From Super Poincaré to Weighted Log-Sobolev and Entropy-Cost Inequalities *

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Abstract

We derive weighted log-Sobolev inequalities from a class of super Poincaré inequalities. As an application, the Talagrand inequality with larger distances are obtained. In particular, on a complete connected Riemannian manifold, we prove that the log^{δ}-Sobolev inequality with $\delta \in (1, 2)$ implies the $L^{2/(2-\delta)}$ -transportation cost inequality

 $W^{\rho}_{2/(2-\delta)}(f\mu,\mu)^{2/(2-\delta)} \le C\mu(f\log f), \ \ \mu(f) = 1, f \ge 0$

for some constant C > 0, and they are equivalent if the curvature of the corresponding generator is bounded below. Weighted log-Sobolev and entropy-cost inequalities are also derived for a large class of probability measures on \mathbb{R}^d .

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1 Introduction

Let (E, ρ) be a Polish space and μ a probability measure on E. For $p \ge 1$ we define the L^p -Wasserstein distance (or the L^p -transportation cost) by

$$W_p^{\rho}(\mu_1,\mu_2) := \left\{ \inf_{\pi \in \mathscr{C}(\mu_1,\mu_2)} \int_{E \times E} \rho(x,y)^p \pi(\mathrm{d}x,\mathrm{d}y) \right\}^{1/p}$$

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for probability measures μ_1, μ_2 on E, where $\mathscr{C}(\mu_1, \mu_2)$ is the class of probability measures on $E \times E$ with marginal distributions μ_1 and μ_2 .

According to [4, Corollary 4],

$$W_{p}^{\rho}(f\mu,\mu)^{2p} \leq C\mu(f\log f), \quad f \geq 0, \mu(f) = 1$$

holds for some C > 0 provided $\mu(e^{\lambda \rho(o,\cdot)^{2p}}) < \infty$ for some $\lambda > 0$, where $o \in E$ is a fixed point. See also [8] for p = 1. Furthermore, it is easy to derive from [14, Theorem 1.15] that for any $q \in [1, 2p)$, there exists C > 0 such that

(1.1)
$$W_q^{\rho}(f\mu,\mu)^{2p} \le C\mu(f\log f), \quad f \ge 0, \mu(f) = 1$$

if and only if $\mu(e^{\lambda\rho(o,\cdot)^{2p}}) < \infty$ for some $\lambda > 0$. In general, however, this concentration of μ does not imply (1.1) for q = 2p. Indeed, there exist a plentiful examples where $\mu(e^{\lambda\rho(o,\cdot)^2}) < \infty$ for some $\lambda > 0$ but there is no any constant C > 0 such that the Talagrand inequality

(1.2)
$$W_2^{\rho}(f\mu,\mu)^2 \le C\mu(f\log f), \quad f \ge 0, \mu(f) = 1$$

holds, see e.g. [1] for examples with $\mu(e^{\lambda\rho(o,\cdot)^2}) < \infty$ for some $\lambda > 0$ but the Poincaré inequality does not hold, which is weaker than (1.2) (see [17, Section 7] or [2, Section 4.1]).

Therefore, to derive (1.1) with q = 2p, one needs something stronger than the corresponding concentration of μ . In fact, it is now well known in the literature that, the Talagrand inequality follows from the log-Sobolev inequality for a class of local Dirichlet forms, see [21, 17, 2, 25, 20] and references within.

In this paper, we aim to derive (1.1) with q = 2p, i.e.

(1.3)
$$W_{2p}^{\rho}(f\mu,\mu)^{2p} \le C\mu(f\log f), \quad f \ge 0, \mu(f) = 1,$$

by using functional inequalities stronger than the log-Sobolev one.

To this end, in Section 2 we study the weighted log-Sobolev inequality

$$\mu(f^2 \log f^2) \le C\mu(\alpha \circ \rho(o, \cdot)\Gamma(f, f)), \quad \mu(f^2) = 1$$

for a positive function $\alpha(r) \to 0$ as $r \to \infty$ and a nice square field Γ . Combining this with known results on log-Sobolev and the Talagrand inequality, we derive (1.2) with the original distance ρ replaced by a larger one, which is induced by the weighted square field $\alpha \circ \rho(o, \cdot)\Gamma$. In particular, we have the following result on a Riemannian manifold.

Let M be a connected complete Riemannian manifold, and $\mu(dx) = e^{V(x)}dx$ a probability measure on M for some $V \in C(M)$. We shall use the following super Poincaré inequality (see [23])

(1.4)
$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > 0$$

to establish the corresponding weighted log-Sobolev inequality

(1.5)
$$\mu(f^2 \log f^2) \le C\mu(\alpha \circ \rho(o, \cdot) |\nabla f|^2), \quad \mu(f^2) = 1.$$

By [25, Theorem 1.1], (1.5) implies

(1.6)
$$W_2^{\rho_{\alpha}}(f\mu,\mu)^2 \le C\mu(f\log f), \quad f \ge 0, \mu(f^2) = 1,$$

where ρ_{α} is the Riemannian distance induced by the metric

(1.7)
$$\langle X, Y \rangle' := \frac{1}{\alpha \circ \rho(o, x)} \langle X, Y \rangle, \quad X, Y \in T_x M, \ x \in M.$$

The main result of the paper is the following.

Theorem 1.1. Assume that (1.4) holds for some positive decreasing $\beta \in C((0,\infty))$ such that

$$\eta(s) := \left(\log(2s)\right) \left(1 \wedge \beta^{-1}(s/2)\right), \quad s \ge 1$$

is bounded, where $\beta^{-1}(s) := \inf\{t \ge 0 : \beta(t) \le s\}$. Then (1.5) holds for some C > 0 and

$$\alpha(s) := \sup_{t \ge \mu(\rho(o, \cdot) \ge s-2)^{-1}} \eta(t), \quad s \ge 0.$$

Consequently, (1.6) holds.

The following consequences show that the above result is sharp in specific situations.

Corollary 1.2. Let $\delta \in (1, 2)$. (a) (1.4) with $\beta(r) = \exp[c(1 + r^{-1/\delta})]$ implies (1.5) with $\alpha(s) := (1 + \rho(o, \cdot))^{-2(\delta - 1)/(2 - \delta)}$

and (1.6) with $\rho_{\alpha}(x, y)$ replaced by

$$\rho(x,y)(1+\rho(o,x)\vee\rho(o,y))^{(\delta-1)/(2-\delta)}.$$

Consequently, it implies

(1.8)
$$W^{\rho}_{2/(2-\delta)}(f\mu,\mu)^{2/(2-\delta)} \le C\mu(f\log f), \quad \mu(f) = 1, f \ge 0$$

for some constant C > 0.

(b) If $V \in C^2(M)$ with $\operatorname{Ric} - \operatorname{Hess}_V$ bounded below, then the following are equivalent to each other:

- (1) (1.4) with $\beta(r) = \exp[c(1 + r^{-1/\delta})]$ for some constant c > 0;
- (2) (1.5) with $\alpha(s) := (1 + \rho(o, \cdot))^{-2(\delta 1)/(2 \delta)}$ for some C > 0;
- (3) (1.6) for some C > 0 and $\rho_{\alpha}(x, y)$ replaced by $\rho(x, y)(1 + \rho(o, x) \vee \rho(o, y))^{(\delta 1)/(2 \delta)}$;
- (4) (1.8) for some C > 0;
- (5) $\mu(\exp[\lambda\rho(o,\cdot)^{2/(2-\delta)}]) < \infty$ for some $\lambda > 0$.

We remark that (1.4) with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some c > 0 is equivalent to the following \log^{δ} -Sobolev inequality mentioned in the abstract (see [23, 24, 13, 26] for more general results on (1.4) and the *F*-Sobolev inequality)

$$\mu(f^2 \log^{\delta}(1+f^2)) \le C_1 \mu(|\nabla f|^2) + C_2, \quad \mu(f^2) = 1.$$

Since due to [24, Corollary 5.3] if (1.4) holds with $\beta(r) = \exp[c(1+r^{-1/\delta})]$ for some $\delta > 2$ then M has to be compact, as a complement to Corollary 1.2 we consider the critical case $\delta = 2$ in the next Corollary.

Corollary 1.3. (1.4) with $\beta(r) = \exp[c(1 + r^{-1/2})]$ for some c > 0 implies (1.5) with $\alpha(s) := e^{-c_1 s}$ for some $c_1 > 0$ and (1.6) with $\rho_{\alpha}(x, y)$ replaced by

$$\rho(x, y) e^{c_2[\rho(o, x) \lor \rho(o, y)]} > e^{c_3 \rho(x, y)} - 1$$

for some $c_2, c_3 > 0$. If Ric – Hess_V is bounded below, they are all equivalent to the concentration $\mu(\exp[e^{\lambda \rho(o, \cdot)}]) < \infty$ for some $\lambda > 0$.

Example 1.1. Let Ric be bounded below. Let $V \in C(M)$ be such that $V + a\rho(o, \cdot)^{\theta}$ is bounded for some a > 0 and $\theta \ge 2$. By [23, Corollaries 2.5 and 3.3], (1.4) holds for $\delta = 2(\theta - 1)/\theta$. Then Corollary 1.2 implies

$$W^{\rho}_{\theta}(f\mu,\mu)^{\theta} \le C\mu(f\log f), \quad f \ge 0, \mu(f) = 1$$

for some constant C > 0.

In this inequality θ could not be replaced by any larger number, since $W^{\rho}_{\theta} \ge W^{\rho}_{1}$ and by Proposition 3.1 below for any $p \ge 1$ the inequality

$$W_1^{\rho}(f\mu,\mu)^p \le C\mu(f\log f), \quad f \ge 0, \mu(f) = 1$$

implies $\mu(e^{\lambda \rho(o,\cdot)^p}) < \infty$ for some $\lambda > 0$, which fails when $p > \theta$ for μ specified above.

Example 1.2. In the situation of Example 1.1 but let $V + \exp[\sigma\rho(o, \cdot)]$ be bounded for some $\sigma > 0$. Then by [23, Corollaries 2.5 and 3.3], (1.4) holds with $\beta(r) = \exp[c(1+r^{-1/2})]$ for some c > 0. Hence, by Corollary 1.3,

(1.9)
$$\inf_{\pi \in \mathscr{C}(\mu, f\mu)} \int_{M \times M} \rho(x, y)^2 e^{c_1 \rho(x, y)} \pi(\mathrm{d}x, \mathrm{d}y) \le C\mu(f \log f), \quad f \ge 0, \mu(f) = 1$$

holds for some $c_1, C > 0$.

On the other hand, it is easy to see from Jensen's inequality that the left hand side is larger than

$$(\exp[c_2 W_1^{\rho}(\mu, f\mu)] - 1)^2$$

for some $c_2 > 0$. So, by Proposition 3.1 below (1.9) implies $\mu(\exp[\exp(\lambda\rho(o, \cdot))]) < \infty$ holds for any $\lambda > 0$, which is the exact concentration property of the given measure μ .

In the next section we study the super Poincaré and the weighted log-Sobolev inequality in an abstract framework, and complete proofs of the above results are presented in Section 3. Finally, weighted log-Sobolev and transportation cost inequalities are also studied for probability measures on \mathbb{R}^d by using concentrations.

2 From super Poincaré to weighted log-Sobolev inequalities

We shall work with a diffusion framework as in [1]. Let (E, \mathscr{F}, μ) be a separable complete probability space, and let $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ be a conservative symmetric local Dirichlet form on $L^2(\mu)$ with domain $\mathscr{D}(\mathscr{E})$ in the following sense. Let \mathscr{A} be a dense subspace of $\mathscr{D}(\mathscr{E})$ under the $\mathscr{E}_1^{1/2}$ -norm $(\mathscr{E}_1(f, f) = ||f||_2^2 + \mathscr{E}(f, f))$ which is composed of bounded functions, stable under products and composition with Lipschitz functions on \mathbb{R} . Let $\Gamma : \mathscr{A} \times \mathscr{A} \to \mathscr{M}_b$ be a bilinear mapping, where \mathscr{M}_b is the set of all bounded measurable functions on E, such that

$$\begin{array}{l} (1) \ \Gamma(f,f) \geq 0 \ \text{and} \ \mathscr{E}(f,g) = \mu(\Gamma(f,g)) \ \text{for} \ f,g \in \mathscr{A}; \\ (2) \ \Gamma(\phi \circ f,g) = \phi'(f)\Gamma(f,g) \ \text{for} \ f,g \in \mathscr{A} \ \text{and} \ \phi \in C_b^{\infty}(\mathbb{R}); \\ (3) \ \Gamma(fg,h) = g\Gamma(f,h) + f\Gamma(g,h) \ \text{for} \ f,g,h \in \mathscr{A} \ \text{with} \ fg \in \mathscr{A} \end{array}$$

It is easy to see that the positivity and the bilinear property imply $\Gamma(f,g)^2 \leq \Gamma(f,f)\Gamma(g,g)$ for all $f,g \in \mathscr{A}$. For simplicity we set below $\Gamma(f,f) = \Gamma(f)$ and $\mathscr{E}(f,f) = \mathscr{E}(f)$.

We shall denote by \mathscr{A}_{loc} the set of functions f such that for any integer n, the truncated function $f_n = \min(n, \max(f, -n))$ is in \mathscr{A} . For such functions, the bilinear map Γ automatically extends and shares the same properties than for functions in \mathscr{A} .

Next, let $\rho \in \mathscr{A}_{loc}$ be positive such that $\Gamma(\rho, \rho) \leq 1$. We shall start from the super Poincaré inequality

(2.1)
$$\mu(f^2) \le r \mathscr{E}(f, f) + \beta(r)\mu(|f|)^2, \quad r > 0.$$

To derive the desired weighted log-Sobolev inequality

(2.2)
$$\mu(f^2 \log f^2) \le C\mu(\Gamma(f, f)\alpha \circ \varrho), \quad \mu(f^2) = 1,$$

we shall also need the following Poincaré inequality

(2.3)
$$\mu(f^2) \le C_0 \mathscr{E}(f, f) + \mu(f)^2$$

for some $C_0 > 0$. Here and in what follows, the reference function f is taken from \mathscr{A} .

Theorem 2.1. Assume (2.3) holds for some $C_0 > 0$. Then (2.1) implies (2.2) for some constant C > 0 and α given in Theorem 1.1.

Proof. (a) Let $\Phi(s) = \mu(\varrho \ge s)$ which decreases to zero as $s \to \infty$. We may take $r_0 > 0$ such that

(2.4)
$$r_0(1 + \sup_{s \ge 1} \eta(s)) \le \frac{1}{32}$$

and

(2.5)
$$\beta^{-1}(e^{r_0^{-1}}/4) \le 1.$$

For a fixed number $r \in (0, r_0]$ we define

$$h_n = \left(\left(\varrho - \Phi^{-1} (2e^{-r^{-1}}) - n \right)_+ \land 1 \right) \left(\left(n + 2 + \Phi^{-1} (2e^{-r^{-1}}) - \varrho \right)_+ \land 1 \right),$$

$$\delta_n = \left(\log \frac{2}{\Phi(n + \Phi^{-1} (2e^{-r^{-1}}))} \right) \beta^{-1} \left(\frac{1}{2\Phi(n + \Phi^{-1} (2e^{-r^{-1}}))} \right),$$

$$B_n = \{ n \le \varrho - \Phi^{-1} (2e^{-r^{-1}}) \le n + 2 \}, \quad n \ge 0.$$

Then

(2.6)
$$\sum_{n=0}^{\infty} h_n^2 \ge \frac{1}{2} \mathbb{1}_{\{\rho \ge 1 + \Phi^{-1}(2e^{-r^{-1}})\}}.$$

By (2.1) and noting that

$$\mu(|f|h_n)^2 \le \mu(f^2h_n^2)\mu(\varrho > n + \Phi^{-1}(2e^{-r^{-1}})) \le \mu(f^2h_n^2)\Phi(n + \Phi^{-1}(2e^{-r^{-1}})),$$

we have

$$\sum_{n=0}^{\infty} \mu(f^2 h_n^2) \le \sum_{n=0}^{\infty} \left\{ r_n \mu \big(\Gamma(fh_n, fh_n) \big) + \beta(r_n) \mu(|f|h_n)^2 \right\}$$
$$\le \sum_{n=0}^{\infty} \left\{ \frac{2r_n}{\delta_n} \mu(\Gamma(f, f) \delta_n 1_{B_n}) + 2r_n \mu(f^2 1_{B_n}) + \beta(r_n) \Phi(n + \Phi^{-1}(2e^{-r^{-1}})) \mu(f^2 h_n^2) \right\}$$

for $r_n > 0$. Since by (2.5) and the definition of α

$$\alpha(s) \ge \delta_n \text{ for } s \ge n + 2 + \Phi^{-1}(2e^{-r^{-1}}),$$

letting $r_n = \delta_n r$ we obtain

(2.7)
$$\sum_{n=0}^{\infty} \mu(f^2 h_n^2) \le \sum_{n=0}^{\infty} \left\{ 2r\mu(\Gamma(f, f)\alpha \circ \varrho \mathbf{1}_{B_n}) + 2r\delta_n\mu(f^2 \mathbf{1}_{B_n}) + \beta(r\delta_n)\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))\mu(f^2 h_n^2) \right\}.$$

Noting that

$$A := r \log \frac{2}{\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))} \ge r \log \frac{2}{\Phi(\Phi^{-1}(2e^{-r^{-1}}))} = 1,$$

we have

$$\beta(\delta_n r) = \beta \left(A \beta^{-1} \left(\frac{1}{2\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))} \right) \right) \le \frac{1}{2\Phi(n + \Phi^{-1}(2e^{-r^{-1}}))}.$$

Thus, by (2.7) and (2.4) and the fact that $\delta_n \leq \sup \eta$, we arrive at

$$\sum_{n=0}^{\infty} \mu(f^2 h_n^2) \le \sum_{n=0}^{\infty} \left\{ 2r\mu(\Gamma(f, f)\alpha \circ \varrho \mathbf{1}_{B_n}) + \frac{1}{8}\mu(f^2) + \frac{1}{2}\sum_{n=0}^{\infty} \mu(f^2 h_n^2) \right\}.$$

It follows from this and (2.6) that

(2.8)
$$\mu(f^2 1_{\{\varrho \ge 1 + \Phi^{-1}(2e^{-r^{-1}})\}}) \le 8r\mu(\Gamma(f, f)\alpha \circ \varrho) + \frac{1}{2}\mu(f^2).$$

(b) On the other hand, since α is decreasing

$$\begin{split} &\mu(f^2 \mathbf{1}_{\{\varrho \leq 1+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})\}}) \leq \mu(f^2 \{(2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})-\varrho)^2_+ \wedge 1\}) \\ &\leq 2s\mu(\Gamma(f,f) \mathbf{1}_{\{\varrho \leq 2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}})\}}) + 2s\mu(f^2) + \beta(s)\mu(|f|)^2 \\ &\leq \frac{2s}{\alpha(2+\Phi^{-1}(2\mathrm{e}^{-r^{-1}}))}\mu(\Gamma(f,f)\alpha \circ \varrho) + 2s\mu(f^2) + \beta(s)\mu(|f|)^2, \quad s > 0. \end{split}$$

Taking

$$s = r\alpha(2 + \Phi^{-1}(2e^{-r^{-1}})) \le \frac{1}{32}$$

due to (2.4), we obtain

$$\mu(f^{2}1_{\{\varrho \leq 1+\Phi^{-1}(2e^{-r^{-1}})\}}) \leq 2r\mu(\Gamma(f,f)\alpha \circ \varrho) + \frac{1}{16}\mu(f^{2}) + \beta(r\alpha(2+\Phi^{-1}(2e^{-r^{-1}})))\mu(|f|)^{2}.$$

Since by (2.5) and the definition of α

$$r\alpha \left(2 + \Phi^{-1}(2e^{-r^{-1}})\right) \ge \left(r\log\frac{2}{\Phi(\Phi^{-1}(2e^{-r^{-1}}))}\right)\beta^{-1}\left(\frac{1}{2\Phi(\Phi^{-1}(2e^{-r^{-1}}))}\right)$$
$$= \beta^{-1}\left(\frac{e^{r^{-1}}}{4}\right),$$

we obtain

$$\mu(f^2 \mathbf{1}_{\{\varrho \leq 1 + \Phi^{-1}(2e^{-r^{-1}})\}}) \leq 2r\mu(\Gamma(f, f)\alpha \circ \varrho) + \frac{1}{16}\mu(f^2) + \frac{e^{r^{-1}}}{4}\mu(|f|)^2.$$

Combining this with (2.8) we conclude that

$$\mu(f^2) \le 40r\mu(\Gamma(f, f)\alpha \circ \varrho) + e^{r^{-1}}\mu(|f|)^2, \quad r \in (0, r_0].$$

Therefore, there exists a constant c > 0 such that

(2.9)
$$\mu(f^2) \le r\mu(\Gamma(f, f)\alpha \circ \varrho) + e^{c(1+r^{-1})}\mu(|f|)^2, \quad r > 0.$$

According to e.g. [24, Corollary 1.3], this is equivalent to the defective weighted log-Sobolev inequality

(2.10)
$$\mu(f^2 \log f^2) \le C_1 \mu(\Gamma(f, f) \alpha \circ \varrho) + C_2, \quad \mu(f^2) = 1.$$

(c) Finally, for any f with $\mu(f) = 0$, it follows from (2.3) that

$$\begin{split} \mu(f^2) &\leq \mu(f^2\{(1+R-\varrho)^2_+ \wedge 1\}) + \|f\|^2_{\infty}\mu(\varrho \geq R) \\ &\leq 2C_0\mu(\Gamma(f,f)1_{\{\varrho \leq 1+R\}}) + (2C_0+1)\|f\|^2_{\infty}\mu(\varrho \geq R) + \mu(f\{(\varrho - R)_+ \wedge 1\})^2 \\ &\leq \frac{2C_0}{\alpha(1+R)}\mu(\Gamma(f,f)\alpha \circ \varrho) + 2(C_0+1)\|f\|^2_{\infty}\mu(\varrho \geq R), \quad R > 0. \end{split}$$

Since $\mu(\varrho \ge R) \to 0$ as $R \to \infty$, the weighted weak Poincaré inequality

$$\mu(f^2) \leq \tilde{\beta}(r)\mu(\Gamma(f,f)\alpha \circ \varrho) + r\|f\|_{\infty}^2, \quad r > 0, \mu(f) = 0$$

holds for some positive function $\tilde{\beta}$ on $(0, \infty)$. By [19, Propsosition 1.3], this and (2.9) implies the weighted Poincaré inequality

$$\mu(f^2) \le C' \mu(\Gamma(f, f) \alpha \circ \varrho) + \mu(f)^2$$

for some constant C' > 0. Combining this with (2.10) we obtain the desired weighted log-Sobolev inequality (2.2).

3 Proofs of Theorem 1.1 and Corollaries

Proof of Theorem 1.1. Since α is bounded, the completeness of the original metric implies that of the weighted one given by (1.7). So, (1.6) follows from (1.5) due to [25, Theorem 1.1] with $p \to 2$. Thus, by Theorem 2.1 with E = M and $\Gamma(f, f) = |\nabla f|^2$, it suffices to prove that (1.4) implies the Poincaré inequality (2.3) for some $C_0 > 0$. Due to [23] the super Poincaré inequality (1.4) implies that the spectrum of L is discrete. Moreover, since M is connected, the corresponding Dirichlet form is irreducible so that 0 is a simple eigenvalue. Therefore, L possesses a spectral gap, which is equivalent to the desired Poincaré inequality.

To complete the proof of Corollary 1.2, in the spirit of [16, 3] we introduce below a deviation inequality induced by the L^1 -transportation cost inequality.

Proposition 3.1. Let $\tilde{\rho}: M \times M \to [0, \infty)$ be measurable. For any r > 0 and measurable set $A \subset M$ with $\mu(A) > 0$, let

$$A_r = \{ x \in M : \tilde{\rho}(x, y) \ge r \text{ for some } y \in A \}, \quad r > 0.$$

If

(3.1)
$$W_1^{\tilde{\rho}}(f\mu,\mu) \le \Phi \circ \mu(f\log f), \quad f \ge 0, \mu(f) = 1$$

holds for some positive increasing $\Phi \in C([0,\infty))$, then

(3.2)
$$\mu(A_r) \le \exp\left[-\Phi^{-1}(r - \Phi \circ \log \mu(A)^{-1})\right], \quad r > \Phi \circ \log \mu(A)^{-1},$$

where $\Phi^{-1}(r) := \inf\{s \ge 0 : \Phi(s) \ge r\}, \ r \ge 0.$

Proof. It suffices to prove for $\mu(A_r) > 0$. In this case, letting $\mu_A = \mu(\cdot \cap A)/\mu(A)$ and $\mu_{A_r} = \mu(\cdot \cap A_r)/\mu(A_r)$, we obtain from (3.1) that

$$r \le W_1^{\tilde{\rho}}(\mu_A, \mu_{A_r}) \le W_1^{\tilde{\rho}}(\mu_A, \mu) + W_1^{\tilde{\rho}}(\mu_{A_r}, \mu) \le \Phi \circ \log \mu(A)^{-1} + \Phi \circ \log \mu(A_r)^{-1}.$$

This completes the proof.

Proof of Corollary 1.2. (a) Let $\beta(r) = e^{c(1+r^{-1/\delta})}$ for some c > 0 and $\delta > 1$. It is easy to see that

$$1 \wedge \beta^{-1}(s/2) \le c_1 \log^{-\delta}(2s), \quad s \ge 1$$

holds for some constant $c_1 > 0$. Next, by [24, Corollary 5.3], (1.4) with this specific function β implies

$$\mu(\rho(o, \cdot) \ge s - 2) \le c_2 \exp[-c_3 s^{2/(2-\delta)}], \quad s \ge 0$$

for some constants $c_2, c_3 > 0$. Therefore,

(3.3)
$$\alpha(s) \le c_4(1+s)^{-2(\delta-1)/(2-\delta)}, \quad s \ge 0$$

holds for some constant $c_4 > 0$.

On the other hand, for any $x_1, x_2 \in M$ let $i \in \{1, 2\}$ such that $\rho(o, x_i) = \rho(o, x_1) \vee \rho(o, x_2)$. Define

$$f(x) = \left(\rho(x, x_i) \land \frac{\rho(o, x_i)}{2}\right) (1 + \rho(o, x_i))^{(\delta - 1)/(2 - \delta)}, \quad x \in \mathbb{R}^d.$$

Then

$$\begin{aligned} \alpha \circ \rho(o, \cdot) |\nabla f|^2 &\leq c_4 (1 + \rho(o, \cdot))^{-2(\delta - 1)/(2 - \delta)} |\nabla f|^2 \\ &\leq c_4 \mathbb{1}_{\{\rho(o, x_i)/2 \leq \rho(o, \cdot) \leq 3\rho(o, x_i)/2\}} (1 + \rho(o, \cdot))^{-2(\delta - 1)/(2 - \delta)} (1 + \rho(o, x_i))^{2(\delta - 1)/(2 - \delta)} \leq c_5 \end{aligned}$$

for some constant $c_5 > 0$. Since by the triangle inequality $\rho(o, x_i) \ge \frac{1}{2}\rho(x_1, x_2)$, this implies that the intrinsic distance ρ_{α} satisfies

$$\rho_{\alpha}(x_1, x_2)^2 \ge \frac{|f(x_1) - f(x_2)|^2}{c_5}$$

$$\ge c_6 \rho(x_1, x_2)^2 (1 + \rho(o, x_1) \lor \rho(o, x_2))^{2(\delta - 1)/(2 - \delta)} \ge c_7 \rho(x_1, x_2)^{2/(2 - \delta)}$$

for some constant $c_6, c_7 > 0$. Hence the proof of (a) is completed by Theorem 1.1.

(b) Now, assume that

$$\operatorname{Ric} - \operatorname{Hess}_V \ge -K$$

for some $K \ge 0$. By (a) and Proposition 3.1, which ensures the implication from (4) to (5), it suffices to deduce (1) from (5). Let

$$h(r) = \mu(e^{r\rho(o,\cdot)^2}), \quad r > 0.$$

By [24, Theorem 5.7], the super Poincaré inequality (1.4) holds with

(3.4)
$$\beta(r) := c_0 \inf_{0 < r_1 < r} r_1 \inf_{s > 0} \frac{1}{s} h(2K + 12s^{-1}) e^{s/r_1 - 1}, \quad r > 0$$

for some constant $c_0 > 0$. Since for any $\lambda > 0$ there exists $c(\lambda) > 0$ such that

 $rt^{2} \leq \lambda t^{2/(2-\delta)} + c(\lambda)r^{1/(\delta-1)}, \quad r > 0,$

it follows from (5) that

$$h(r) \le c_1 \exp[c_1 r^{1/(\delta-1)}], \quad r > 0$$

for some constants $c_1 > 0$. Therefore,

$$\beta(r) \le c_2 \inf_{0 < r_1 < r} r_1 \inf_{s > 0} \frac{1}{s} \exp[c_2 s^{-1/(\delta - 1)} + s/r_1], \quad r > 0$$

for some $c_2 > 0$. Taking $s = r^{(\delta-1)/\delta}$ and $r_1 = r$, we conclude that

$$\beta(r) \le e^{c(1+r^{-1/\delta})}, \quad r > 0$$

for some c > 0. Thus, (1) holds.

Proof of Corollary 1.3. The proof is similar to that of Corollary 1.2 by noting that (1.4) with $\beta(r) = \exp[c(1+r^{-1/2})]$ implies $\mu(\rho(o,\cdot) \ge s) \le \exp[-ce^{c_1s}]$ for some $c_1 > 0$, see [24, Corollary 5.3].

4 Weighted log-Sobolev and transportation cost inequalities on \mathbb{R}^d

Our main purpose of this section is to establish the weighted log-Sobolev inequality for an arbitrary probability measure using the concentration of this measure. We shall also prove the HWI inequality introduced in [2] for the corresponding weighted Dirichlet form. The main point is to find square fields (resp. cost functions) for a given probability measure to satisfy the log-Sobolev inequality (resp. the Talagrand transportation cost inequality).

So, the line of our study is exactly opposed to existed references in the literature, see e.g. [9, 10, 11] and references within, which provided conditions on the reference measure such that the log-Sobolev (resp. transportation cost) inequality holds for a given square field (resp. the corresponding cost function).

The basic idea of the study comes from Caffarelli [5] which says that for any probability measure $\mu(dx) := e^{V(x)} dx$ on \mathbb{R}^d , there exists a convex function ψ on \mathbb{R}^d such that $\nabla \psi$ pushes μ forward to the standard Gaussian measure γ ; that is, letting

$$y(x) := \nabla \psi(x), \quad x \in \mathbb{R}^d,$$

which is one-to-one, one has $\gamma = \mu \circ y^{-1}$. Furthermore, $\nabla \psi$ is uniquely determined and $\operatorname{Hess}_{\psi}$ is non-degenerate with

$$\det(\operatorname{Hess}_{\psi}) = (2\pi)^{d/2} \mathrm{e}^{V + |\nabla \psi|^2/2}.$$

Let

$$\rho(x_1, x_2) := |y(x_1) - y(x_2)|, \quad x_1, x_2 \in \mathbb{R}^d.$$

Let W_2 be the L^2 -Wasserstein distance induced by the usual Euclidiean metric. Due to Talagrand [21]

(4.1)
$$W_2(\gamma, f^2\gamma)^2 \le 2\gamma(f^2\log f^2), \quad \gamma(f^2) = 1.$$

Since $\pi \in \mathscr{C}(\mu \circ y^{-1}, (f^2 \circ y^{-1})\mu \circ y^{-1})$ if and only if $\pi \circ (y \otimes y) \in \mathscr{C}(\mu, f^2\mu)$, we obtain from (4.1) and the change of variables theorem that

$$W_2^{\rho}(\mu, f^2\mu)^2 = W_2(\gamma, (f^2 \circ y^{-1})\gamma)^2 \le 2\gamma(f^2 \circ y^{-1}\log f^2 \circ y^{-1}) = 2\mu(f^2\log f^2), \quad \mu(f^2) = 1.$$

Similarly, since

$$\nabla(f \circ y^{-1}) = (Dy^{-1})(\nabla f) \circ y^{-1} = [(Dy) \circ y^{-1}]^{-1}(\nabla f) \circ y^{-1} = [(\operatorname{Hess}_{\psi})^{-1}\nabla f] \circ y^{-1},$$

where $Dy := (\partial_i y_j)_{d \times d}$, by Gross' log-Sobolev inequality for γ (see [12]) we obtain

$$\mu(f^2 \log f^2) \le 2\mu(|(\mathrm{Hess}_{\psi})^{-1} \nabla f|^2), \quad f \in C_0^{\infty}(\mathbb{R}^d), \mu(f^2) = 1.$$

On the other hand, however, since the transportation $\nabla \psi$ is normally inexplicit, it is hard to estimate the distance ρ and the matrix Hess_{ψ} . So, to derive transportation and log-Sobolev inequalities with explicit distances and Dirichlet forms, we shall construct, instead of $\nabla \psi$, an explicit map using the concentration of μ , which transports the measure into the standard Gaussian measure with a perturbation. In many cases this perturbation is bounded and hence, does not make much trouble to derive the desired inequalities.

4.1 Main results

In this subsection we provide an explicit positive function α and an explicit distance ρ on \mathbb{R}^d such that the log-Sobolev inequality

(4.2)
$$\mu(f^2 \log f^2) \le 2\mu(\alpha |\nabla f|^2), \quad f \in C_0^\infty(\mathbb{R}^d), \ \mu(f^2) = 1$$

and the transportation-cost inequality

(4.3)
$$W_2^{\rho}(\mu, f^2\mu)^2 \le 2\mu(f^2\log f^2), \quad \mu(f^2) = 1$$

hold. In a special case, we are also able to present the HWI inequality stronger than (4.2).

Let us first consider a probability measure $\mu(dx) := e^{V(x)} dx$ on $[\delta, \infty)$ for some $\delta \in [-\infty, \infty)$, where $[-\infty, \infty)$ is regarded as \mathbb{R} . Let

$$\Phi_{\delta}(r) := \frac{1}{c_{\delta}} \int_{\delta}^{r} e^{-s^{2}/2} ds, \quad \varphi(r) := \mu([\delta, r)) = \int_{\delta}^{r} e^{V(x)} dx, \quad r \ge \delta,$$

where $c_{\delta} := \int_{\delta}^{\infty} e^{-x^2/2} dx$ is the normalization.

Theorem 4.1. Let $\mu(dx) := 1_{[\delta,\infty)}(x)e^{V(x)}dx$ be a probability measure on $[\delta,\infty)$. For the above defined Φ_{δ} and φ , (4.2) and (4.3) hold with \mathbb{R}^d replaced by $[\delta,\infty)$ for

$$\begin{aligned} \alpha &:= \left(\frac{\Phi_{\delta}' \circ \Phi_{\delta}^{-1} \circ \varphi}{\varphi'}\right)^2,\\ \rho(x, y) &:= |\Phi_{\delta}^{-1} \circ \varphi(x) - \Phi_{\delta}^{-1} \circ \varphi(y)|, \quad x, y \ge \delta. \end{aligned}$$

Furthermore,

(4.4)
$$\mu(f^2 \log f^2) + W_2^{\rho}(\mu, f^2 \mu)^2 \le 2\sqrt{2\mu(\alpha f'^2)}W_2^{\rho}(\mu, f^2 \mu), \quad f \in C_0^{\infty}([\delta, \infty)), \mu(f^2) = 1.$$

The inequality (4.4), linking the Wasserstein distance, the relative entropy and the energy, is called the HWI inequality in [2] and [18].

To extend this result to \mathbb{R}^d for $d \geq 2$, we consider the polar coordinate $(r, \theta) \in [0, \infty) \times \mathbb{S}^{d-1}$, where \mathbb{S}^{d-1} is the unit sphere in \mathbb{R}^d with the induced metric. Then μ can be represented as

$$d\mu = c(d)r^{d-1}e^{V(r\theta)}drd\theta =: G(r,\theta)drd\theta,$$

where $d\theta$ is the normalized volume measure on \mathbb{S}^{d-1} , and c(d)/d equals to the volume of the unit ball in \mathbb{R}^d . Let $B(0, r) := \{x \in \mathbb{R}^d : |x| < r\}$ and

$$\Phi_0(r) := \int_{B(0,r)} \frac{\mathrm{e}^{-|x|^2/2} \mathrm{d}x}{(2\pi)^{d/2}}, \quad r \ge 0,$$

$$h(\theta) := \int_0^\infty s^{d-1} \mathrm{e}^{V(s\theta)} \mathrm{d}s, \quad \theta \in \mathbb{S}^{d-1},$$

$$\varphi_\theta(r) := \frac{1}{h(\theta)} \int_0^r s^{d-1} \mathrm{e}^{V(s\theta)} \mathrm{d}s, \quad \theta \in \mathbb{S}^{d-1}, r \ge 0.$$

Since $\mu(\mathbb{R}^d) = 1$, we have $h(\theta) \in (0, \infty)$ for a.e. $\theta \in \mathbb{S}^{d-1}$. We shall prove that the map

$$x \mapsto \Phi_0^{-1} \circ \varphi_{\frac{x}{|x|}}(|x|) \frac{x}{|x|}$$

transports μ into a Gaussian measure with density $h \circ \theta$. Thus, to derive the desired inequalities for μ , we need a regularity property of this transportation specified in the following result.

Theorem 4.2. Let $r(x) := |x|, \theta(x) := \frac{x}{|x|}, x \in \mathbb{R}^d$. If $C(h) := \sup_{\theta_1, \theta_2 \in \mathbb{S}^{d-1}} \frac{h(\theta_1)}{h(\theta_2)} < \infty$, then (4.3) holds for

$$\rho(x_1, x_2) := C(h)^{-1/2} |(\Phi_0^{-1} \circ \varphi_\theta(r)\theta)(x_1) - (\Phi_0^{-1} \circ \varphi_\theta(r)\theta)(x_2)|, \quad x_1, x_2 \in \mathbb{R}^d.$$

If moreover $\varphi_{\theta}(r)$ is differentiable in θ then (4.2) holds for

$$\alpha := C(h) \inf_{\varepsilon > 0} \max \left\{ \frac{(1+\varepsilon)r^2}{(\Phi_0^{-1} \circ \varphi_\theta(r))^2}, \ \frac{(\Phi_0' \circ \Phi_0^{-1} \circ \varphi_\theta(r))^2}{(\varphi_{\theta'}(r))^2} + \frac{(1+\varepsilon^{-1})|\nabla_\theta \varphi_\theta(r)|^2}{(\varphi_{\theta'}(r)\Phi_0^{-1} \circ \varphi_\theta(r))^2} \right\}.$$

If, in particular, h is constant (it is the case if V(x) depends only on |x|), then the following HWI inequality holds:

(4.5)
$$\mu(f^2 \log f^2) + W_2^{\rho}(\mu, f^2 \mu)^2 \le 2\sqrt{2\mu(\alpha |\nabla f|^2)} W_2^{\rho}(\mu, f^2 \mu), \quad f \in C_0^{\infty}(\mathbb{R}^d), \mu(f^2) = 1,$$

for

$$\alpha := \max\left\{\frac{r^2}{(\Phi_0^{-1} \circ \varphi(r))^2}, \ \frac{(\Phi_0' \circ \Phi_0^{-1} \circ \varphi(r))^2}{(\varphi'(r))^2}\right\}$$

and $\varphi = \varphi_{\theta}$ is independent of θ .

Note that if V is locally bounded and $\zeta(r) := \sup_{|x|=r} V(x)$ satisfies $\int_0^\infty r^{d-1} e^{\zeta(r)} dr < \infty$, then $C(h) < \infty$. Thus, Theorem 4.2 applies to a large number of probability measures. In particular, we have the following concrete result.

Corollary 4.3. Let V be differentiable such that $\mu(dx) := e^{V(x)} dx$ is a probability measure and

(4.6)
$$-c_1|x|^{\delta-1} \le \langle \nabla V(x), \nabla |x| \rangle \le -c_2|x|^{\delta-1}$$

holds for some constants δ , c_1 , $c_2 > 0$ and large |x|. If there exists a constant $c_3 > 0$ such that

$$(4.7) |\nabla_{\theta} V| \le c_3,$$

where ∇_{θ} is the gradient on \mathbb{S}^{d-1} at point θ , then there exists a constant c > 0 such that

(4.8)
$$\mu(f^2 \log f^2) \le c\mu((1+|\cdot|)^{2-\delta} |\nabla f|^2), \quad f \in C_0^{\infty}(\mathbb{R}^d), \mu(f^2) = 1.$$

Consequently,

(4.9)
$$W_2^{\tilde{\rho}}(\mu, f^2\mu)^2 \le c'\mu(f^2\log f^2), \quad \mu(f^2) = 1$$

holds for some constant c' > 0 and

$$\tilde{\rho}(x,y) := \frac{|x-y|}{(1+|x|\vee|y|)^{1-\delta/2}}, \quad x,y \in \mathbb{R}^d.$$

Remark. (a) The inequalities presented in Corollary 4.3 are sharp in the sense that (4.9) (and hence also (4.8)) implies $\mu(e^{\lambda r^{\delta}}) < \infty$ for some $\lambda > 0$, which is the exact concentration of μ . This follows from [3, Corollary 3.2] and the fact that $\tilde{\rho}(0, x) \approx |x|^{\delta/2}$ for large |x|.

(b) When V is strictly concave, the matrix

$$\Lambda[v_1, v_2] := \int_0^1 s(-\text{Hess}_V)((1-s)v_1 + sv_2) ds$$

is strictly positive definite for any $v_1, v_2 \in \mathbb{R}^d$. It is proved by Kolesnikov (see [15, Corollary 3.1]) that

(4.10)
$$\mu(f^2 \log f^2) \le \int_{\mathbb{R}^d} \langle \Lambda[T_f, \cdot]^{-1} \nabla f, \nabla f \rangle \mathrm{d}\mu, \quad f \in C_0^\infty(\mathbb{R}^d), \mu(f^2) = 1,$$

where $x \mapsto T_f(x)$ is the optimal transport of $f^2\mu$ to μ . In particular, for $V(x) := -|x|^{\delta} + c$ with $\delta > 2$ and a constant c, [15, Example 3.2] implies (4.8) for even smooth function f^2 . But Corollary 4.3 works for more general V and all smooth function f. (c) Recently, Gentil, Guillin and Miclo [9] (see [10, 11] for further study) established a Talagrand type inequality for $V(x) = -|x|^{\delta} + c$ with $\delta \in [1, 2]$ and a constant c. Precisely, there exist constants a, D > 0 such that

(4.11)
$$\inf_{\pi \in \mathscr{C}(\mu, f^2\mu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} L_{a,D}(x-y)\pi(\mathrm{d}x, \mathrm{d}y) \le D\mu(f^2\log f^2), \quad \mu(f^2) = 1,$$

where

$$L_{a,D}(x) := \begin{cases} \frac{|x|^2}{2}, & \text{if } |x| \le a, \\ \frac{a^{2-\delta}}{\delta} |x|^{\delta} + \frac{a^2(\delta-2)}{2\delta}, & \text{otherwise.} \end{cases}$$

Since $L_{a,D}(x-y) \geq \varepsilon \tilde{\rho}(x,y)^2$ for some constant $\varepsilon > 0$, this inequality implies (4.9) for $\delta \in [1,2]$. But (4.11) is yet unavailable for $\delta \notin [1,2]$ while (4.9) holds for more general V. In particular, if $\delta > 2$ then (4.9) with $\tilde{\rho}(x,y) \geq c(|x-y| \vee |x-y|^{\delta/2})$ for some c > 0, which is new in the literature.

4.2 Proofs

We first briefly prove for the one-dimensional case (i.e. Theorem 4.1), then extend the argument to high dimensions. It turns out, comparing with the one-dimensional case, that the difficulty point of the proof for high dimensions comes from the angle part. So, a restriction concerning the angle part was made in Theorem 4.2.

Proof of Theorem 4.1. Let $y(x) := \Phi_{\delta}^{-1} \circ \varphi(x), x \ge \delta$. We have

$$\frac{\mathrm{d}\mu}{\mathrm{d}y} = \frac{\mathrm{d}\mu}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{e}^{V(x)} \frac{\mathrm{d}\varphi^{-1} \circ \Phi_{\delta}(y)}{\mathrm{d}y}$$
$$= \frac{\mathrm{e}^{V(x)}\Phi_{\delta}'(y)}{\varphi' \circ \varphi^{-1} \circ \Phi_{\delta}(y)} = \frac{\mathrm{e}^{V(x)}\Phi_{\delta}'(y)}{\varphi'(x)} = \Phi_{\delta}'(y).$$

Therefore, μ is the standard Gaussian measure under the new coordinate $y \in [\delta, \infty)$. In other words, one has

$$\gamma(\mathrm{d}x) := (\mu \circ y^{-1})(\mathrm{d}x) = Z \mathbb{1}_{[\delta,\infty)}(x) \mathrm{e}^{-x^2/2} \mathrm{d}x,$$

where Z is the normalization constant. By the HWI inequality proved in [2, 17, 18] and the Gross log-Sobolev inequality which implies the Talagrand inequality, we have

(4.12)
$$\gamma(g^2 \log g^2) + W_2(\gamma, g^2 \gamma)^2 \le 2\sqrt{2\gamma((g')^2)} W_2(\gamma, g^2 \gamma), W_2(\gamma, g^2 \gamma)^2 \le 2\gamma(g^2 \log g^2), \quad \gamma(g^2) = 1.$$

We remark that although the HWI and Gross's log-Sobolev inequalities are stated in the above references for the global Gaussian measure, they are also true on a regular convex domain Ω , since the stronger gradient estimate

$$|\nabla P_t f| \le e^{-t} P_t |\nabla f|, \quad f \in C_b^1(\Omega)$$

holds for the Neumann heat semigroup on Ω (cf. [22] and references within).

For any $f \in C_0^1([\delta, \infty))$ with $\mu(f^2) = 1$, let $g := f \circ y^{-1}$. We have

$$\frac{\mathrm{d}g}{\mathrm{d}x} = (f' \circ y^{-1})\frac{\mathrm{d}y^{-1}}{\mathrm{d}x} = \frac{f' \circ y^{-1}}{y' \circ y^{-1}} = (f' \circ y^{-1})\left(\frac{\Phi_{\delta}' \circ \Phi_{\delta}^{-1} \circ \varphi}{\varphi'}\right) \circ y^{-1}.$$

Since $\gamma = \mu \circ y^{-1}$, this and (4.12) imply (4.3) and (4.4). Finally, (4.2) is implied by (4.4).

Proof of Theorem 4.2. Let (r, θ) be the polar coordinate introduced in Section 2, and let ∇_{θ} denote the gradient operator on \mathbb{S}^{d-1} for the standard metric induced by the Euclidean metric on \mathbb{R}^d . By the orthogonal decomposition of the gradient, we have

(4.13)
$$\nabla f = (\partial_r f) \frac{\partial}{\partial r} + r^{-1} \nabla_\theta f, \quad |\nabla f|^2 = (\partial_r f)^2 + r^{-2} |\nabla_\theta f|^2.$$

Let us introduce a new polar coordinate (\bar{r}, θ) , where

$$\bar{r}(r,\theta) := \Phi_0^{-1} \circ \varphi_\theta(r), \quad r \ge 0, \theta \in \mathbb{S}^{d-1}.$$

We have

$$d\mu := G(r,\theta) dr d\theta = \frac{G(r,\theta)}{\partial_r \bar{r}} d\bar{r} d\theta = c(d)h(\theta)\Phi'_0(\bar{r}) d\bar{r} d\theta = c(d)h(\theta)d\mu_0,$$

where $d\mu_0 := \Phi'_0(\bar{r}) d\bar{r} d\theta$ is the standard Gaussian measure under the new polar coordinate (\bar{r}, θ) . Thus, letting

$$y(x) := \bar{r}(x)\theta(x) = \Phi_0^{-1} \circ \varphi_{\frac{x}{|x|}}(|x|)\theta(x), \quad x \in \mathbb{R}^d,$$

we have

$$(\mu \circ y^{-1})(\mathrm{d}x) = c(d)h(x/|x|)(\mu_0 \circ y^{-1})(\mathrm{d}x) = c(d)h(x/|x|)\gamma(\mathrm{d}x),$$

where γ is the standard Gaussian measure on \mathbb{R}^d . By Gross' log-Sobolev inequality one has

$$\gamma(g^2 \log g^2) \le 2\gamma(|\nabla g|^2), \quad g \in C_0^{\infty}(\mathbb{R}^d), \mu_0(g^2) = 1.$$

Thus, by the perturbation of the log-Sobolev inequality (cf. [7]), we have

$$(4.14) \quad (\mu \circ y^{-1})(g^2 \log g^2) \le 2C(h)(\mu \circ y^{-1})(|\nabla g|^2), \quad g \in W^{2,1}(\gamma), (\mu \circ y^{-1})(g^2) = 1.$$

Moreover, by [2, Corollary 3.1], (4.14) implies

(4.15)
$$W_2(\mu \circ y^{-1}, g^2 \mu \circ y^{-1})^2 \le 2C(h)(\mu \circ y^{-1})(g^2 \log g^2), \quad (\mu \circ y^{-1})(g^2) = 1.$$

This implies (4.3) for the desired distance ρ by using the change of variables theorem as explained above.

Similarly, to prove (4.2) we intend apply (4.14) for $g := f \circ y^{-1}$, where $f \in C_0^{\infty}(\mathbb{R}^d)$ with $\mu(f^2) = 1$. Since $y^{-1} = (\varphi_{\theta}^{-1} \circ \Phi_0(r), \theta)$ under the polar coordinate, by the chain rule we have

$$\nabla_{\theta}(f \circ y^{-1}) = \nabla_{\theta}f(\varphi_{\theta}^{-1} \circ \Phi_{0}(r), \theta) = \left((\nabla_{\theta}f) \circ y^{-1} + (\partial_{r}f) \circ y^{-1} \right) \nabla_{\theta}\varphi_{\theta}^{-1} \circ \Phi_{0}(r).$$

But $\varphi_{\theta} \circ \varphi_{\theta}^{-1} \circ \Phi_0 = \Phi_0$ implies

$$(\nabla_{\theta}\varphi_{\theta})(\varphi_{\theta}^{-1}\circ\Phi_{0}(r))+\varphi_{\theta}'\circ\varphi_{\theta}^{-1}\circ\Phi_{0}(r)\cdot\nabla_{\theta}(\varphi_{\theta}^{-1}\circ\Phi_{0}(r))=0,$$

where $(\nabla_{\theta}\varphi_{\theta})(\varphi_{\theta}^{-1}\circ\Phi_{0}(r)):=\nabla_{\theta}\varphi_{\theta}(s)|_{s=\varphi_{\theta}^{-1}\circ\Phi_{0}(r)}$, we arrive at

$$(4.16) \qquad \begin{aligned} |\nabla_{\theta}(f \circ y^{-1})|^{2} \\ &\leq (1+\varepsilon)(\partial_{r}f)^{2} \circ y^{-1} \Big(\frac{|\nabla_{\theta}\varphi_{\theta}(r)|(\varphi_{\theta}^{-1} \circ \Phi_{0}(r))}{\varphi_{\theta}' \circ \varphi_{\theta}^{-1} \circ \Phi_{0}(r)} \Big)^{2} + (1+\varepsilon^{-1})|\nabla_{\theta}f|^{2} \circ y^{-1} \\ &= (1+\varepsilon)(\partial_{r}f)^{2} \circ y^{-1} \Big(\frac{|\nabla_{\theta}\varphi_{\theta}(r)|}{\varphi_{\theta}'(r)} \Big)^{2} \circ y^{-1} + (1+\varepsilon^{-1})|\nabla_{\theta}f|^{2} \circ y^{-1} \end{aligned}$$

for any $\varepsilon > 0$.

On the other hand,

$$\partial_r(f \circ y^{-1}) = (\partial_r f) \circ y^{-1} \frac{\Phi'_0(r)}{\varphi_{\theta'} \circ \varphi_{\theta}^{-1} \circ \Phi_0(r)}$$

Since

(4.17)
$$r = \Phi_0^{-1} \circ \varphi_\theta(r(y^{-1})) = \Phi_0^{-1} \circ \varphi_\theta(r) \circ y^{-1},$$

we have

$$\Phi_0'(r) = \left(\Phi_0' \circ \Phi_0^{-1} \circ \varphi_\theta(r)\right) \circ y^{-1}, \quad \varphi_\theta' \circ \varphi_\theta^{-1} \circ \Phi_0(r) = \varphi_\theta'(r) \circ y^{-1}.$$

Thus,

$$|\partial_r (f \circ y^{-1})|^2 = \left\{ (\partial_r f) \frac{\Phi'_0 \circ \Phi_0^{-1} \circ \varphi_\theta(r)}{\varphi_{\theta'}(r)} \right\}^2 \circ y^{-1}.$$

Combining this with (4.13), (4.16) and (4.17), we obtain

$$\begin{split} |\nabla(f \circ y^{-1})|^2 &= (\partial_r (f \circ y^{-1}))^2 + r^{-2} |\nabla_\theta (f \circ y^{-1})|^2 \\ &\leq \left\{ (\partial_r f) \frac{\Phi'_0 \circ \Phi_0^{-1} \circ \varphi_\theta (r)}{\varphi_{\theta'}(r)} \right\}^2 \circ y^{-1} \\ &+ (\Phi_0^{-1} \circ \varphi_\theta (r))^{-2} \circ y^{-1} \left\{ (1 + \varepsilon) (\partial_r f)^2 \Big(\frac{|\nabla_\theta \varphi_\theta (r)|}{\varphi_{\theta'}(r)} \Big)^2 + (1 + \varepsilon^{-1}) |\nabla_\theta f|^2 \right\} \circ y^{-1} \\ &= (\partial_r f)^2 \circ y^{-1} \left\{ \frac{(\Phi'_0 \circ \Phi_0^{-1} \circ \varphi_\theta (r))^2}{(\varphi_{\theta'}(r))^2} + \frac{(1 + \varepsilon) |\nabla_\theta \varphi_\theta (r)|^2}{(\varphi_{\theta'}(r))^2 (\Phi_0^{-1} \circ \varphi_\theta (r))^2} \right\} \circ y^{-1} \\ &+ (r \circ y^{-1})^{-2} |\nabla_\theta f|^2 \circ y^{-1} \Big(\frac{(1 + \varepsilon^{-1}) r^2}{(\Phi_0^{-1} \circ \varphi_\theta (r))^2} \Big) \circ y^{-1} \\ &\leq |\nabla f|^2 \circ y^{-1} \max \left\{ \frac{(1 + \varepsilon^{-1}) r^2}{(\Phi_0^{-1} \circ \varphi_\theta (r))^2}, \frac{(\Phi'_0 \circ \Phi_0^{-1} \circ \varphi_\theta (r))^2}{(\varphi_{\theta'}(r))^2} + \frac{(1 + \varepsilon) |\nabla_\theta \varphi_\theta (r)|^2}{(\varphi_{\theta'}(r))^2 (\Phi_0^{-1} \circ \varphi_\theta (r))^2} \right\} \circ y^{-1} \end{split}$$

for any $\varepsilon > 0$. Therefore,

(4.18)
$$|\nabla(f \circ y^{-1})|^2 \le (\alpha |\nabla f|^2) \circ y^{-1}$$

and hence (4.2) follows from (4.14) by letting $g = f \circ y^{-1}$.

Finally, if h is constant then $\mu \circ y^{-1}$ is the standard Gaussian measure. Hence, by [2, Theorem 4.3] one has

$$\begin{split} &W_2(\mu \circ y^{-1}, (f^2 \circ y^{-1})\mu \circ y^{-1})^2 + (\mu \circ y^{-1})(f^2 \circ y^{-1}\log f^2 \circ y^{-1}) \\ &\leq 2\sqrt{2(\mu \circ y^{-1})(|\nabla (f \circ y^{-1})|^2)} \, W_2(\mu \circ y^{-1}, (f^2 \circ y^{-1})\mu \circ y^{-1}). \end{split}$$

By combining this with (4.18) we prove (4.5).

Proof of Corollary 4.3. Since there exists a constant $c_0 > 0$ such that

$$\Phi_0'(r) = c_0 r^{d-1} e^{-r^2/2} = \begin{cases} \Theta(r^{d-1}) & \text{as } r \to 0, \\ \Theta(r(1 - \Phi_0(r))) & \text{as } r \to \infty, \end{cases}$$

where $f = \Theta(g)$ means that the two positive functions f and g are asymptotically bounded by each other up to constants, there exists a constant $c \ge 1$ such that

$$\frac{1}{c}\Phi_0'(r) \le \min\{r, r^{d-1}\}(1 - \Phi_0(r)) \le c\Phi_0'(r), \quad r \ge 0.$$

Equivalently,

(4.19)
$$\frac{1}{c}\Phi'_0 \circ \Phi_0^{-1}(r) \le \min\{\Phi_0^{-1}(r), \Phi_0^{-1}(r)^{d-1}\}(1-r) \le c\Phi'_0 \circ \Phi_0^{-1}(r), r \in [0,1).$$

Next, it is easy to see from (4.6) that

(4.20)
$$\Phi_0^{-1} \circ \varphi_\theta(r) = \begin{cases} \Theta(r^{\delta/2}) & \text{as } r \to \infty, \\ \Theta(r) & \text{as } r \to 0, \end{cases}$$

and

(4.21)
$$\frac{1 - \varphi_{\theta}(r)}{\varphi_{\theta}'(r)} = \frac{\int_{r}^{\infty} s^{d-1} \mathrm{e}^{V(s\theta)} \mathrm{d}s}{r^{d-1} \mathrm{e}^{V(r\theta)}} \le cr^{1-\delta}$$

for some constant c > 0 and all $r \ge 1$. Combining (4.19), (4.20) and (4.21) we obtain

(4.22)
$$\max\left\{\frac{r^2}{(\Phi_0^{-1} \circ \varphi_\theta(r))^2}, \frac{(\Phi_0' \circ \Phi_0^{-1} \circ \varphi_\theta(r))^2}{(\varphi_{\theta'}(r))^2}\right\} \le c(1+r)^{2-\delta}$$

for some constant c > 0.

If (4.7) holds then

$$|\nabla_{\theta}\varphi_{\theta}(r)| = |\nabla_{\theta}(1 - \varphi_{\theta}(r)| \le c_4 \min\left\{r^d, \int_r^{\infty} s^{d-1} \mathrm{e}^{V(s\theta)} \mathrm{d}s\right\},\$$

so that due to (4.20) and (4.21)

$$\frac{|\nabla_{\theta}\varphi_{\theta}(r)|^{2}}{(\varphi_{\theta}'(r))^{2}(\Phi_{0}^{-1}\circ\varphi_{\theta}(r))^{2}} \leq c_{5} \left(\frac{\min\{r^{d}, \int_{r}^{\infty} s^{d-1}\mathrm{e}^{V(s\theta)}\mathrm{d}s\}}{(r1_{\{r<1\}} + r^{\delta/2}1_{\{r\geq1\}})r^{d-1}\mathrm{e}^{V(r\theta)}}\right)^{2} \leq c_{6}(1+r)^{2-3\delta}$$

for some constants $c_5, c_6 > 0$. Combining this with (4.22) and Theorem 4.2, we prove (4.8).

Finally, for any $x_1, x_2 \in \mathbb{R}^d$ let $i \in \{1, 2\}$ such that $|x_i| = |x_1| \vee |x_2|$. Similarly to the proof of Corollary 1.2, define

$$f(x) = \frac{|x - x_i| \wedge \frac{|x_i|}{2}}{(1 + |x_i|)^{1 - \delta/2}}, \quad x \in \mathbb{R}^d.$$

Then

$$\Gamma(f,f) := (1+|\cdot|)^{2-\delta} |\nabla f|^2 \le \frac{1_{\{|x_i|/2 \le |\cdot| \le 3|x_i|/2\}} (1+|\cdot|)^{2-\delta}}{(1+|x_i|)^{2-\delta}} \le C(\delta)$$

for some constant $C(\delta) > 0$. Since $|x_i| \ge \frac{1}{2}|x_1 - x_2|$, this implies that the intrinsic distance ρ induced by Γ satisfies

$$\rho(x_1, x_2)^2 \ge \frac{|f(x_1) - f(x_2)|^2}{C(\delta)} \ge C_1(\delta)\tilde{\rho}(x_1, x_2)^2$$

for some constant $C_1(\delta) > 0$, and hence is complete. Thus, by [25, Theorem 1.1] or [26, Theorem 6.3.3], (4.9) follows from (4.8).

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