are $\pm$ 1 or 0.

where $\Sigma^{ab}_{\mu\nu} = \frac{1}{2}\ e^a_{[\mu}\ e^b_{\nu]}$ and $R_{\mu\nu}{}^{ab}(\omega) = \partial_{[\mu}\omega_{\nu]}{}^{ab} + \omega_{[\mu}{}^{ac}\ \omega_{\nu]}{}^{cb}$. The second term containing the parameter η is the Holst action with $\tilde{R}_{\mu\nu}{}^{ab} = \frac{1}{2}\epsilon^{abcd}R_{\mu\nu cd}$ and η^{-1} is the Immirzi parameter [2]. For $\eta = -i$, the action (1) leads to the self-dual Ashtekar canonical formalism for gravity in terms of complex $SU(2)$ connection [3]. For real η, this action allows a Hamiltonian formulation [1, 4] in terms of real $SU(2)$ connection which coincides with that of Barbero [5] for $\eta = 1$.

In the first order formalism, equations of motion are obtained by varying the Hilbert-Palatini-Holst action (1) with respect to the connection $\omega_\mu{}^{ab}$ and tetrad e^a_μ fields independently. Variation with respect to $\omega_\mu{}^{ab}$ leads to the standard no-torsion equation: $D_{[\mu}(\omega)\ e^a_{\nu]} = 0$, which can be solved for the connection in terms of tetrad fields in the usual way: $\omega = \omega(e)$ where the standard spin connection is:

$$\omega_\mu{}^{ab}(e) = \frac{1}{2}\left[e^{\nu a}\partial_{[\mu}e^b_{\nu]} - e^{\nu b}\partial_{[\mu}e^a_{\nu]} - e^{\rho a}e^{\sigma b}\partial_{[\rho}e^c_{\sigma]}e^c_\mu\right] \tag{2}$$

Variation of action (1) with respect to the tetrad e^a_μ leads to the usual Einstein equation: $R_a{}^\mu - \frac{1}{2}\ e^\mu_a\ R = 0$. Thus, adding the Holst action to Hilbert-Palatini action as in Eqn.(1) does not change the equations of motion of the theory. Notice that for $\omega = \omega(e)$, the Holst term in the Lagrangian density is identically zero: $e\ \Sigma^{\mu\nu}_{ab}\ \tilde{R}_{\mu\nu}{}^{ab}(\omega(e)) = = \frac{1}{2}\ \epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}(\omega(e)) = 0$, due to the cyclicity property $R_{[\mu\nu\alpha]\beta}(\omega(e)) = 0$.

While classical equations of motion do not depend on the Immirzi parameter, non-perturbative physical effects depending on this parameter are expected to appear in quantum gravity.

Inclusion of spin 1/2 fermions into the Holst's generalised Hilbert-Palatini action (1) has been done recently by Perez and Rovelli and also by Freidel, Minic and Takeuchi[6]. This has been achieved by minimal coupling of the fermion through a term $-(1/2)\ (\bar{\lambda}\ \gamma^\mu D_\mu(\omega)\lambda - \overline{D_\mu(\omega)\lambda}\ \gamma^\mu\lambda)$ into the action (1) without changing the Holst term. This indeed does change equations of motion leading to dependence on the Immirzi parameter even at classical level. However, as shown by Mercuri [7], it is possible to modify the Holst action in presence of Dirac fermions so that the classical equations motion stay independent of the Immirzi parameter. To do this, to the Einstein-Cartan action[2]:

$$S_{GF} = \frac{1}{2}\int d^4x\ e\ \left[\Sigma^{\mu\nu}_{ab}R_{\mu\nu}{}^{ab}(\omega) - \bar{\lambda}\gamma^\mu D_\mu(\omega)\lambda + \overline{D_\mu(\omega)\lambda}\ \gamma^\mu\lambda\right], \tag{3}$$

[2] In our conventions all the Dirac gamma matrices are hermitian $(\gamma^a)^\dagger = \gamma^a$, $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\delta^{ab}$ and $\gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$, $(\gamma_5)^2 = +1$ and $\sigma_{ab} = \frac{1}{2}\gamma_{[a}\gamma_{b]}$. For Majorana fermions $\bar{\psi} = \psi^T C$ where C is the charge conjugation matrix with properties $C^\dagger C = CC^\dagger = 1$, $C^T = -C$, $C\gamma_a C^{-1} = -\gamma_a^T$.

we add a modified Holst term introducing a non-minimal coupling for the fermion:

$$S_{HolstF} = \frac{i\eta}{2}\int d^4x\ e\left[\Sigma^{\mu\nu}_{ab}\tilde{R}_{\mu\nu}{}^{ab}(\omega) - \bar{\lambda}\gamma_5\gamma^\mu D_\mu(\omega)\lambda - \overline{D_\mu(\omega)\lambda}\ \gamma_5\gamma^\mu\lambda\right] \tag{4}$$

Variation of the total action $S_{GF}+S_{HolstF}$ with respect to the connection field $\omega_\mu{}^{ab}$ yields the standard torsion equation as an equation of motion:

$$D_{[\mu}(\omega)e^a_{\nu]} = 2\ T_{\mu\nu}{}^a(\lambda) \equiv \frac{1}{2e}e^{a\alpha}\ \epsilon_{\mu\nu\alpha\beta}\ \bar{\lambda}\gamma_5\gamma^\beta\lambda \tag{5}$$

This can be solved as:

$$\omega_{\mu ab} = \omega_{\mu ab}(e,\lambda) \equiv \omega_{\mu ab}(e) + \kappa_{\mu ab}(\lambda) \tag{6}$$

where $\omega(e)$ the spin connection of pure gravity (2) and the contorsion tensor is given by (general relation between torsion and contorsion is $2\ T_{\mu\nu}{}^\lambda = -\kappa_{[\mu\nu]}{}^\lambda$):

$$\kappa_{\mu ab}(\lambda) = -\frac{1}{4}\ e^c_\mu\ \epsilon_{abcd}\ \bar{\lambda}\gamma_5\gamma^d\lambda \tag{7}$$

It is straight forward to check that the fermionic Holst Lagrangian density (4) above is a total derivative for connection $\omega(e,\lambda) = \omega(e)+\kappa(\lambda)$ given by (6) and (7). Mercuri has made an interesting observation[7] that the modified Holst action $S_{HolstF}[\omega(e,\lambda)]$ can be cast in a form involving the Nieh-Yan invariant density and divergence of an axial current density in the following manner:

$$S_{HolstF}\left[\omega(e,\lambda)\right] = -\frac{i\eta}{2}\int d^4x\left[I_{NY} + \partial_\mu J^\mu(\lambda)\right] \tag{8}$$

where $J_\mu(\lambda) = e\ \bar{\lambda}\gamma_5\gamma_\mu\lambda$ and the Nieh-Yan invariant density in general is [8]:

$$I_{NY} = \epsilon^{\mu\nu\alpha\beta}\left[\ T_{\mu\nu}{}^a\ T_{\alpha\beta a} - \frac{1}{2}\ \Sigma^{ab}_{\mu\nu}\ R_{\alpha\beta ab}(\omega)\ \right] \tag{9}$$

For the present case, notice that $\epsilon^{\mu\nu\alpha\beta}T_{\mu\nu}{}^a(\lambda)T_{\alpha\beta a}(\lambda)$ is identically zero for the explicit torsion expression of Eqn.(5) and hence Nieh-Yan invariant density is simply $-\ (1/2)\ \epsilon^{\mu\nu\alpha\beta}\ \Sigma^{ab}_{\mu\nu}\ R_{\alpha\beta ab}(\omega(e,\lambda))$. In general the Nieh-Yan topological invariant density is just the divergence of pseudo-trace axial vector constructed from torsion:

$$I_{NY} = \epsilon^{\mu\nu\alpha\beta}\ \ \partial_\mu T_{\nu\alpha\beta} \tag{10}$$

This allows us to see that the modified Holst Lagrangian density is indeed a total derivative when the connection equation of motion (6 and 7) is used:

$$S_{HolstF}\left[\omega(e,\lambda)\right] = \frac{i\eta}{4}\int d^4x\ \partial_\mu J^\mu(\lambda) = -\frac{i\eta}{6}\int d^4x\ \epsilon^{\mu\nu\alpha\beta}\partial_\mu T_{\nu\alpha\beta}(\lambda)$$

where we have used the fact that $2\ \epsilon^{\mu\nu\alpha\beta}\ T_{\nu\alpha\beta}(\lambda) = -3\ J^{\mu}(\lambda)$.

Next variations of the total action $S_{GF} + S_{HolstF}$ with respect to tetrad field e^a_μ and fermion λ lead to same equations of motion as those obtained from the variations of gravity-fermion action S_{GF} alone, making these classical equations of motion independent of Immirzi parameter.

Coupling of higher spin fermions to gravity also requires a special consideration in presence of Holst term. For example, we could consider the supergravity theories which contain spin 3/2 fermions. If we add the original Holst term of Eqn.(1) without any modifications to the standard actions of these theories in the manner done by Perez and Rovelli [6] for spin 1/2 fermions, the equations of motion obtained from the resulting actions will indeed develop dependence on Immirzi parameter indicating violation of supersymmetry. It is worthwhile to ask if there are any possible modifications of the Holst term which preserve the original supergravity equations of motion. In the following we shall discuss such modifications of Holst action which when added to the standard $N = 1, 2, 4$ supergravity actions will leave supergravity equations of motion unchanged and thereby preserve supersymmetry. In addition we shall also see that in each of these cases, for the connection satisfying connection equation of motion the modified Holst action can be written in an analogous form as written by Mercuri for spin 1/2 fermions (8).

2 N = 1 supergravity with Holst action

The simplest supersymmetric generalisation of Einstein gravity is $N = 1$ supergravity [9] which is described by a spin $\frac{3}{2}$ Majorana spinor, the gravitino ψ_μ, and the tetrad field e^a_μ. Generalised supergravity action containing the modified Holst term for this theory is given by:

$$S_1 \ = \ S_{SG1} \ + \ S_{SHolst1} \tag{11}$$

where the supergravity action is:

$$S_{SG1} \ = \ \frac{1}{2}\int d^4x \ \left[\ e \ \Sigma^{\mu\nu}_{ab} \ R_{\mu\nu}{}^{ab}(\omega) \ - \ \epsilon^{\mu\nu\alpha\beta} \ \bar{\psi}_\mu\gamma_5\gamma_\nu D_\alpha(\omega)\psi_\beta\right] \tag{12}$$

and supersymmetric Holst action as introduced by Tsuda [10] is:

$$S_{SHolst1} \ = \ \frac{i\eta}{2}\int d^4x \ \left[\ e \ \Sigma^{\mu\nu}_{ab} \ \tilde{R}_{\mu\nu}{}^{ab}(\omega) \ - \ \epsilon^{\mu\nu\alpha\beta} \ \bar{\psi}_\mu\gamma_\nu D_\alpha(\omega)\psi_\beta\right] \tag{13}$$

Again for $\eta \ = \ -i$, action (11) is the $N = 1$ supersymmetric generalisation of the Ashtekar chiral action.

Variation of action S_1 with respect to connection $\omega_\mu{}^{ab}$ leads to the standard torsion equation of $N = 1$ supergravity:

$$D_{[\mu}(\omega)\ e^a_{\nu]} \ = \ 2\ T_{\mu\nu}{}^a(\psi) \ \equiv \ \frac{1}{2}\ \bar{\psi}_\mu \gamma^a \psi_\nu \tag{14}$$

which in turn is solved by

$$\omega_\mu{}^{ab} \ = \ \omega_\mu{}^{ab}(e, \psi) \equiv \omega_\mu{}^{ab}(e) \ + \ \kappa_\mu{}^{ab}(\psi) \tag{15}$$

where $\omega(e)$ is the pure gravity spin-connection given by (2) and the contorsion tensor is

$$\kappa_{\mu\alpha\beta}(\psi) \ = \ \frac{1}{4}\ \left[\ \bar{\psi}_\alpha \gamma_\mu \psi_\beta \ + \ \bar{\psi}_\mu \gamma_\alpha \psi_\beta \ - \ \bar{\psi}_\mu \gamma_\beta \psi_\alpha \ \right] \tag{16}$$

Next, supersymmetric Holst Lagrangian density(13) is a total derivative for $\omega = \omega(e, \psi)$. It can also be cast in the form as in (8) involving Nieh-Yan topological invariant density and divergence of an axial current density as:

$$S_{SHolst1}\left[\omega(e, \psi)\right] \ = \ - \ \frac{i\eta}{2} \int d^4x \left[I_{NY} \ + \ \partial_\mu J^\mu(\psi)\right] \tag{17}$$

where now we have the gravitino axial vector current density $J^\mu(\psi) = \frac{1}{2}\ \epsilon^{\mu\nu\alpha\beta} \bar{\psi}_\nu \gamma_\alpha \psi_\beta$. Here also, Fierz rearrangement implies $\epsilon^{\mu\nu\alpha\beta} T_{\mu\nu a}(\psi) T_{\alpha\beta}{}^a(\psi) = 0$ for the torsion given by (14) and hence the Nieh-Yan density is simply $-\ (1/2)\ \epsilon^{\mu\nu\alpha\beta}\ \Sigma^{ab}_{\alpha\beta}\ R_{\mu\nu ab}(\omega(e, \psi))$. Using the general property of the Nieh-Yan topological invariant density given in Eqn.(10), it follows that the modified Holst Lagrangian density for the connection $\omega(e, \psi)$ is a total derivative:

$$S_{SHolst1}[\omega(e, \psi)] = \ - \ \frac{i\eta}{4} \int d^4x\ \partial_\mu J^\mu(\psi) = \frac{i\eta}{2} \int d^4x\ \epsilon^{\mu\nu\alpha\beta} \partial_\mu T_{\nu\alpha\beta}(\psi)$$

This is to be contrasted with the pure gravity case above where Holst Lagrangian density is exactly zero for $\omega = \omega(e)$.

When substitution $\omega = \omega(e, \psi)$ is made into the variation of super-Holst action (13) with respect to gravitino ψ_μ and tetrad e^a_μ fields, we obtain integrals over total derivatives and hence these do not contribute to the equations of motion which come entirely from the variations of supergravity action S_{SG1} (12). Thus addition of super-Holst action (13) to supergravity action (12) does not change the standard equations of motion of $N = 1$ supergravity.

3 N = 2 super-Holst action

Next level supersymmetric generalisation of Einstein gravity is the $N = 2$ supergravity [11]. Besides the tetrad fields e^a_μ and their two super-partner gravitinos whose chiral projections are ψ^I_μ and $\psi_{I\mu}$, $I = 1, 2$ $(\gamma_5\psi^I_\mu = +\psi^I_\mu$ and $\gamma_5\psi_{I\mu} = -\psi_{I\mu})$, this theory also contains an Abelian gauge field A_μ. The action for this theory is given by [11]:

$$\begin{aligned} S_{SG2} \;&=\; \int d^4x\; e\; [\; \frac{1}{2}\; \Sigma^{\mu\nu}_{ab}\; R_{\mu\nu}{}^{ab}(\omega) \;-\; \frac{1}{4}\; F_{\mu\nu}\; F^{\mu\nu} \\ &\qquad -\; \frac{1}{2e}\; \epsilon^{\mu\nu\alpha\beta} \left(\bar{\psi}^I_\mu \gamma_\nu D_\alpha(\omega)\psi_{I\beta} - \bar{\psi}_{I\mu}\gamma_\nu D_\alpha(\omega)\psi^I_\beta\right) \\ &\qquad +\; \frac{1}{2\sqrt{2}}\; \bar{\psi}^I_\mu \psi^J_\nu \epsilon_{IJ}(F^{+\mu\nu} + \hat{F}^{+\mu\nu}) \\ &\qquad +\; \frac{1}{2\sqrt{2}}\; \bar{\psi}_{I\mu}\psi_{J\nu}\epsilon^{IJ}(F^{-\mu\nu} + \hat{F}^{-\mu\nu})\Big] \end{aligned} \tag{18}$$

where super-covariant field strength is

$$\hat{F}_{\mu\nu} \;=\; \partial_{[\mu}A_{\nu]} \;-\; \frac{1}{\sqrt{2}}\; \left(\bar{\psi}^I_\mu\psi^J_\nu\epsilon_{IJ} \;+\; \bar{\psi}_{I\mu}\psi_{J\nu}\epsilon^{IJ}\right)$$

and self-(antiself-)dual field strengths are $F^{\pm}_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} \pm {}^*F_{\mu\nu})$ and star dual * is given by $^*F_{\mu\nu} = \frac{1}{2e}\; \epsilon_{\mu\nu\alpha\beta}\; F^{\alpha\beta}$.

We generalise the $N = 2$ supergravity action (18) by adding a modified Holst term to obtain the new action as:

$$S_2 \;=\; S_{SG2} \;+\; S_{SHolst2} \tag{19}$$

where the super-Holst action is

$$\begin{aligned} S_{SHolst2} \;=\; i\eta \int d^4x\; e\; &\left[\frac{1}{2}\; \Sigma^{\mu\nu}_{ab}\; \tilde{R}_{\mu\nu}{}^{ab}(\omega) \;-\; \frac{1}{4e}\; \epsilon^{\mu\nu\alpha\beta}\bar{\psi}^I_\mu\psi^J_\nu\; \bar{\psi}_{I\alpha}\psi_{J\beta}\right. \\ &\left. -\; \frac{1}{2e}\; e^{\mu\nu\alpha\beta}\left(\bar{\psi}^I_\mu\gamma_\nu D_\alpha(\omega)\psi_{I\beta} \;+\; \bar{\psi}_{I\mu}\gamma_\nu D_\alpha(\omega)\psi^I_\beta\right)\right] \end{aligned} \tag{20}$$

Notice that this $N = 2$ super-Holst action has an additional four-gravitino term as compared to similar $N = 1$ super-Holst action (13). This term plays an important role as shall be seen in what follows. Also, in this modified Holst action, there are only fields that couple to the connection field ω in the original supergravity action; no terms involving the gauge field A_μ are included. This modified Holst action as it is does have the desired property of leaving the original supergravity equations unaltered. To see this, we vary the generalised total action S_2 (19) with respect to the connection $\omega_\mu{}^{ab}$ to obtain:

$$-\frac{1}{2}\int d^4x\; \epsilon^{\mu\nu\alpha\beta}\left[D_\mu(\omega)\Sigma^{ab}_{\alpha\beta} + e^a_\mu\bar{\psi}^I_\alpha\gamma^b\psi_{I\beta}\right]\left(\frac{1}{2}\epsilon_{abcd} + i\eta\delta_{ac}\delta_{bd}\right)\delta\omega_\nu{}^{cd} \;=\; 0$$

which implies:

$$\epsilon^{\mu\nu\alpha\beta}\ D_\mu(\omega)\Sigma^{ab}_{\alpha\beta} \ = \ - \ \frac{1}{2}\ \epsilon^{\mu\nu\alpha\beta} e^{[a}_\mu \bar{\psi}^I_\alpha \gamma^{b]} \psi_{I\beta}$$

which in turn leads to the standard torsion equation of $N = 2$ supergravity:

$$D_{[\mu}(\omega)e^a_{\nu]} \ = \ 2\ T_{\mu\nu}{}^a(\psi) \equiv \frac{1}{2}\ \left(\bar{\psi}^I_\mu \gamma^a \psi_{I\nu} \ + \ \bar{\psi}_{I\mu}\gamma^a\psi^I_\nu\right)$$

whose solution is given by:

$$\omega_\mu{}^{ab} \ = \ \omega_\mu{}^{ab}(e,\psi) \ \equiv \ \omega_\mu{}^{ab}(e) \ + \ \kappa_\mu{}^{ab}(\psi) \tag{21}$$

Here $\omega_\mu{}^{ab}(e)$ is the usual torsion-free spin-connection (2) and contorsion tensor of $N = 2$ supergravity is:

$$\kappa_{\mu\alpha\beta}(\psi) \ = \ \frac{1}{4}\left[\bar{\psi}^I_\alpha\gamma_\mu\psi_{I\beta} \ + \ \bar{\psi}^I_\mu\gamma_\alpha\psi_{I\beta} \ - \ \bar{\psi}^I_\mu\gamma_\beta\psi_{I\alpha} \ + \ c.c.\right] \tag{22}$$

Thus, despite the additional super-Holst term $S_{SHolst2}$ in the total action S_2 above, the connection equations (21, 22) obtained are the standard $N = 2$ supergravity equations.

Next for this connection $\omega(e,\psi)$, the super-Holst Lagrangian density (20) is a total derivative. To see this, notice that:

$$\begin{aligned} -\ \frac{1}{2}\ \epsilon^{\mu\nu\alpha\beta}&\left[\bar{\psi}^I_\mu\gamma_\nu D_\alpha(\omega(e,\psi))\psi_{I\beta} + \bar{\psi}_{I\mu}\gamma_\nu D_\alpha(\omega(e,\psi))\psi^I_\beta + \frac{1}{2}\bar{\psi}^I_\mu\psi^J_\nu\bar{\psi}_{I\alpha}\psi_{J\beta}\right] \\ &= \ - \ \frac{1}{2}\left[\partial_\mu J^\mu(\psi) + \epsilon^{\mu\nu\alpha\beta}T_{\mu\nu a}T_{\alpha\beta}{}^a\right] \end{aligned} \tag{23}$$

where the axial current density $J^\mu(\psi) \ = \ \epsilon^{\mu\nu\alpha\beta}\bar{\psi}^I_\nu\gamma_\alpha\psi_{I\beta}$. To obtain this relation we have made use of $2\ T_{\mu\nu}{}^\lambda \ = \ - \ \kappa_{[\mu\nu]}{}^\lambda = \frac{1}{2}\ \bar{\psi}^I_{[\mu}\gamma^\lambda\psi_{\nu]I}$ and the identity:

$$-\ \frac{1}{2}\ \epsilon^{\mu\nu\alpha\beta}\ T_{\mu\nu a}(\psi)T_{\alpha\beta}{}^a(\psi) \ = \ \frac{1}{4}\ \epsilon^{\mu\nu\alpha\beta}\ \bar{\psi}^I_\mu\psi^J_\nu\bar{\psi}_{I\alpha}\psi_{J\beta}$$

which can be checked easily using the explicit expression for the torsion and a simple Fierz rearrangement. Clearly the four-gravitino term in the left hand side of Eqn.(23) which has its origin the the four-gravitino term in the super-Holst action (20) is important to obtain the desired form of this equation.

Here also for the connection $\omega(e,\psi)$ given by (21) and (22), the super-Holst Lagrangian density can be written in a special form in terms of the Nieh-Yan invariant density and divergence of an axial current density as:

$$S_{SHolst2}\left[\omega(e,\psi)\right] \ = \ - \ \frac{i\eta}{2}\int d^4x\ \left[\ I_{NY} \ + \ \partial_\mu J^\mu(\psi)\ \right] \tag{24}$$

Again using the general property of the Nieh-Yan invariant density relating it to a derivative of torsion (10), we find that super-Holst Lagrangian density is a total derivative for connection $\omega(e,\psi)$:

$$S_{SHolst2}\left[\omega(e,\psi)\right] = -\frac{i\eta}{4}\int d^4x\ \partial_\mu J^\mu(\psi) = \frac{i\eta}{2}\int d^4x\ \epsilon^{\mu\nu\alpha\beta}\ \partial_\mu T_{\nu\alpha\beta}(\psi) \quad (25)$$

where we have used the fact that $2\epsilon^{\mu\nu\alpha\beta}T_{\nu\alpha\beta}(\psi) = -J^\mu(\psi)$.

Not only the connection equation of $N=2$ supergravity is unchanged by adding the super-Holst action (20), other equations of motion are also not modified. For example, to check this explicitly, substituting $\omega = \omega(e,\psi) = \omega(e) + \kappa(\psi)$ into the variation of super-Holst Lagrangian density $\mathcal{L}_{SHolst2}$ (20) with respect to gravitino field ψ^I_μ leads to:

$$\begin{aligned}&\left[\delta\psi^I_\mu\ \frac{\delta\mathcal{L}_{SHolst2}}{\delta\psi^I_\mu}\right]_{\omega=\omega(e,\psi)} \\ &\quad = \ -\ \frac{i\eta}{2}\ \epsilon^{\mu\nu\alpha\beta}\ \left[\ \delta\bar{\psi}^I_\mu\gamma_\nu D_\alpha(\omega(e))\psi_{I\beta}\ +\ \bar{\psi}_{I\mu}\gamma_\nu D_\alpha(\omega(e))\delta\psi^I_\beta \right.\\ &\qquad\qquad \left. +\ \delta\bar{\psi}^I_\mu\gamma^b\psi_{I\beta}\ \kappa_{\alpha\nu b}\ +\ \delta\bar{\psi}^I_\mu\psi^J_\nu\ \bar{\psi}_{I\alpha}\psi_{J\beta}\ \right]\end{aligned}$$

where the last two terms can be checked to cancel against each other by using the explicit expression for the $N=2$ contorsion tensor (22) and a Fierz rearrangement. Again we notice that the presence of the four-gravitino term in the $N=2$ super-Holst action (20) is important for this cancellation to happen. Now the first two terms in the right hand side of above equation combine into a total derivative:

$$\left[\delta\psi^I_\mu\ \frac{\delta\mathcal{L}_{SHolst2}}{\delta\psi^I_\mu}\right]_{\omega=\omega(e,\psi)} = -\ \frac{i\eta}{2}\ \epsilon^{\mu\nu\alpha\beta}\ \partial_\mu\left(\delta\bar{\psi}^I_\nu\gamma_\alpha\psi_{I\beta}\right)$$

Hence this variation does not contribute to the gravitino equation of motion; only contributions to the variation of total action S_2 of Eqn.(19) come from the supergravity action S_{SG2} (18) yielding the standard supergravity equations.

Similar conclusion holds for the other equation of motion obtained by varying the tetrad field e^a_μ. This can be seen explicitly from

$$\left[\delta e^a_\mu\ \frac{\delta}{\delta e^a_\mu}\ \left(e\ \Sigma^{\mu\nu}_{ab}\ \tilde{R}^{ab}_{\mu\nu}(\omega)\right)\right]_{\omega=\omega(e,\psi)} = 2\ \epsilon^{\mu\nu\alpha\beta}\left[\nabla_\mu\kappa_{\alpha\beta\lambda} + \kappa_{\mu\beta}{}^\sigma\ \kappa_{\alpha\sigma\lambda}\right]\ e^\lambda_b\delta e^b_\nu$$

and

$$\begin{aligned}&-\ \left[\delta e^a_\mu\ \frac{\delta}{\delta e^a_\mu}\ \left(\epsilon^{\mu\nu\alpha\beta}\left(\bar{\psi}^I_\mu\gamma_\nu D_\alpha(\omega)\psi_{I\beta}\ +\ \bar{\psi}_{I\mu}\gamma_\nu D_\alpha(\omega)\psi^I_\beta\right)\right)\right]_{\omega=\omega(e,\psi)} \\ &\qquad = \epsilon^{\mu\nu\alpha\beta}\ \left[\ \nabla_\mu\left(\bar{\psi}^I_\alpha\gamma_\lambda\psi_{I\beta}\right)\ -\ \bar{\psi}^I_\mu\gamma^\sigma\psi_{I\beta}\ \kappa_{\alpha\lambda\sigma}\right]\ e^\lambda_b\delta e^b_\nu\end{aligned}$$

From the expression for contorsion tensor (22) notice that $\epsilon^{\mu\nu\alpha\beta}\left(\bar{\psi}^I_\alpha \gamma_\lambda \psi_{I\beta}\right)$ $= \ -2\ \epsilon^{\mu\nu\alpha\beta}\ \kappa_{\alpha\beta\lambda}$ and $e^{\mu\nu\alpha\beta}\bar{\psi}^I_\mu \gamma^\sigma \psi_{I\beta}\ \kappa_{\alpha\lambda\sigma} \ = \ 2\ \epsilon^{\mu\nu\alpha\beta} \kappa_{\mu\beta}{}^{\sigma}\ \kappa_{\alpha\sigma\lambda}$, so that adding above two equations yields:

$$\left[\delta e^a_\mu\ \frac{\delta \mathcal{L}_{SHolst2}}{\delta e^a_\mu}\right]_{\omega=\omega(e,\psi)} \ = \ 0$$

Again the δe^a_μ variation of total action S_2 obtains contributions only from the supergravity action (18) leading to the standard supergravity equation of motion. Also, since the super-Holst action $S_{SHolst2}$ (20) does not depend on the gauge field, the last equation of motion obtained by varying A_μ comes from the supergravity action S_{SG2} (18).

4 N = 4 supergravity

Now we shall consider the generalisation of Holst action to the case of $N = 4$ supergravity [12]. This theory, in its $SU(4)$ version, describes four spin 3/2 Majorana gravitinos whose chiral projections ψ^I_μ and $\psi_{I\mu}$ $(I = 1, 2, 3, 4)$ with $\gamma_5 \psi^I_\mu = +\psi^I_\mu$ and $\gamma_5 \psi_{I\mu} = -\psi_{I\mu}$ transform as 4 and $\bar{4}$ representations of $SU(4)$ and four Majorana spin 1/2 fermions whose chiral projections Λ^I and Λ_I with $\gamma_5 \Lambda^I = -\Lambda^I$ and $\gamma_5 \Lambda_I = +\Lambda_I$ also transform as 4 and $\bar{4}$ respectively. Bosonic fields of the theory include the tetrad fields e^a_μ and six complex vector fields $A_{\mu IJ}$ (antisymmetric in IJ) and their $SU(4)$ dual $\bar{A}_\mu{}^{IJ} \ = \ \left(A_{\mu IJ}\right)^* \ = \ \frac{1}{2}\ \epsilon^{IJKL}\ A_{\mu KL}$. In addition, there are scalar fields that parametrise the coset manifold $SU(1,\ 1)/U(1)$. These are represented as a doublet of $SU(1,\ 1)$ complex scalar fields $\phi_A = (\phi_1,\ \phi_2)$ and their $SU(1,\ 1)$ dual $\phi^A = \eta^{AB}\ \phi^*_B = (\phi^*_1,\ -\phi^*_2)$ subject to the condition $\phi^A\ \phi_A$ $\equiv \phi^*_1\ \phi_1 - \phi^*_2\ \phi_2 = 1$. The equations of motion of this theory exhibit an $SU(1,\ 1)$ invariance, though its action does not. The action is given by [12]:

$$\begin{aligned} S_{SG4} \ \ = \ \ & \int d^4x\ e\left[\frac{1}{4}\ R(\omega,e) - \frac{1}{2e}\epsilon^{\mu\nu\alpha\beta}\bar{\psi}^I_\mu \gamma_\nu \mathcal{D}_\alpha(\omega)\psi_{I\beta} - \frac{1}{2}\bar{\Lambda}^I\gamma^\mu \mathcal{D}_\mu(\omega)\Lambda_I \right. \\ & \qquad - \ \frac{1}{2}c_\mu \bar{c}^\mu \ - \ \frac{1}{8}\left(\frac{\phi^1-\phi^2}{\Phi}\right) F^+_{IJ\mu\nu}\bar{F}^{+IJ\mu\nu} \\ & + \frac{1}{2\sqrt{2}\Phi}\bar{\psi}^I_\mu \psi^J_\nu\left(F^{+\mu\nu}_{IJ} + \hat{F}^{+\mu\nu}_{IJ}\right) - \frac{1}{2\Phi^*}\bar{\Lambda}^I\gamma_\mu \psi^J_\nu\left(F^{-\mu\nu}_{IJ} + \hat{F}^{-\mu\nu}_{IJ}\right) \\ & \left. - \frac{1}{\sqrt{2}}\ \bar{\Lambda}^I\gamma^\mu\gamma^\nu\left(c_\nu + \frac{1}{2\sqrt{2}}\bar{\psi}^J_\nu \Lambda_J\right)\psi_{I\mu} \ + \ c.\ c.\right] \end{aligned} \tag{26}$$

where $\Phi \equiv (\phi^1 + \phi^2)$ and $\Phi^* \equiv (\phi_1 - \phi_2)$ and covariant derivatives $\mathcal{D}$ are:

$$\mathcal{D}_\mu(\omega)\Lambda_I = \left(D_\mu(\omega) + (3i/2)a_\mu\right)\Lambda_I, \qquad \mathcal{D}_\mu(\omega)\Lambda^I = \left(D_\mu(\omega) - (3i/2)a_\mu\right)\Lambda^I$$
$$\mathcal{D}_\alpha(\omega)\psi^I_\beta = \left(D_\alpha(\omega) + (i/2)a_\alpha\right)\psi^I_\beta, \qquad \mathcal{D}_\alpha(\omega)\psi_{I\beta} = \left(D_\alpha(\omega) - (i/2)a_\alpha\right)\psi_{I\beta}$$

and the $SU(1,1)$ invariant vectors a_μ, c_μ and $\bar{c}_\mu$ are:

$$a_\mu \;=\; i\phi_A \partial_\mu \phi^A, \qquad c_\mu \;=\; \epsilon_{AB}\; \phi^A \partial_\mu \phi^B, \qquad \bar{c}_\mu \;=\; \epsilon^{AB}\; \phi_A \partial_\mu \phi_B$$

The field strengths $F_{\mu\nu IJ} \;=\; \partial_{[\mu} A_{\nu] IJ}$ and $\bar{F}^{IJ}_{\mu\nu} \;=\; \partial_{[\mu} A^{IJ}_{\nu]}$ are supercovariantized as:

$$\begin{aligned}
\hat{F}^{\mu\nu}_{IJ} \;&=\; F^{\mu\nu}_{IJ} \;-\; \frac{1}{2\sqrt{2}}\; \Phi \left(\bar{\psi}^{[\mu}_{[I} \psi^{\nu]}_{J]} + \sqrt{2}\; \epsilon_{IJKL} \bar{\psi}^{K[\mu} \gamma^{\nu]} \Lambda^L \right) \\
&\quad -\; \frac{1}{2\sqrt{2}}\; \Phi^* \left(\epsilon_{IJKL} \bar{\psi}^{K[\mu} \psi^{\nu]L} + \sqrt{2}\; \bar{\psi}^{[\mu}_{[I} \gamma^{\nu]} \Lambda_{J]} \right) \\
\hat{\bar{F}}^{IJ}_{\mu\nu} \;&=\; \bar{F}^{IJ}_{\mu\nu} \;-\; \frac{1}{2\sqrt{2}}\; \Phi^* \left(\bar{\psi}^{[I}_{[\mu} \psi^{J]}_{\nu]} + \sqrt{2}\; \epsilon^{IJKL}\; \bar{\psi}_{K[\mu} \gamma_{\nu]} \Lambda_L \right) \\
&\quad -\; \frac{1}{2\sqrt{2}}\; \Phi \left(\epsilon^{IJKL}\; \bar{\psi}_{K[\mu} \psi_{\nu]L} + \sqrt{2}\; \bar{\psi}^{[I}_{[\mu} \gamma_{\nu]} \Lambda^{J]} \right)
\end{aligned}$$

To the $N=4$ supergravity action (26) we add a appropriately modified Holst term:

$$S_4 \;=\; S_{SG4} \;+\; S_{SHolst4} \tag{27}$$

where the $N=4$ super-Holst action is given by:

$$\begin{aligned}
S_{SHolst4} \;=\; i\eta \int d^4x\; e\; \Bigg[&\frac{1}{2}\; \Sigma^{\mu\nu}_{ab}\; \tilde{R}_{\mu\nu}{}^{ab}(\omega) \\
&- \frac{1}{2e}\; \epsilon^{\mu\nu\alpha\beta}\; \left(\bar{\psi}^I_\mu \gamma_\nu \mathcal{D}_\alpha(\omega) \psi_{I\beta} + \bar{\psi}_{I\mu} \gamma_\nu \mathcal{D}_\alpha(\omega) \psi^I_\beta \right) \\
&- \frac{1}{2}\; \left(\bar{\Lambda}_I \gamma^\mu \mathcal{D}_\mu(\omega) \Lambda^I - \bar{\Lambda}^I \gamma^\mu \mathcal{D}_\mu(\omega) \Lambda_I \right) \\
&- \frac{1}{4e}\; \epsilon^{\mu\nu\alpha\beta}\; \bar{\psi}^I_\mu \psi^J_\nu\; \bar{\psi}_{I\alpha} \psi_{J\beta} \\
&- \frac{1}{4e}\; \epsilon^{\mu\nu\alpha\beta}\; \bar{\Lambda}^I \gamma_\mu \psi^J_\nu\; \bar{\Lambda}_I \gamma_\alpha \psi_{J\beta} \Bigg]
\end{aligned} \tag{28}$$

Here only those fields which are coupled to connection ω in the supergravity action are involved and not others like the gauge fields $A_{\mu IJ}$, $\bar{A}_\mu^{\;IJ}$ and scalar fields ϕ^A which do not have any coupling to ω. Also in addition to the four-gravitino term, which is also present in the super-Holst action for $N=2$ supergravity, we have an additional four-fermion term involving two gravitinos and two Λ's. Both these terms are important to achieve the desired result that equations of motion of $N=4$ supergravity theory are not modified in presence of this super-Holst term.

Variation of total action S_4 (27) with respect to the connection $\omega_\mu^{\;ab}$ leads to:

$$\begin{aligned}
\int d^4x \Bigg[\; \epsilon^{\mu\nu\alpha\beta}\; &\left(D_\beta(\omega) \Sigma^{cd}_{\mu\nu} - \frac{1}{2}\; \bar{\psi}^I_\mu e^{[c}_\nu \gamma^{d]} \psi_{I\beta} \right) \\
&- e\; \bar{\Lambda}_I e^{\alpha[c} \gamma^{d]} \Lambda^I \Bigg] \left(\frac{1}{2} \epsilon_{abcd} + i\eta \delta_{ac} \delta_{bd} \right) \delta\omega_\alpha^{\;ab} \;=\; 0
\end{aligned}$$

This implies the standard torsion equation of $N = 4$ supergravity:

$$D_{[\mu}(\omega)e^a_{\nu]} = 2T_{\mu\nu}{}^a = 2[T_{\mu\nu}{}^a(\psi)+T_{\mu\nu}{}^a(\Lambda)] \equiv \frac{1}{2}\bar{\psi}^I_{[\mu}\gamma^a\psi_{\nu]I}+\frac{1}{2e}e^{a\alpha}\epsilon_{\mu\nu\alpha\beta}\bar{\Lambda}_I\gamma^\beta\Lambda^I \tag{29}$$

which is solved by

$$\omega_{\mu ab} \;=\; \omega_{\mu ab}(e,\psi,\Lambda) \;\equiv\; \omega_{\mu ab}(e) \;+\; \kappa_{\mu ab} \tag{30}$$

where $\omega_{\mu ab}(e)$ is the standard pure gravitational spin-connection given by (2) and $N = 4$ contorsion tensor κ has contributions from both the gravitinos ψ and fermions Λ:

$$\begin{aligned}
\kappa_{\mu\alpha\beta} \;&=\; \kappa_{\mu\alpha\beta}(\psi) \;+\; \kappa_{\mu\alpha\beta}(\Lambda)\\
\kappa_{\mu\alpha\beta}(\psi) \;&=\; \frac{1}{4}\left[\bar{\psi}^I_\alpha\gamma_\mu\psi_{I\beta} \;+\; \bar{\psi}^I_\mu\gamma_\alpha\psi_{I\beta} \;-\; \bar{\psi}^I_\mu\gamma_\beta\psi_{I\alpha} \;+\; c.c.\right]\\
\kappa_{\mu\alpha\beta}(\Lambda) \;&=\; -\;\frac{1}{4e}\,\epsilon_{\mu\alpha\beta\sigma}\,\bar{\Lambda}_I\gamma^\sigma\Lambda^I
\end{aligned} \tag{31}$$

Like in the earlier cases of $N = 1$ and $N = 2$ supergravity, for the connection $\omega = \omega(e,\psi,\Lambda) = \omega(e) + \kappa(\psi,\Lambda)$ super-Holst Lagrangian density $\mathcal{L}_{SHolst4}$ (28) is a total derivative. To demonstrate that this is so, notice that:

$$\begin{aligned}
&-\frac{1}{2}\Big[\;\epsilon^{\mu\nu\alpha\beta}\;\left(\bar{\psi}^I_\mu\gamma_\nu\mathcal{D}_\alpha(\omega)\psi_{I\beta} + \bar{\psi}_{I\mu}\gamma_\nu\mathcal{D}_\alpha(\omega)\psi^I_\beta\right)\\
&\qquad +\; e\;\left(\bar{\Lambda}_I\gamma^\mu\mathcal{D}_\mu(\omega)\Lambda^I - \bar{\Lambda}^I\gamma^\mu\mathcal{D}_\mu(\omega)\Lambda_I\right)\Big]_{\omega=\omega(e,\psi,\Lambda)}\\
&\qquad -\;\frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\left[\bar{\psi}^I_\mu\psi^J_\nu\;\bar{\psi}_{I\alpha}\psi_{J\beta} + \bar{\Lambda}^I\gamma_\mu\psi^J_\nu\;\bar{\Lambda}_I\gamma_\alpha\psi_{J\beta}\right]\\
&= -\;\frac{1}{2}\;\left[\partial_\mu\left(J^\mu(\psi) + J^\mu(\Lambda)\right)\;+\;\epsilon^{\mu\nu\alpha\beta}T_{\mu\nu a}T_{\alpha\beta}{}^a\right]
\end{aligned} \tag{32}$$

where $J^\mu(\psi) = \epsilon^{\mu\nu\alpha\beta}\;\bar{\psi}^I_\nu\gamma_\alpha\psi_{I\beta}$ and $J^\mu(\Lambda) = e\bar{\Lambda}_I\gamma^\mu\Lambda^I$. Here we have used $2T_{\mu\nu}{}^\lambda(\psi) = -\kappa_{[\mu\nu]}{}^\lambda(\psi)$, $T_{\mu\nu\alpha}(\Lambda) = -\kappa_{\mu\nu\alpha}(\Lambda)$ and identities $e\bar{\Lambda}_I\gamma^\alpha\Lambda^I\;\kappa_\mu{}^\mu{}_\alpha(\psi)$ $= 2\;\epsilon^{\mu\nu\alpha\beta}T_{\mu\nu}{}^a(\psi)T_{\alpha\beta a}(\Lambda)$, $\epsilon^{\mu\nu\alpha\beta}\;T_{\mu\nu a}(\Lambda)T_{\alpha\beta}{}^a(\Lambda) = 0$ and the following relation obtained by Fierz rearrangements:

$$\begin{aligned}
-\frac{1}{2}\;\epsilon^{\mu\nu\alpha\beta}\;T_{\mu\nu a}T_{\alpha\beta}{}^a &= -\frac{1}{2}\;\epsilon^{\mu\nu\alpha\beta}\left[T_{\mu\nu a}(\psi)T_{\alpha\beta}{}^a(\psi) + 2\;T_{\mu\nu a}(\psi)T_{\alpha\beta}{}^a(\Lambda)\right]\\
&= \frac{1}{4}\;\epsilon^{\mu\nu\alpha\beta}\;\left[\bar{\psi}^I_\mu\psi^J_\nu\;\bar{\psi}_{I\alpha}\psi_{J\beta} + \bar{\Lambda}^I\gamma_\mu\psi^J_\nu\;\bar{\Lambda}_I\gamma_\alpha\psi_{J\beta}\right]
\end{aligned} \tag{33}$$

Notice that the two four-fermion terms of the super-Holst action (28) have played an important role in allowing us to write the equation (32). Now substituting this equation into the super-Holst action (28), we find that super-Holst action for $\omega = \omega(e,\psi,\Lambda)$ takes the same special form as in the earlier cases:

$$S_{SHolst4}\left[\omega(e,\psi,\Lambda)\right] \;=\; -\;\frac{i\eta}{2}\int d^4x\left[I_{NY} \;+\; \partial_\mu J^\mu(\psi,\Lambda)\right] \tag{34}$$

where $J_\mu(\psi,\Lambda) \equiv J_\mu(\psi) + J_\mu(\Lambda)$. It is important to note that this axial vector density $J_\mu(\psi,\Lambda)$ is not the conserved axial current of the $N = 4$ theory; in fact the conserved current density associated with the axial $U(1)$ invariance of the theory is $\mathcal{J}_\mu = J_\mu(\psi) + 3\ J_\mu(\Lambda)$.

Now for the Nieh-Yan invariant density we use

$$\begin{aligned} I_{NY} = \epsilon^{\mu\nu\alpha\beta}\partial_\mu T_{\nu\alpha\beta} &= \epsilon^{\mu\nu\alpha\beta}\partial_\mu[T_{\nu\alpha\beta}(\psi) + T_{\nu\alpha\beta}(\Lambda)] \\ &= -\frac{1}{2}\ \partial_\mu[J^\mu(\psi) + 3J^\mu(\Lambda)] \end{aligned}$$

where we have used the facts: $2\epsilon^{\mu\nu\alpha\beta}T_{\nu\alpha\beta}(\psi) = -J^\mu(\psi)$, $\quad 2\epsilon^{\mu\nu\alpha\beta}T_{\nu\alpha\beta}(\Lambda) = -3J^\mu(\Lambda)$. This thus leads us to:

$$\begin{aligned} S_{SHolst4}\left[\omega(e,\psi,\Lambda)\right] = &- \frac{i\eta}{4}\int d^4x\ \partial_\mu\left[J^\mu(\psi)\ -\ J^\mu(\Lambda)\right] \\ = &\ \frac{i\eta}{2}\int\ d^4x\ \epsilon^{\mu\nu\alpha\beta}\ \partial_\mu\left[T_{\nu\alpha\beta}(\psi) - \frac{1}{3}\ T_{\nu\alpha\beta}(\Lambda)\right] \end{aligned} \qquad (35)$$

Next to check explicitly that the other equations of motion are not changed in this case too, consider for example, the Λ_I-variation of the super-Holst Lagrangian density $\mathcal{L}_{SHolst4}$ from Eqn.(28):

$$\begin{aligned} \delta\Lambda_I\frac{\delta\mathcal{L}_{SHolst4}}{\delta\Lambda_I} \ = \ -\frac{i\eta}{2}\ e\ \Big[&\Big(\ \delta\bar{\Lambda}_I\gamma^\mu\mathcal{D}_\mu(\omega)\Lambda^I - \bar{\Lambda}^I\gamma^\mu\mathcal{D}_\mu(\omega)\delta\Lambda_I\Big) \\ &- \frac{1}{2}\ \left(\bar{\psi}^I_\mu\gamma^\mu\psi_{I\nu} + \bar{\psi}_{I\mu}\gamma^\mu\psi^I_\nu\right)\delta\bar{\Lambda}_J\gamma^\nu\Lambda^J\Big] \end{aligned}$$

where, in writing the second term on the right hand side, we have used the Fierz rearrangement

$$e\ \left(\bar{\psi}^I_\mu\gamma^\mu\psi_{I\nu} + \bar{\psi}_{I\mu}\gamma^\mu\psi^I_\nu\right)\ \bar{\Lambda}_J\gamma^\nu\Lambda^J \ = \ -\ \epsilon^{\mu\nu\alpha\beta}\ \bar{\Lambda}^I\gamma_\mu\psi^J_\nu\ \bar{\Lambda}_I\gamma_\alpha\psi_{J\beta}$$

Now substituting $\omega = \omega(e,\psi,\Lambda)$ from (30) we obtain:

$$\begin{aligned} \left[\delta\Lambda_I\frac{\delta\mathcal{L}_{SHolst4}}{\delta\Lambda_I}\right]_{\omega=\omega(e,\psi,\Lambda)} = -\frac{i\eta}{2}e\Big[&\ \delta\bar{\Lambda}_I\gamma^\mu\mathcal{D}_\mu(\omega(e))\Lambda^I\ - \bar{\Lambda}^I\gamma^\mu\mathcal{D}_\mu(\omega(e))\delta\Lambda_I \\ &+\ \delta\bar{\Lambda}_I\gamma^\nu\Lambda^I\left(\kappa_{\mu\ \nu}^{\ \mu} - \frac{1}{2}\left(\bar{\psi}^I_\mu\gamma^\mu\psi_{I\nu} + \bar{\psi}_{I\mu}\gamma^\mu\psi^I_\nu\right)\right)\Big] \end{aligned}$$

Using (31) for $N = 4$ contorsion tensor, last two terms cancel leaving the first two terms which combine into a total derivative:

$$\left[\delta\Lambda_I\frac{\delta\mathcal{L}_{SHolst4}}{\delta\Lambda_I}\right]_{\omega=\omega(e,\psi,\Lambda)} \ = \ -\ \frac{i\eta}{2}\ \partial_\mu\left(e\delta\bar{\Lambda}_I\gamma^\mu\Lambda^I\right)$$

Similarly, variation of super-Holst Lagrangian density (28) with respect to the gravitino ψ^I_μ and tetrad e^a_μ fields are:

$$\left[\delta\psi^I_\mu\ \frac{\delta\mathcal{L}_{SHolst4}}{\delta\psi^I_\mu}\right]_{\omega=\omega(e,\psi,\Lambda)} \ = \ -\ \frac{i\eta}{2}\ \epsilon^{\mu\nu\alpha\beta}\ \partial_\mu\left(\delta\bar{\psi}^I_\nu\gamma_\alpha\psi_{I\beta}\right)$$

$$\left[\delta e_\mu^a \ \frac{\delta \mathcal{L}_{SHolst4}}{\delta e_\mu^a}\right]_{\omega=\omega(e,\psi,\Lambda)} \ = \ 0$$

Thus clearly all the equations of motion obtained by varying the modified supergravity action S_4 (27) are the same as those obtained by varying the supergravity action S_{SG4} (26) alone; addition of the super-Holst action $S_{SHolst4}$ (28) does not change these classical equations of motion. These are indeed independent of the Immirzi parameter.

5 Concluding remarks

We have extended the Holst action for pure gravity with Immirzi parameter as its associated coupling constant to the case of supergravity theories. This has been done in a manner that the equations of motion of supergravity theories are not changed by such modifications of the original Holst action. This ensures that supersymmetry is preserved and Immirzi parameter does not play any role in the classical equations of motion. This is unlike the case studied by Perez and Rovelli and also by Freidel, Minic and Takeuchi [6] where a spin 1/2 fermion is minimally coupled to gravity in presence of original Holst action without any modification. In such a situation, the equations of motion do develop dependence on Immirzi parameter.

For each of $N = 1, 2, 4$ supergravity theories we find that the modified Holst Lagrangian density becomes a total derivative when we use the connection equation of motion $\omega = \omega(e, ...) = \omega(e) + \kappa(...)$ where ellipsis indicates the various fermions which introduce torsion in the theory. This total derivative takes a special form analogous to the one described by Mercuri for the case of spin 1/2 fermions (8). It is given in terms of Nieh-Yan invariant density and divergence of an axial fermion current density:

$$S_{Holst}[\omega = \omega(e, ...)] \ = \ - \ \frac{i\eta}{2} \int d^4x \ [\ I_{NY} \ + \ \partial_\mu J^\mu(...) \ ] \qquad (36)$$

The Nieh-Yan topological density is the divergence of pseudo-trace axial vector associated with torsion: $I_{NY} \ = \ \partial_\mu \ [\epsilon^{\mu\nu\alpha\beta} \ T_{\nu\alpha\beta}]$.

It is important to emphasise that the modified Holst action on its own does not have this special form (36) and reduces to this form only for the connection that satisfies the connection equation of motion.

For arbitrary real values of Immirzi parameter η^{-1}, the Holst action allows a canonical formulation of pure gravity [1, 4] in terms of a real Ashtekar-Barbero $SU(2)$ connection. For modified Holst action for the gravity coupled to a spin 1/2 fermion, this has also been done [7]. Extension of such a canonical formulation to $N = 1$ supergravity has been presented by Tsuda in [10]. In the same spirit, for the modified Holst actions (20) and (28) for $N = 2$ and $N = 4$ supergravity theories, a similar generalised Hamiltonian formulation can be developed. Care needs to taken in this analysis to fix the gauge after the proper constraint analysis is performed [13].

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