

# POSITIVE MASS THEOREMS FOR ASYMPTOTICALLY DE SITTER SPACETIMES

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ABSTRACT. We use planar coordinates as well as hyperbolic coordinates to separate the de Sitter spacetime into two parts. These two ways of cutting the de Sitter give rise to two different spatial infinities. For spacetimes which are asymptotic to either half of the de Sitter spacetime, we are able to provide definitions of the total energy, the total linear momentum, the total angular momentum, respectively. And we prove two positive mass theorems, corresponding to these two sorts of spatial infinities, for spacelike hypersurfaces whose mean curvatures are bounded by certain constant from above.

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## 1. INTRODUCTION

The positive mass theorem plays a fundamental role in general relativity. For asymptotically flat spacetimes, the positive mass theorem was firstly proved by Schoen and Yau [20, 21, 22], and then by Witten [26] using a different method. Anti-de Sitter spacetimes are characterized by Killing spinors with imaginary Killing constants, which have certain nice properties. Witten's method was extended successfully to asymptotically Anti-de Sitter spacetimes and the positive mass theorem was proved completely and rigorously in this case [25, 10, 31, 17, 28]. As recent cosmological observations indicated that our universe possibly has a positive cosmological constant, the positive mass theorem for asymptotically de Sitter spacetimes gains much more importance. In spacetimes with a positive cosmological constant, a conserved quantity was defined in [1]. Certain discussions on its positivity and relevant issues can be found in [1, 16, 23, 24, 3, 11, 13].

In particular, the conformal mass associated with the time-like conformal Killing vector of the de Sitter spacetime was defined in planar coordinates for asymptotically de Sitter spacetimes and its positivity was discussed in [16].

The de Sitter spacetime is a maximally symmetric space with positive constant curvature. It is covered by global coordinates where each time slice is a 3-sphere with constant curvature and has no spatial infinity. One does not know how to define the total energy-momentum on this slice. However, half of the de Sitter spacetime can be covered by planar coordinates and each time slice is a 3-Euclidean space up to a conformal factor, whose second fundamental form is proportional to the metric. Using singular coordinate transformations, this half de Sitter spacetime can be covered by static coordinates where cosmological horizon occurs and time/distance switch from one to another when crossing the horizon.

It seems thus possible to build certain positive mass theorem for half of the de Sitter spacetime. However, as pointed out by Witten [27], there is no positive conserved energy in de Sitter spacetime, and the corresponding Killing vector fields to the Lorentzian generators are timelike in some region of de Sitter spacetime and spacelike in some other region (see also [1]). This indicates that there should not be the positive mass theorem in the standard sense without extra restriction. Mathematically, de Sitter spacetimes are characterized by Killing spinors with real Killing constants. When one tries to generalize Witten's argument to asymptotically de Sitter spacetimes, certain essential mathematical difficulties occur. To overcome these and to avoid using real Killing spinors, we need to reduce an initial data set in an asymptotically half de Sitter spacetime in planar coordinates to a standard asymptotically flat initial data set. The total energy, the total linear momentum and the total angular momentum can be defined via this asymptotically flat initial data and the positive mass theorem can be proved in this situation. The same philosophy applies to hyperbolic coordinates which cover a different half of the de Sitter spacetime. By using the hyperbolic positive mass theorem proved in [31, 28], we can prove the positive mass theorem in this case. Thus we get two different half-cuts using planar coordinates and hyperbolic coordinates and obtain two positive mass theorems, corresponding to the asymptotically flat and asymptotically hyperbolic positive mass theorems respectively. We emphasize that two different half-cuts of the de Sitter spacetime give rise to two different spatial infinities, therefore, two different ways to measure the total energy-momentum. It will be interesting to see how they relate to each other.

The definition of energy-momentum in planar coordinates in this paper is slightly different from the conformal mass defined by Kastor and Traschen [16] (see also [24]). Let  $g_0$  and  $K_0$  be the metric and the second fundamental form of the time slice in half de Sitter spacetime in planar coordinates. For a  $\mathcal{P}$ -asymptotically de Sitter initial data set  $(M, g, K)$ ,  $g - g_0$  on ends is used to define the total energy, the new tensor  $h = K - \sqrt{\frac{\Lambda}{3}}g$  is used to define the

total linear momentum (as well as the total angular momentum) here. But  $g - g_0$  and  $K - K_0$  on ends are used to define the conformal mass in [16]. We emphasize the conformal factor  $\mathcal{P}$  must be constant along spacelike hypersurfaces in the definition of  $\mathcal{P}$ -asymptotically de Sitter initial data sets. If the conformal factor  $\mathcal{P}$  is constant along the spacelike hypersurface and mean curvature of the spacelike hypersurface is bounded from above by certain constant, then the dominant energy condition for  $\mathcal{P}$ -asymptotically de Sitter spacetimes implies that for associated asymptotically flat spacetimes. Thus we can transfer to the asymptotically flat case and get the positivity by using the positive mass theorem for asymptotically flat spacetimes [20, 21, 22, 26, 29]. In [16], Witten's method was generalized directly to asymptotically de Sitter spacetimes in order to prove the positivity of the conformal mass under the assumption that the Dirac-Witten equation has solutions which are, at least through the first correction, eigenspinors of  $\gamma^{\hat{t}}$  (e.g., under (44), page 5911 [16]). In this case it does not need to use the positive mass theorem for asymptotically flat spacetimes.

We remark that geometrical/physical properties of de Sitter spacetime are quite different in planar coordinates and in static coordinates. For example, de Sitter spacetime has an apparent horizon  $\{r = \lambda\}$  in static coordinates. Inspired by certain physical properties in asymptotically de Sitter spacetimes in static coordinates, some conjectured that any such spacetime with mass greater than the mass of de Sitter has a cosmological singularity [4]. However, in planar coordinates, we can construct examples which contradict to this conjecture, at least in the "local" version. By a theorem of Corvino [12], there exists an asymptotically flat, scalar flat metric  $\bar{g}_t$  in  $\mathbb{R}^3$  with positive mass, which is identically Schwarzschild  $(1 + \frac{e^{-\frac{t}{\lambda}} m}{2r})^4 \bar{g}$  ( $m > 0$ ) near infinity. Now the initial data set  $(\mathbb{R}^3, e^{\frac{2t}{\lambda}} \bar{g}_t, \lambda^{-1} e^{\frac{2t}{\lambda}} \bar{g}_t)$  satisfies constraint equations and provides Cauchy data for vacuum Einstein fields equations with positive cosmological constant, which evolves into a nontrivial vacuum spacetime identical to Schwarzschild de Sitter in planar coordinates near spatial infinity. By the short time existence, this spacetime is smooth and free of singularity for short time and therefore provides a counterexample to the conjecture in this sense.

The paper is organized as follows. In Section 2, we discuss various coordinate systems for de Sitter spacetimes. Most of them are well-known (cf. [14]). In Section 3, we give definitions of  $\mathcal{P}$ -asymptotically de Sitter initial data sets and the total energy, the total linear momentum and the total angular momentum for these initial data sets. In Section 4, we prove the positive mass theorem for a  $\mathcal{P}$ -asymptotically de Sitter initial data set when its mean curvature is bounded from above by certain constant. In Section 5, we use hyperbolic coordinates to derive a positive mass theorem for an  $\mathcal{H}$ -asymptotically de Sitter initial data set. In Section 6, we discuss mean curvatures of time slices in certain asymptotically de Sitter spacetimes. In Section 7, we compute the total angular momentum for certain time slices in the Kerr-de Sitter spacetime.

## 2. DE SITTER SPACETIME

The de Sitter spacetime with cosmological constant  $\Lambda > 0$  is a hypersurface embedded into 5-dimensional Minkowski spacetime  $\mathbb{R}^{1,4}$

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \frac{3}{\Lambda} \quad (2.1)$$

with the induced metric (c.f. [14]). It is a maximally symmetric space with constant curvature. There are several other coordinates widely used for de Sitter spacetime. In particular, one has the planar (inflationary) coordinates  $(t, x^i)$  and the static coordinates  $(\vec{t}, \vec{r}, \vec{\theta}, \vec{\psi})$ . Denote

$$\Lambda = \frac{3}{\lambda^2}, \quad \lambda > 0. \quad (2.2)$$

The de Sitter spacetime (2.1) can be covered by the following global coordinates:

$$\begin{aligned} X^0 &= \lambda \sinh \frac{\vec{t}}{\lambda}, \\ X^1 &= \lambda \cosh \frac{\vec{t}}{\lambda} \sin \vec{r} \sin \vec{\theta} \cos \vec{\psi}, \\ X^2 &= \lambda \cosh \frac{\vec{t}}{\lambda} \sin \vec{r} \sin \vec{\theta} \sin \vec{\psi}, \\ X^3 &= \lambda \cosh \frac{\vec{t}}{\lambda} \sin \vec{r} \cos \vec{\theta}, \\ X^4 &= \lambda \cosh \frac{\vec{t}}{\lambda} \cos \vec{r}, \end{aligned} \quad (2.3)$$

where  $-\infty < \vec{t} < \infty$ ,  $0 \leq \vec{r} \leq \pi$ ,  $0 \leq \vec{\theta} \leq \pi$  and  $0 \leq \vec{\psi} \leq 2\pi$ . In global coordinates, the de Sitter metric is

$$\tilde{g}_{dS} = -d\vec{t}^2 + \lambda^2 \cosh^2 \frac{\vec{t}}{\lambda} (d\vec{r}^2 + \sin^2 \vec{r} (d\vec{\theta}^2 + \sin^2 \vec{\theta} d\vec{\psi}^2)) \quad (2.4)$$

And  $\vec{t}$ -slices are 3-spheres, which do not have spatial infinity.

The planar coordinates are defined as follows:

$$\begin{aligned} X^0 &= \lambda \sinh \frac{t}{\lambda} + \frac{\lambda}{2} \sum_{i=1}^3 \left(\frac{x^i}{\lambda}\right)^2 e^{\frac{t}{\lambda}}, \\ X^i &= x^i e^{\frac{t}{\lambda}} \quad (i = 1, 2, 3), \\ X^4 &= -\lambda \cosh \frac{t}{\lambda} + \frac{\lambda}{2} \sum_{i=1}^3 \left(\frac{x^i}{\lambda}\right)^2 e^{\frac{t}{\lambda}}, \end{aligned} \quad (2.5)$$

where  $-\infty < t < \infty$ ,  $-\infty < x_i < \infty$ . Since

$$X^0 - X^4 = \lambda e^{\frac{t}{\lambda}} > 0,$$

(2.5) covers only the upper-half de Sitter spacetime. And the lower-half part is covered by

$$\begin{aligned} X^0 &= -\lambda \sinh \frac{t}{\lambda} - \frac{\lambda}{2} \sum_{i=1}^3 \left(\frac{x^i}{\lambda}\right)^2 e^{\frac{t}{\lambda}}, \\ X^i &= x^i e^{\frac{t}{\lambda}} \quad (i = 1, 2, 3), \\ X^4 &= \lambda \cosh \frac{t}{\lambda} - \frac{\lambda}{2} \sum_{i=1}^3 \left(\frac{x^i}{\lambda}\right)^2 e^{\frac{t}{\lambda}}. \end{aligned} \tag{2.6}$$

When  $t$  goes from  $-\infty$  to  $\infty$ ,  $X^0 - X^4$  goes from 0 to  $\infty$  on the upper-half de Sitter spacetime while  $X^0 - X^4$  goes from 0 to  $-\infty$  on the lower-half part. However, de Sitter spacetime can not be fully covered by two planar coordinates since the hypersurface

$$X^0 - X^4 = 0$$

is excluded. In planar coordinates, the de Sitter metric is

$$\tilde{g}_{dS} = -dt^2 + e^{\frac{2t}{\lambda}} ((dx^1)^2 + (dx^2)^2 + (dx^3)^2). \tag{2.7}$$

The second fundamental form  $K$  of a  $t$ -slice is

$$K = \frac{1}{\lambda} g \tag{2.8}$$

with respect to the timelike unit normal  $\partial_t$ . In Minkowski spacetime  $\mathbb{R}^{1,4}$ ,  $\partial_t$  is upward in the upper-half de Sitter spacetime and downward in the lower-half part.

The static coordinates are defined as follows:

$$\begin{aligned} X^0 &= \sqrt{\lambda^2 - \bar{r}^2} \sinh \frac{\bar{t}}{\lambda}, \\ X^1 &= \bar{r} \sin \bar{\theta} \cos \bar{\psi}, \\ X^2 &= \bar{r} \sin \bar{\theta} \sin \bar{\psi}, \\ X^3 &= \bar{r} \cos \bar{\theta}, \\ X^4 &= -\sqrt{\lambda^2 - \bar{r}^2} \cosh \frac{\bar{t}}{\lambda}, \end{aligned} \tag{2.9}$$

where  $-\infty < \bar{t} < \infty$ ,  $0 < \bar{r} < \lambda$ ,  $0 \leq \bar{\theta} \leq \pi$  and  $0 \leq \bar{\psi} \leq 2\pi$ . Since

$$\begin{aligned} X^0 - X^4 &= \sqrt{\lambda^2 - \bar{r}^2} e^{\frac{\bar{t}}{\lambda}} > 0, \\ X^0 + X^4 &= -\sqrt{\lambda^2 - \bar{r}^2} e^{-\frac{\bar{t}}{\lambda}} < 0, \end{aligned}$$

(2.9) covers only a quarter of the de Sitter spacetime. The static coordinates for  $\bar{r} > \lambda$  are defined as follows:

$$\begin{aligned} X^0 &= \sqrt{\bar{r}^2 - \lambda^2} \cosh \frac{\bar{t}}{\lambda}, \\ X^1 &= \bar{r} \sin \bar{\theta} \cos \bar{\psi}, \\ X^2 &= \bar{r} \sin \bar{\theta} \sin \bar{\psi}, \\ X^3 &= \bar{r} \cos \bar{\theta}, \\ X^4 &= -\sqrt{\bar{r}^2 - \lambda^2} \sinh \frac{\bar{t}}{\lambda}. \end{aligned} \tag{2.10}$$

Since

$$\begin{aligned} X^0 - X^4 &= \sqrt{\bar{r}^2 - \lambda^2} e^{\frac{\bar{t}}{\lambda}} > 0, \\ X^0 + X^4 &= \sqrt{\bar{r}^2 - \lambda^2} e^{-\frac{\bar{t}}{\lambda}} > 0, \end{aligned}$$

(2.10) covers another quarter of the de Sitter spacetime. (2.9) and (2.10) cover the region which is covered by (2.5) excluding

$$X^0 + X^4 = 0$$

in planar coordinates. When  $\bar{t}$  goes from  $-\infty$  to  $\infty$ ,  $X^0 - X^4$  goes from 0 to  $\infty$  on the upper-half de Sitter spacetime while  $X^0 + X^4$  goes from  $-\infty$  to 0 in region  $\{\bar{r} < \lambda\}$  and it goes from  $\infty$  to 0 in region  $\{\bar{r} > \lambda\}$ .

Static coordinates of the lower-half de Sitter  $X^0 - X^4 < 0$  can be defined in a similar way. These four static coordinates cover the de Sitter spacetime excluding

$$X^0 \mp X^4 = 0$$

which correspond to  $\bar{r} = \lambda$  (cosmological horizons) for fixed  $\bar{t}$ . The de Sitter metric is

$$\tilde{g}_{dS} = -\left(1 - \frac{\bar{r}^2}{\lambda^2}\right) d\bar{t}^2 + \frac{d\bar{r}^2}{1 - \frac{\bar{r}^2}{\lambda^2}} + \bar{r}^2 (d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\psi}^2) \tag{2.11}$$

in the static coordinates.

The hyperbolic coordinates  $(T, R, \theta, \psi)$  are defined as follows:

$$\begin{aligned} X^0 &= \lambda \sinh \frac{T}{\lambda} \cosh \frac{R}{\lambda}, \\ X^1 &= \lambda \sinh \frac{T}{\lambda} \sinh \frac{R}{\lambda} \sin \theta \cos \psi, \\ X^2 &= \lambda \sinh \frac{T}{\lambda} \sinh \frac{R}{\lambda} \sin \theta \sin \psi, \\ X^3 &= \lambda \sinh \frac{T}{\lambda} \sinh \frac{R}{\lambda} \cos \theta, \\ X^4 &= -\lambda \cosh \frac{T}{\lambda}, \end{aligned} \tag{2.12}$$

where  $-\infty < T < \infty$ ,  $0 < R < \infty$ ,  $0 \leq \theta \leq \pi$  and  $0 \leq \psi \leq 2\pi$ . The hyperbolic coordinates cover half the de Sitter spacetime

$$X^4 < -\lambda.$$

The de Sitter metric in the hyperbolic coordinates is

$$\tilde{g}_{dS} = -dT^2 + \sinh^2 \frac{T}{\lambda} \left( dR^2 + \lambda^2 \sinh^2 \frac{R}{\lambda} (d\theta^2 + \sin^2 \theta d\psi^2) \right). \quad (2.13)$$

The second fundamental form  $K$  of a  $T$ -slice

$$K = \frac{1}{\lambda} \coth \frac{T}{\lambda} g. \quad (2.14)$$

Now we give the relation between planar coordinates and static coordinates. Let planar coordinates

$$\begin{aligned} x^1 &= r \sin \theta \cos \psi, \\ x^2 &= r \sin \theta \sin \psi, \\ x^3 &= r \cos \theta. \end{aligned} \quad (2.15)$$

By equating (2.5) to (2.9), and (2.5) to (2.10), we obtain in the region  $\bar{r} < \lambda$  of the upper-half de Sitter spacetime,

$$\begin{aligned} \bar{t} &= t - \frac{\lambda}{2} \ln \left( 1 - \frac{r^2 e^{\frac{2t}{\lambda}}}{\lambda^2} \right), \\ \bar{r} &= r e^{\frac{t}{\lambda}}, \\ \bar{\theta} &= \theta, \\ \bar{\psi} &= \psi. \end{aligned} \quad (2.16)$$

In the region  $\bar{r} > \lambda$  of the upper-half de Sitter spacetime, we have

$$\begin{aligned} \bar{t} &= t - \frac{\lambda}{2} \ln \left( \frac{r^2 e^{\frac{2t}{\lambda}}}{\lambda^2} - 1 \right), \\ \bar{r} &= r e^{\frac{t}{\lambda}}, \\ \bar{\theta} &= \theta, \\ \bar{\psi} &= \psi. \end{aligned} \quad (2.17)$$

Thus,  $t$ -slices in planar coordinates are not  $\bar{t}$ -slices in static coordinates and vice versa.

### 3. TOTAL ENERGY-MOMENTA

It is not a general fact that the positive mass theorem holds true for spacetimes with positive cosmological constant. However, we find that, in certain

special case, the mass is indeed nonnegative. From discussions in the previous section, we realize that cosmological horizons occur if static coordinates are used to cover the de Sitter spacetime. Although the hypersurface

$$X^0 - X^4 = 0$$

is excluded, planar coordinates are interesting in physics. In its early stage, the universe is generally believed to be through a phase of inflation, which can be described by a de Sitter spacetime in planar coordinates. On the other hand, recent cosmological observations indicate that the expansion of the universe is accelerating, so the future of our universe may well be again described by a de Sitter spacetime in these coordinates. In this section, it will be used to study the positive mass theorem for  $\mathcal{P}$ -asymptotically de Sitter spacetimes in a mathematically rigorous and complete way.

Suppose  $(N^{1,3}, \tilde{g})$  is a spacetime which satisfies the Einstein equations with a positive cosmological constant  $\Lambda$

$$\tilde{R}_{\alpha\beta} - \frac{\tilde{R}}{2}\tilde{g}_{\alpha\beta} + \Lambda\tilde{g}_{\alpha\beta} = T_{\alpha\beta} \quad (3.1)$$

where  $0 \leq \alpha, \beta \leq 3$ .  $N^{1,3}$  satisfies the dominant energy condition if, for any timelike vector  $W$ ,

- (i)  $T_{uv}W^uW^v \geq 0$ ;
- (ii)  $T^{uv}W_u$  is a non-spacelike vector.

In a frame, we obtain

$$\begin{aligned} T^{00} &\geq \sqrt{T^{0i}T_i^0}, \\ T^{00} &\geq |T^{\alpha\beta}|. \end{aligned} \quad (3.2)$$

Let  $(M, g, K)$  be a spacelike hypersurface with induced Riemannian metric  $g$  and second fundamental form  $K$ . Choose a frame  $e_\alpha$  in  $N^{1,3}$  such that  $e_0$  is normal to  $M$  and  $e_i$  is tangential to  $M$ . By the Gauss and Codazzi equations,

$$\begin{aligned} T_{00} &= \frac{1}{2}(R + (tr_g K)^2 - |K|_g^2 - 2\Lambda), \\ T_{0i} &= \nabla^j (K_{ij} - g_{ij} tr_g K), \end{aligned} \quad (3.3)$$

where  $\nabla_i$  is the covariant derivative and  $R$  is the scalar curvature of  $g$ .

The tensor

$$h = K - \sqrt{\frac{\Lambda}{3}}g. \quad (3.4)$$

will be used to define the total momenta.

Inspired by the metric and the second fundamental form (2.7), (2.8) of a  $t$ -slice in de Sitter spacetime, we can define  $\mathcal{P}$ -asymptotically de Sitter initial data sets.

**Definition 3.1.** An initial data set  $(M, g, K)$  is  $\mathcal{P}$ -asymptotically de Sitter of order  $\tau > \frac{1}{2}$  if there is a compact set  $M_c$  such that  $M - M_c$  is the disjoint union of a finite number of subsets  $M_1, \dots, M_k$  - called the “ends” of  $M$  - each diffeomorphic to  $\mathbb{R}^3 - B_r$  where  $B_r$  is the closed ball of radius  $r$  with center at the coordinate origin. And  $g$  and  $h$  are of the forms

$$\begin{aligned} g &= \mathcal{P}^2 \bar{g}, \\ h &= \mathcal{P} \bar{h} \end{aligned} \tag{3.5}$$

along  $M$ , where  $\mathcal{P}$  is a certain spacetime function which is a positive constant along  $M$  and tensors  $\bar{g}, \bar{h}$  satisfy the following conditions

$$\bar{g} - \check{g} \in C_{-\tau}^{2,\alpha}(M), \bar{h} \in C_{-\tau-1}^{0,\alpha}(M), \text{tr}_{\bar{g}}(\bar{h}) \in C_{-\tau-1}^{1,\alpha}(M) \tag{3.6}$$

for certain  $\tau > \frac{1}{2}$ ,  $0 < \alpha < 1$ , where  $\check{g}$  is the standard metric of  $\mathbb{R}^3$ ,  $C_{-\tau}^{k,\alpha}$  is weighted Hölder spaces (c.f. [6]).

Denote the metric  $\check{g}_{\mathcal{P}} = \mathcal{P}^2 \check{g}$ . As  $(M, \bar{g}, \bar{h})$  is asymptotically flat in the standard sense [6, 29], the total energy, the total linear momentum and the total angular momentum can be defined in the standard way. Let  $\{x^i\}$  be natural coordinates of  $\mathbb{R}^3$ ,  $\bar{g}_{ij} = \bar{g}(\partial_i, \partial_j)$ ,  $\bar{h}_{ij} = \bar{h}(\partial_i, \partial_j)$  and

$$\tilde{h}_{ij}^z = \frac{1}{2} \epsilon_i^{uv} (\bar{\nabla}_u \rho_z^2) (\bar{h}_{vj} - \bar{g}_{vj} \text{tr}_{\bar{g}}(\bar{h}))$$

where  $z$  is a point in  $M$  and  $\rho_z$  is the distance function beginning at  $z$ . The tensor  $\tilde{h}_{ij}^z$  is trace-free and measures the rotation of the system with respect to  $z$ . It will be referred to as the local angular momentum density tensor at point  $z$ . We suppose that

$$\tilde{h}^z \in C_{-\tau-1}^{0,\alpha}(M), \epsilon^{kij} \bar{\nabla}_k (\tilde{h}_{ij}^z - \tilde{h}_{ji}^z) \in L_{\frac{q}{2}, -\tau-2}, \bar{\nabla}^i (\tilde{h}_{ij}^z - \tilde{h}_{ji}^z) \in L_{\frac{q}{2}, -\tau-2} \tag{3.7}$$

for some  $q > 3$ , where  $L_{p,\tau}$  is weighted Sobolev spaces (c.f. [6]). We further assume

$$R \in L^1(M), T_{0i} \in L^1(M), \bar{\nabla}^i \tilde{h}_{ji}^z \in L^1(M). \tag{3.8}$$

Under (3.6), (3.7) and (3.8) the total energy  $\bar{E}_l$ , the total linear momentum  $\bar{P}_{lk}$  and the total angular momentum  $\bar{J}_{lk}(z)$  with respect to some point  $z$  of the end  $M_l$  are defined as follows [2, 19, 29]

$$\begin{aligned} \bar{E}_l &= \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_{r,l}} (\partial_j \bar{g}_{ij} - \partial_i \bar{g}_{jj}) * dx^i, \\ \bar{P}_{lk} &= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_{r,l}} (\bar{h}_{ki} - \bar{g}_{ki} \text{tr}_{\bar{g}}(\bar{h})) * dx^i, \\ \bar{J}_{lk}(z) &= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_{r,l}} \tilde{h}_{ki}^z * dx^i. \end{aligned} \tag{3.9}$$

Definitions of  $\bar{E}_l, \bar{P}_{lk}, \bar{J}_{lk}(z)$  are independent of the choice of local coordinates on ends [6, 9, 29, 30].

**Definition 3.2.** The total energy  $E_l$ , the total linear momentum  $P_{lk}$  and the total angular momentum  $J_{lk}$  of the end  $M_l$  are defined as

$$E_l = \mathcal{P}\bar{E}_l, \quad P_{lk} = \mathcal{P}^2\bar{P}_{lk}, \quad J_{lk}(z) = \mathcal{P}^2\bar{J}_{lk}(z) \quad (3.10)$$

for a  $\mathcal{P}$ -asymptotically de Sitter initial data set  $(M, g, K)$ .

**Remark 3.1.** Certain functional spaces are used in the definition of  $\mathcal{P}$ -asymptotically de Sitter initial data sets so that the positivity follows from the positive mass theorem proved in [29] for asymptotically flat spacetimes.

**Remark 3.2.** Denote  $\{\check{e}_i = \mathcal{P}^{-1}\partial_i\}$  and  $\{\check{e}^i = \mathcal{P}dx^i\}$ . It is easy to check that

$$E_l = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_{r,l}} [\check{e}_j(g(\check{e}_i, \check{e}_j)) - \check{e}_i(g(\check{e}_j, \check{e}_i))] *_{\check{g}_{\mathcal{P}}} \check{e}^i,$$

$$P_l(\check{e}_k) = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_{r,l}} [h(\check{e}_k, \check{e}_i) - g(\check{e}_k, \check{e}_i)tr_g(h)] *_{\check{g}_{\mathcal{P}}} \check{e}^i.$$

**Definition 3.3.** A future/past apparent horizon in a  $\mathcal{P}$ -asymptotically de Sitter initial data set  $(M, g, K)$  is a 2-sphere  $\Sigma$  whose trace of the second fundamental form  $H_\Sigma$  satisfies

$$\pm H_\Sigma = tr_{g_\Sigma}(h|_\Sigma) = tr_{g_\Sigma}(K|_\Sigma) - 2\sqrt{\frac{\Lambda}{3}}. \quad (3.11)$$

**Remark 3.3.** Since it is spacelike,  $\Sigma$  admits two smooth nonvanishing outward pointing null normal vector fields  $V_\pm$  with  $V_+$  future directed and  $V_-$  past directed. Let  $S_\pm$  be the smooth null hypersurfaces near  $\Sigma$  generated by the null geodesics with initial tangents  $V_\pm$ . Then

$$\theta_\pm = div_\Sigma V_\pm = H_\Sigma \pm tr_g(K|_\Sigma)$$

are the null expansion of  $S_\pm$  respectively which measure the overall outward expansion of the future and past going light rays emanating from  $\Sigma$ .  $\Sigma$  is a physical future/past apparent horizon if  $\theta_\pm = 0$  respectively. Thus the physical apparent horizon is not the apparent horizon defined by (3.11) (or by (5.6) in Section 5).

In planar coordinates, the Schwarzschild-de Sitter metric can be written as (the McVittie form, cf. [16])

$$\tilde{g}_{Sch} = -\frac{(1 - \frac{m}{2Ar})^2}{(1 + \frac{m}{2Ar})^2} dt^2 + A^2(1 + \frac{m}{2Ar})^4 \delta_{ij} dx^i dx^j$$

where  $A(t) = e^{\frac{t}{\lambda}}$ . It is easy to check that, for a  $t$ -slice,

$$E = m, \quad P_k = 0, \quad J_k(z) = 0.$$

Furthermore, if  $m > 0$ ,  $\{r = \frac{m}{2A}\}$  is a minimal 2-sphere in a  $t$ -slice, therefore the trace of its second fundamental form satisfies (3.11) and it is an apparent horizon.

## 4. POSITIVE MASS THEOREM

Suppose  $(M, g, K)$  is a  $\mathcal{P}$ -asymptotically de Sitter initial data set. Let  $(M, \bar{g}, p)$  be a generalized asymptotically flat initial data set of order  $\tau$ ,  $1 \geq \tau > \frac{1}{2}$ , in the sense of [29], which means that  $p$  is an arbitrary and not necessarily symmetric 2-tensor. Suppose  $(M, \bar{g}, p)$  satisfies the following conditions: there exists certain compact set  $\bar{M}_c \supset M_c$  such that the anti-symmetric part  $p^a$ ,  $tr_{\bar{g}}(p)$  are bounded on  $\bar{M}_c$ . Furthermore,

$$p \in C_{-\tau-1}^{0,\alpha}(M - \bar{M}_c), tr_{\bar{g}}(p) \in W_{-\tau-1}^{1,\frac{q}{2}}(M), \{d\theta, d^*\theta\} \in L_{\frac{q}{2}, -\tau-2}(M)$$

where  $\theta$  is the associated 2-form of  $p^a$ . Suppose  $(M, \bar{g}, p)$  has possibly a finite number of future/past apparent horizons  $\Sigma_i$ , each  $\Sigma_i$  is a 2-sphere whose trace of the second fundamental form  $\bar{H}_{\Sigma_i}$  satisfies

$$\pm \bar{H}_{\Sigma_i} = tr_{\bar{g}_{\Sigma_i}}(p|_{\Sigma_i}). \quad (4.1)$$

**Proposition 4.1.** *Conditions (3.11) and (4.1) are equivalent for  $p = \bar{h}$ .*

*Proof:* Note that  $g = \mathcal{P}^2 \bar{g}$ ,  $h = \mathcal{P} \bar{h}$ , and  $\mathcal{P}$  is constant along  $M$  which also produces a rescaling with the same factor on  $\Sigma$ . The respective mean curvatures of  $\Sigma$  thus have the relation

$$H_g = \mathcal{P}^{-1} H_{\bar{g}}.$$

On the other hand,

$$tr_g(h) = \mathcal{P}^{-2} tr_{\bar{g}}(\mathcal{P} \bar{h}) = \mathcal{P}^{-1} tr_{\bar{g}}(\bar{h}).$$

Therefore the proposition is proved. Q.E.D.

For  $(M, \bar{g}, p)$ , the total energy is defined the same as  $\bar{E}_l$  in (3.9), the total “linear” momentum is defined as

$$\bar{P}_k = \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_{r,l}} (p_{ki} - \bar{g}_{ki} tr_{\bar{g}}(p)) * dx^i.$$

However, unlike the case of symmetric 2-tensor  $h$ , the total “linear” momentum contains both translation and rotation. Denote

$$\mu = \frac{1}{2} (\bar{R} + (tr_{\bar{g}}(p))^2 - |p|_{\bar{g}}^2), \omega_j = \bar{\nabla}^i p_{ji} - \bar{\nabla}_j tr_{\bar{g}}(p), \chi_j = 2\bar{\nabla}^i (p_{ij} - p_{ji}).$$

$(M, \bar{g}, p)$  satisfies the generalized dominant energy condition if

$$\mu \geq \max \{ |\omega|_{\bar{g}}, |\omega + \chi|_{\bar{g}} \}. \quad (4.2)$$

We employ the following condition on the trace of the second fundamental form of  $M$  along  $M$

$$tr_g(K) \leq \sqrt{3\Lambda}. \quad (4.3)$$

**Proposition 4.2.** *Under the assumption (4.3), the first one of the dominant energy condition (3.2) implies (4.2) for symmetric  $p = \bar{h}$ .*

*Proof:* Let  $\{\bar{e}_i\}$  be an orthonormal frame of  $\bar{g}$  and set  $K_{ij} = K(\bar{e}_i, \bar{e}_j)$ . Thus  $\{e_i = \mathcal{P}^{-1}\bar{e}_i\}$  forms an orthonormal basis of  $g$ . Straightforward computation yields

$$\begin{aligned}\mu &= \frac{1}{2} \left( \bar{R} + \mathcal{P}^{-2} (\text{tr}_{\bar{g}}(K) - \frac{3\mathcal{P}^2}{\lambda})^2 - \mathcal{P}^{-2} |K - \frac{\mathcal{P}^2}{\lambda} \bar{g}|_{\bar{g}}^2 \right) \\ &= \mathcal{P}^2 T_{00} + \frac{6\mathcal{P}^2}{\lambda^2} - \frac{2\mathcal{P}^2}{\lambda} \text{tr}_g(K)\end{aligned}$$

Under (4.3) and (3.2), we have

$$\begin{aligned}\mu &\geq \mathcal{P}^2 \sqrt{\sum_i (\nabla_{e_j} K(e_i, e_j) - \nabla_{e_i} K(e_j, e_j))^2} \\ &= \mathcal{P}^{-1} \sqrt{\sum_i (\bar{\nabla}_{\bar{e}_j} K_{ij} - \bar{\nabla}_{\bar{e}_i} K_{jj})^2} \\ &= \sqrt{\sum_i (\bar{\nabla}_j \bar{h}_{ij} - \bar{\nabla}_i \bar{h}_{jj})^2}.\end{aligned}$$

Q.E.D.

Note that  $\check{g}_{ij} = \delta_{ij}$ ,  $\check{g}_{\mathcal{P}ij} = \mathcal{P}^2 \delta_{ij}$ . Recall the positive mass theorem proved in [29].

**Theorem 4.1** (Zhang). *Let  $(M, \bar{g}, p)$  be a generalized asymptotically flat initial data set of order  $1 \geq \tau > \frac{1}{2}$  which has possibly a finite number of apparent horizons. If the generalized dominant energy condition (4.2) holds, then for each end  $M_l$ ,*

$$\bar{E}_l \geq |\bar{P}_l|_{\check{g}}. \quad (4.4)$$

*If equality holds in (4.4) for some end  $M_{l_0}$ , then  $M$  has only one end. Furthermore, if  $E_{l_0} = 0$  and  $\bar{g}$  is  $C^2$ ,  $p$  is  $C^1$ , then the following equations hold on  $M$*

$$\bar{R}_{ijkl} + p_{ik}p_{jl} - p_{il}p_{jk} = 0, \quad \bar{\nabla}_i p_{jk} - \bar{\nabla}_j p_{ik} = 0, \quad \bar{\nabla}^i (p_{ij} - p_{ji}) = 0. \quad (4.5)$$

By Theorem 4.1, or the original positive mass theorem [20, 21, 22, 26, 18, 29], we obtain

**Theorem 4.2.** *Let  $(M, g, K)$  be a  $\mathcal{P}$ -asymptotically de Sitter initial data set of order  $1 \geq \tau > \frac{1}{2}$  which has possibly a finite number of apparent horizons in spacetime  $(N^{1,3}, \tilde{g})$  with positive cosmological constant  $\Lambda > 0$ . Suppose  $N^{1,3}$  satisfies the dominant energy condition (3.2). If (4.3) holds along  $M$ , then for each end  $M_l$ ,*

$$E_l \geq |P_l|_{\check{g}_{\mathcal{P}}}. \quad (4.6)$$

*If  $E_l = 0$  for some end  $M_{l_0}$ , then*

$$(M, g, K) \equiv (\mathbb{R}^3, \mathcal{P}^2 \check{g}, \sqrt{\frac{\Lambda}{3}} \mathcal{P}^2 \check{g}). \quad (4.7)$$

Moreover, the mean curvature achieves the equality in (4.3) and the space-time  $(N^{1,3}, \tilde{g})$  is de Sitter along  $M$ . In particular,  $(N^{1,3}, \tilde{g})$  is globally de Sitter in the planar coordinates if it is globally hyperbolic.

*Proof:* Let  $\{\bar{e}_i\}$  be an orthonormal frame of  $\bar{g}$ , then  $\{e_i = \mathcal{P}^{-1}\bar{e}_i\}$  forms an orthonormal basis of  $g$ . Since

$$|\bar{P}_l|_{\tilde{g}} = \mathcal{P}^{-1}|P_l|_{\tilde{g}_{\mathcal{P}}},$$

the positivity of mass (4.6) is a straightforward consequence of Theorem 4.1. When  $E_l = 0$ , the first equality in (4.5) implies that

$$\langle \bar{R}(\bar{e}_i, \bar{e}_j)\bar{e}_l, \bar{e}_k \rangle_{\bar{g}} = -\bar{h}(\bar{e}_i, \bar{e}_k)\bar{h}(\bar{e}_j, \bar{e}_l) + \bar{h}(\bar{e}_i, \bar{e}_l)\bar{h}(\bar{e}_j, \bar{e}_k).$$

By the Gauss equation of  $(M, g, K)$  in  $(N^{1,3}, \tilde{g})$ , one has

$$\begin{aligned} \tilde{R}_{ijkl} &= \langle R(e_i, e_j)e_l, e_k \rangle_g + K(e_i, e_k)K(e_j, e_l) - K(e_i, e_l)K(e_j, e_k) \\ &= \mathcal{P}^{-2} \left( -\bar{h}(\bar{e}_i, \bar{e}_k)\bar{h}(\bar{e}_j, \bar{e}_l) + \bar{h}(\bar{e}_i, \bar{e}_l)\bar{h}(\bar{e}_j, \bar{e}_k) \right) \\ &\quad + K(e_i, e_k)K(e_j, e_l) - K(e_i, e_l)K(e_j, e_k) \\ &= -h_{ik}h_{jl} + h_{il}h_{jk} + \left(h_{ik} + \frac{1}{\lambda}g_{ik}\right)\left(h_{jl} + \frac{1}{\lambda}g_{jl}\right) \\ &\quad - \left(h_{il} + \frac{1}{\lambda}g_{il}\right)\left(h_{jk} + \frac{1}{\lambda}g_{jk}\right) \\ &= \frac{1}{\lambda}(g_{ik}h_{jl} + g_{jl}h_{ik} - g_{il}h_{jk} - g_{jk}h_{il}) + \frac{1}{\lambda^2}(g_{ik}g_{jl} - g_{il}g_{jk}) \end{aligned}$$

along  $M$ . By the second equality in (4.5) (i.e., Codazzi equations), we obtain

$$\tilde{R}_{0jkl} = 0.$$

Note that the first equality in (4.5) also implies that  $\mu = 0$ . From the computation in the proof of Proposition 4.2, we have

$$T_{00} + \frac{6}{\lambda^2} - \frac{2}{\lambda}tr_g(K) = 0.$$

Thus the dominant energy conditions (3.2) and (4.3) imply

$$\begin{aligned} T_{\alpha\beta} &= 0, \\ tr_g(K) &= \frac{3}{\lambda}. \end{aligned}$$

Therefore  $N^{1,3}$  is vacuum and

$$tr_{\bar{g}}(\bar{h}) = 0. \tag{4.8}$$

The vanishing mass for  $(M, \bar{g}, \bar{h})$  under (4.8) implies that

$$(M, \bar{g}, \bar{h}) \equiv (\mathbb{R}^3, \check{g}, 0).$$

Therefore (4.7) holds and

$$\tilde{R}_{ijkl} = \frac{1}{\lambda^2}(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk})$$

along  $M$ . Finally, we compute  $\tilde{R}_{0j0l}$ .

$$\begin{aligned}\tilde{g}^{00}\tilde{R}_{0j0l} &= -\tilde{g}^{ik}\tilde{R}_{ijkl} + \frac{3}{\lambda^2}\tilde{g}_{jl} \\ &= -\frac{1}{\lambda^2}\tilde{g}^{ik}(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}) + \frac{3}{\lambda^2}\tilde{g}_{jl} \\ &= \frac{1}{\lambda^2}\tilde{g}_{jl}\end{aligned}$$

along  $M$ . Thus  $(N^{1,3}, \tilde{g})$  is de Sitter along  $M$ . In particular, if it is globally hyperbolic, by the theorem of Christodoulou and Klainerman [8], we know that  $(M, \tilde{g}, \tilde{h})$  develops the Minkowski spacetime  $\mathbb{R}^{1,3}$  which implies that  $(N^{1,3}, \tilde{g})$  is de Sitter. Q.E.D.

**Remark 4.1.** *In general, the function  $\mathcal{P}$  in an initial data set  $(M, g, K)$  is a nonconstant function. In this case, the positive mass theorem (e.g. Theorem 4.2) does not hold true.*

We can also apply Theorem 4.1 to obtain certain positive mass theorem including the total angular momentum for asymptotically de Sitter spacetimes.

**Theorem 4.3.** *Let  $(M, g, K)$  be a  $\mathcal{P}$ -asymptotically de Sitter initial data set of order  $1 \geq \tau > \frac{1}{2}$  which has no apparent horizon in spacetime  $(N^{1,3}, \tilde{g})$  with positive cosmological constant  $\Lambda > 0$ . Suppose that there exists a point  $z \in M$  such that the local angular momentum tensor  $\tilde{h}^z$  satisfies (3.7), (3.8). If (4.2) holds for  $p = C_1\tilde{h} + C_2\tilde{h}^z$  where  $C_1$  and  $C_2$  are real constants, then for each end  $M_l$ , we have*

$$E_l \geq |C_1P_l + C_2J_l(z)|_{\tilde{g}_P}. \quad (4.9)$$

If equality holds in (4.9) for some end  $M_{l_0}$ , then  $M$  has only one end. Furthermore, if  $E_{l_0} = 0$  and  $g_{ij}$  is  $C^2$ ,  $p_{ij}$  is  $C^1$ , then (4.5) holds for this  $p$ .

## 5. HYPERBOLIC COORDINATES AND POSITIVE MASS THEOREM

In this section, we use hyperbolic coordinates and the positive mass theorem for asymptotically hyperbolic manifolds proved in [31, 28] to derive a positive mass theorem for asymptotically half de Sitter spacetime. Let

$$\check{g}_{\mathcal{H}} = dR^2 + \lambda^2 \sinh^2 \frac{R}{\lambda} (d\theta^2 + \sin^2 \theta d\psi^2). \quad (5.1)$$

The frame of the metric is

$$\check{e}_{\mathcal{H}}^1 = \frac{\partial}{\partial R}, \check{e}_{\mathcal{H}}^2 = \frac{1}{\lambda \sinh \frac{R}{\lambda}} \frac{\partial}{\partial \theta}, \check{e}_{\mathcal{H}}^3 = \frac{1}{\lambda \sinh \frac{R}{\lambda} \sin \theta} \frac{\partial}{\partial \psi}$$

and the coframe is

$$\check{e}_{\mathcal{H}}^1 = dR, \check{e}_{\mathcal{H}}^2 = \lambda \sinh \frac{R}{\lambda} d\theta, \check{e}_{\mathcal{H}}^3 = \lambda \sinh \frac{R}{\lambda} \sin \theta d\psi.$$

Denote the 4-vector  $n^\nu$  ( $\nu = 0, 1, 2, 3$ )

$$n^0 = 1, \quad n^1 = \sin \theta \cos \psi, \quad n^2 = \sin \theta \sin \psi, \quad n^3 = \cos \theta.$$

**Definition 5.1.** *An initial data set  $(M, g, K)$  is  $\mathcal{H}$ -asymptotically de Sitter of order  $\tau > \frac{3}{2}$  if there is a compact set  $M_c$  such that  $M - M_c$  is the disjoint union of a finite number of subsets  $M_1, \dots, M_k$  - called the “ends” of  $M$  - each diffeomorphic to  $\mathbb{R}^3 - B_R$  where  $B_R$  is the closed ball of radius  $R$  with center at the coordinate origin. Suppose there exists a spacetime function  $T$  which is constant along  $M$ . Let*

$$h = K - \frac{\coth \frac{T}{\lambda}}{\lambda} g. \quad (5.2)$$

And  $g$  and  $h$  are of the forms

$$\begin{aligned} g &= \mathcal{H}^2 \bar{g}, \\ h &= \mathcal{H} \bar{h} \end{aligned} \quad (5.3)$$

along  $M$ , where  $\mathcal{H} = \sinh \frac{T}{\lambda}$ , and

$$\bar{g}(\check{e}_i^{\mathcal{H}}, \check{e}_j^{\mathcal{H}}) - \check{g}(\check{e}_i^{\mathcal{H}}, \check{e}_j^{\mathcal{H}}) = a_{ij}, \quad \bar{h}(\check{e}_i^{\mathcal{H}}, \check{e}_j^{\mathcal{H}}) = \bar{h}_{ij}$$

have the following asymptotic behavior on the end:

$$\begin{aligned} a_{ij} &= O(e^{-\frac{\tau}{\lambda} R}), \quad \check{\nabla}_k^{\mathcal{H}} a_{ij} = O(e^{-\frac{\tau}{\lambda} R}), \quad \check{\nabla}_i^{\mathcal{H}} \check{\nabla}_k^{\mathcal{H}} a_{ij} = O(e^{-\frac{\tau}{\lambda} R}), \\ \bar{h}_{ij} &= O(e^{-\frac{\tau}{\lambda} R}), \quad \check{\nabla}_k^{\mathcal{H}} \bar{h}_{ij} = O(e^{-\frac{\tau}{\lambda} R}), \end{aligned} \quad (5.4)$$

where  $\check{\nabla}^{\mathcal{H}}$  is the Levi-Civita connection of the hyperbolic metric  $\check{g}_{\mathcal{H}}$ . Moreover,

$$\left(\bar{R} + \frac{6}{\lambda^2}\right) e^{\rho_z} \in L^1(M), \quad (\bar{\nabla}^j \bar{h}_{ij} - \bar{\nabla}_i \text{tr}_{\bar{g}}(\bar{h})) e^{\rho_z} \in L^1(M) \quad (5.5)$$

for some  $z \in M$ . Here  $\bar{R}$ ,  $\bar{\nabla}$ ,  $\rho_z$  are scalar curvature, Levi-Civita connection and distance function of  $\bar{g}$  respectively.

**Definition 5.2.** *A future/past apparent horizon in an  $\mathcal{H}$ -asymptotically de Sitter initial data set  $(M, g, K)$  is a 2-sphere  $\Sigma$  whose trace of the second fundamental form  $H_\Sigma$  satisfies*

$$\pm H_\Sigma = \text{tr}_{g_\Sigma}(K|_\Sigma) - 2\sqrt{\frac{\Lambda}{3}} \tanh \frac{T}{2\lambda}. \quad (5.6)$$

Let

$$\tilde{h} = \bar{h} + \frac{1}{\lambda} \bar{g}. \quad (5.7)$$

The condition (5.6) implies that

$$\pm \bar{H}_\Sigma = \text{tr}_{\bar{g}_\Sigma}(\bar{h}|_\Sigma) + 2\sqrt{\frac{\Lambda}{3}} = \text{tr}_{\bar{g}_\Sigma}(\tilde{h}|_\Sigma) \quad (5.8)$$

which shows that  $\Sigma$  is an apparent horizon of  $(M, \bar{g}, \bar{h})$  [31, 28]. Denote

$$\mathcal{E}_l = \check{\nabla}^{\mathcal{H},j} \bar{g}_{1j} - \check{\nabla}_1^{\mathcal{H}} \text{tr}_{\check{g}_{\mathcal{H}}}(\bar{g}) + \frac{1}{\lambda}(a_{22} + a_{33}) + 2(\bar{h}_{22} + \bar{h}_{33}). \quad (5.9)$$

The total energy-momentum of the end  $M_l$  are defined as

$$E_{l\nu}^{\mathcal{H}} = \frac{\mathcal{H}^2}{16\pi} \lim_{R \rightarrow \infty} \int_{S_{R,l}} \mathcal{E}_l n^\nu e^{\frac{R}{\lambda}} \check{e}_{\mathcal{H}}^2 \wedge \check{e}_{\mathcal{H}}^3 \quad (5.10)$$

where  $0 \leq \nu \leq 3$ ,  $S_{R,l}$  is the coordinate sphere of radius  $R$  in the end  $M_l$ .

Now we study the relation between the dominant energy condition of the  $\mathcal{H}$ -asymptotically de Sitter initial data set and its associated asymptotically hyperbolic initial data set. Suppose that  $(M, \bar{g}, \bar{h})$  is an asymptotically hyperbolic initial data set of order  $\tau > \frac{3}{2}$ . Denote

$$\mu = \frac{1}{2}(\bar{R} + (\text{tr}_{\bar{g}}(\tilde{h}))^2 - |\tilde{h}|_{\bar{g}}^2), \quad \omega_j = \bar{\nabla}^i \tilde{h}_{ji} - \bar{\nabla}_j \text{tr}_{\bar{g}}(\tilde{h}).$$

Since

$$\begin{aligned} \text{tr}_{\bar{g}}(\tilde{h}) &= \text{tr}_{\bar{g}}(\bar{h}) + \frac{3}{\lambda} \\ &= \mathcal{H} \text{tr}_g(h) + \frac{3}{\lambda} \\ &= \mathcal{H} \text{tr}_g(K) - \frac{3}{\lambda} \cosh \frac{T}{\lambda} + \frac{3}{\lambda}, \\ |\tilde{h}|_{\bar{g}}^2 &= (\bar{h}_{ij} + \frac{1}{\lambda} \bar{g}_{ij})(\bar{h}_{kl} + \frac{1}{\lambda} \bar{g}_{kl}) \bar{g}^{ik} \bar{g}^{jl} \\ &= \mathcal{H}^2 |h|_g^2 + \frac{2\mathcal{H}}{\lambda} \text{tr}_g(h) + \frac{3}{\lambda^2} \\ &= \mathcal{H}^2 (|K|_g^2 - \frac{2}{\lambda} \coth \frac{T}{\lambda} \text{tr}_g(K) + \frac{3}{\lambda^2} \coth^2 \frac{T}{\lambda}) \\ &\quad + \frac{2\mathcal{H}}{\lambda} \text{tr}_g(K) - \frac{6}{\lambda^2} \cosh \frac{T}{\lambda} + \frac{3}{\lambda^2} \\ &= \mathcal{H}^2 |K|_g^2 + \frac{2}{\lambda} \sinh \frac{T}{\lambda} (1 - \cosh \frac{T}{\lambda}) \text{tr}_g(K) \\ &\quad + \frac{3}{\lambda^2} \cosh^2 \frac{T}{\lambda} - \frac{6}{\lambda^2} \cosh \frac{T}{\lambda} + \frac{3}{\lambda^2}, \end{aligned}$$

we obtain

$$\begin{aligned} \mu &= \mathcal{H}^2 T_{00} + \frac{2}{\lambda} \text{tr}_g(K) \sinh \frac{T}{\lambda} (1 - \cosh \frac{T}{\lambda}) \\ &\quad + \frac{3}{\lambda^2} \sinh^2 \frac{T}{\lambda} + \frac{3}{\lambda^2} \cosh^2 \frac{T}{\lambda} - \frac{6}{\lambda^2} \cosh \frac{T}{\lambda} + \frac{3}{\lambda^2} \\ &= \mathcal{H}^2 T_{00} + \frac{2}{\lambda} (\text{tr}_g(K) \sinh \frac{T}{\lambda} - \frac{3}{\lambda} \cosh \frac{T}{\lambda}) (1 - \cosh \frac{T}{\lambda}), \\ \omega_j &= \mathcal{H}^2 T_{0i}. \end{aligned}$$

Therefore, if

$$\operatorname{tr}_g(K) \sinh \frac{T}{\lambda} \leq \sqrt{3\Lambda} \cosh \frac{T}{\lambda}, \quad (5.11)$$

then the dominant energy condition (3.2) implies that

$$\mu \geq |\omega|_{\tilde{g}}. \quad (5.12)$$

**Theorem 5.1.** *Let  $(M, g, K)$  be an  $\mathcal{H}$ -asymptotically de Sitter initial data set of order  $\tau > \frac{3}{2}$  which has possibly a finite number of apparent horizons in spacetime  $(N^{1,3}, \tilde{g})$  with positive cosmological constant  $\Lambda > 0$ . Suppose  $N^{1,3}$  satisfies the dominant energy condition (3.2). If (5.11) holds along  $M$ , then for each end  $M_l$ ,*

$$E_{l0}^{\mathcal{H}} \geq \sqrt{(E_{l1}^{\mathcal{H}})^2 + (E_{l2}^{\mathcal{H}})^2 + (E_{l3}^{\mathcal{H}})^2}. \quad (5.13)$$

If  $E_{l0}^{\mathcal{H}} = 0$  for some end  $M_{l_0}$ , then

$$(M, g, K) \equiv (\mathbb{H}^3, \sinh^2 \frac{T}{\lambda} \check{g}_{\mathcal{H}}, \sqrt{\frac{\Lambda}{3}} \sinh \frac{T}{\lambda} \cosh \frac{T}{\lambda} \check{g}_{\mathcal{H}}). \quad (5.14)$$

Moreover, the mean curvature achieves the equality in (5.11) and the spacetime  $(N^{1,3}, \tilde{g})$  is de Sitter along  $M$ . In particular,  $(N^{1,3}, \tilde{g})$  is globally de Sitter in the hyperbolic coordinates if it is globally hyperbolic.

*Proof:* The first part of the theorem, i.e., inequality (5.13), is straightforward since the condition (5.12) ensures the hyperbolic positive mass theorem proved in [31, 28]. If  $E_{l0}^{\mathcal{H}} = 0$  for some end  $M_{l_0}$ , then  $\mu = 0$ . This implies that the mean curvature achieves the equality in (5.11). Thus,

$$\begin{aligned} \operatorname{tr}_{\tilde{g}}(\tilde{h}) &= \frac{3}{\lambda}, \\ |\tilde{h}|_{\tilde{g}}^2 &= \mathcal{H}^2 |h|_g^2 + \frac{3}{\lambda^2}. \end{aligned}$$

Therefore

$$\bar{R} = -\operatorname{tr}_{\tilde{g}}(\tilde{h})^2 + |\tilde{h}|_{\tilde{g}}^2 \geq -\frac{6}{\lambda^2}.$$

Under this condition, that  $E_{l0}^{\mathcal{H}} = 0$  gives that

$$(M, \bar{g}) \equiv (\mathbb{H}^3, \check{g}_{\mathcal{H}}).$$

Thus

$$\bar{R} = -\frac{6}{\lambda^2} + \mathcal{H}^2 |h|_g^2 = -\frac{6}{\lambda^2}.$$

We obtain

$$\begin{aligned} h &= 0, \\ K &= \frac{\cosh \frac{T}{\lambda}}{\lambda \sinh \frac{T}{\lambda}} g. \end{aligned}$$

By the Gauss equation,

$$\begin{aligned}\tilde{R}_{ijkl} &= R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk} \\ &= -\frac{1}{\lambda^2 \sinh^2 \frac{T}{\lambda}} (\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}) + \frac{\cosh^2 \frac{T}{\lambda}}{\lambda^2 \sinh^2 \frac{T}{\lambda}} (\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}) \\ &= \frac{1}{\lambda^2} (\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}).\end{aligned}$$

By the Codazzi equations,

$$\tilde{R}_{0jkl} = 0.$$

The dominant energy condition and (5.11) also imply that

$$T_{\alpha\beta} = 0.$$

So

$$\begin{aligned}\tilde{g}^{00}\tilde{R}_{0j0l} &= -\tilde{g}^{ik}\tilde{R}_{ijkl} + \frac{3}{\lambda^2}\tilde{g}_{jl} \\ &= -\frac{1}{\lambda^2}\tilde{g}^{ik}(\tilde{g}_{ik}\tilde{g}_{jl} - \tilde{g}_{il}\tilde{g}_{jk}) + \frac{3}{\lambda^2}\tilde{g}_{jl} \\ &= \frac{1}{\lambda^2}\tilde{g}_{jl}\end{aligned}$$

along  $M$ . Thus  $(N^{1,3}, \tilde{g})$  is de Sitter along  $M$  and we prove the second part of the theorem. Q.E.D.

**Remark 5.1.** *In general, the function  $\mathcal{H}$  in an initial data set  $(M, g, K)$  is a nonconstant function. In this case, the positive mass theorem (e.g. Theorem 5.1) does not hold true.*

**Remark 5.2.** *We can also discuss the total angular momentum along the line of [31] in this case.*

## 6. MEAN CURVATURES OF HYPERSURFACES

In this section, we discuss mean curvatures of  $\mathcal{P}$ -asymptotically de Sitter spacelike hypersurfaces. In particular, we discuss the existence of constant mean curvature spacelike hypersurfaces. This is analogous to the existence of maximal spacelike hypersurfaces in asymptotically flat spacetimes, which was studied extensively by Bartnik, etc (c.f. [5, 7] and references therein). For consideration of length of the paper, we study only the simple case that the spacetime is globally hyperbolic whose existence is implied by Christodoulou and Klainerman's theorem [8]. We will address elsewhere by generalizing the methods in [5, 7] to study the existences of constant mean curvature in general spacetimes.

Let  $\tilde{g}$  be the metric of an asymptotically flat spacetime defined on  $\mathbb{R} \times \mathbb{R}^3$

$$\tilde{g} = -a^2(t, x)dt^2 + g_{ij}(t, x)dx^i dx^j \quad (6.1)$$

with  $a > 0$ . The  $t$ -slice has the metric  $g$  and the second fundamental form

$$k_{ij} = \frac{1}{2a} \partial_t g_{ij}.$$

Suppose  $F(t, x)$  is a smooth function on  $\mathbb{R} \times \mathbb{R}^3$ . Consider a new metric

$$\tilde{g}_\lambda = -a^2(t, x) dt^2 + e^{2F} g_{ij}(t, x) dx^i dx^j \quad (6.2)$$

The  $t$ -slice has the metric  $g_\lambda = e^{2F} g$  and the second fundamental form

$$K_{ij} = e^{2F} k_{ij} + \frac{1}{a} e^{2F} (\partial_t F) g_{ij}.$$

If we take

$$F(t, x) = \frac{1}{3} \int_0^t (\Theta - \text{tr}_g(k)) a(s, x) ds$$

with certain prescribed function  $\Theta(t, x)$ , then the mean curvature of the  $t$ -slice in (6.2) is

$$\text{tr}_{g_\lambda}(K) = \Theta(t, x). \quad (6.3)$$

In their celebrated work [8], Christodoulou and Klainerman proved the global existence of an asymptotically flat metric for (6.1), which is vacuum and foliated by maximal spacelike hypersurfaces, i.e.,  $\text{tr}_g(k) = 0$ . Taking

$$\Theta = \sqrt{3\Lambda}$$

in (6.3), we obtain an existence of constant mean curvature spacelike hypersurfaces. Using estimates on the lapse function [8], we can re-write the metric (6.2) as follows

$$\tilde{g}_\lambda = -a^2(t, x) dt^2 + e^{\frac{2t}{\lambda}} \hat{g}_{ij}(t, x) dx^i dx^j. \quad (6.4)$$

Therefore, the metric  $\tilde{g}_\lambda$  is asymptotically de Sitter. However,  $\tilde{g}_\lambda$  does not satisfy the vacuum Einstein fields equations with positive cosmological constant in general. It will be interesting to extend Christodoulou and Klainerman's work to the current case.

Finally, we note that suitable choices of  $\Theta$  will give rise to spacelike hypersurfaces with mean curvature violating condition (4.3).

## 7. KERR-DE SITTER

In this section, we compute the total angular momentum for suitable time slices in the Kerr-de Sitter spacetime. In the Boyer-Lindquist coordinates  $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\psi})$ , the Kerr-de Sitter metric is

$$\begin{aligned} \tilde{g}_{KdS} = & -\frac{\Delta_{\bar{r}}}{U} \left( d\bar{t} - \frac{a}{\xi} \sin^2 \bar{\theta} d\bar{\psi} \right)^2 + \frac{U}{\Delta_{\bar{r}}} d\bar{r}^2 + \frac{U}{\Delta_{\bar{\theta}}} d\bar{\theta}^2 \\ & + \frac{\Delta_{\bar{\theta}} \sin^2 \bar{\theta}}{U} \left( a d\bar{t} - \frac{(\bar{r}^2 + a^2)}{\xi} d\bar{\psi} \right)^2, \end{aligned}$$

where

$$\begin{aligned}\Delta_{\bar{r}} &= (\bar{r}^2 + a^2)\left(1 - \frac{\bar{r}^2}{\lambda^2}\right) - 2m\bar{r}, \\ \Delta_{\bar{\theta}} &= 1 + \frac{a^2 \cos^2 \bar{\theta}}{\lambda^2}, \\ U &= \bar{r}^2 + a^2 \cos^2 \bar{\theta}, \\ \xi &= 1 + \frac{a^2}{\lambda^2}.\end{aligned}$$

In Boyer-Lindquist coordinates,  $m = 0$  does not imply directly that the metric is de Sitter. Thus, inspired by [15], we employ the following coordinate transformation

$$\begin{aligned}\hat{t} &= \bar{t}, \\ \hat{r} \cos \hat{\theta} &= \bar{r} \cos \bar{\theta}, \\ \left(1 + \frac{a^2}{\lambda^2}\right)\hat{r}^2 &= \bar{r}^2 + a^2 \sin^2 \bar{\theta} + \frac{a^2}{\lambda^2}\bar{r}^2 \cos^2 \bar{\theta}, \\ \hat{\psi} &= \left(1 + \frac{a^2}{\lambda^2}\right)\bar{\psi} - \frac{a}{\lambda^2}\bar{t}.\end{aligned}$$

In coordinates  $(\hat{t}, \hat{r}, \hat{\theta}, \hat{\psi})$ , the Kerr-de Sitter metric can be written as

$$\tilde{g}_{KdS} = -\left(1 - \frac{\hat{r}^2}{\lambda^2}\right)d\hat{t}^2 + \frac{d\hat{r}^2}{1 - \frac{\hat{r}^2}{\lambda^2}} + \hat{r}^2(d\hat{\theta}^2 + \sin^2 \hat{\theta}d\hat{\psi}^2) + a_{\hat{\mu}\hat{\nu}}d\hat{x}^\mu d\hat{x}^\nu$$

where the nonzero  $a_{\hat{\mu}\hat{\nu}}$  are

$$\begin{aligned}a_{\hat{t}\hat{t}} &= \frac{2m\bar{r}}{U}, \\ a_{\hat{t}\hat{\psi}} &= -\frac{2ma\bar{r} \sin^2 \bar{\theta}}{U}, \\ a_{\hat{r}\hat{r}} &= \frac{2m\bar{r}U}{(\Delta_{\bar{r}} + 2m\bar{r})\Delta_{\bar{r}}}, \\ a_{\hat{\psi}\hat{\psi}} &= \frac{2ma^2\bar{r} \sin^4 \bar{\theta}}{U}.\end{aligned}$$

The new coordinates  $(\hat{t}, \hat{r}, \hat{\theta}, \hat{\psi})$  is indeed the static coordinates for the Kerr-de Sitter metric. Now we transfer it into the planar coordinates. Let  $(t, r, \theta, \psi)$  be polar coordinates corresponding to the planar coordinates. The transformations are given as follows:

$$\begin{aligned}\hat{t} &= t - \frac{\lambda}{2} \ln \left|1 - \frac{r^2 A^2}{\lambda^2}\right|, \\ \hat{r} &= Ar, \\ \hat{\theta} &= \theta, \\ \hat{\psi} &= \psi,\end{aligned}$$

where  $A = e^{\frac{t}{\lambda}}$ . In polar coordinates, the Kerr-de Sitter is

$$\tilde{g}_{KdS} = -dt^2 + e^{\frac{2t}{\lambda}}(dr^2 + r^2(d\theta^2 + \sin^2 \theta d\psi^2)) + a_{\mu\nu} dx^\mu dx^\nu$$

and the nonzero components of  $a_{\mu\nu}$  have the following asymptotic behaviors

$$\begin{aligned} a_{tt} &= \frac{2m\lambda^2}{r^3 A^3} B^{-\frac{3}{2}} + O(r^{-4}), \\ a_{tr} &= \frac{2m\lambda^3}{r^4 A^3} B^{-\frac{3}{2}} + O(r^{-5}), \\ a_{t\theta} &= \frac{2ma^2\lambda}{r^3 A^4} B^{-\frac{5}{2}} \sin \theta \cos \theta + O(r^{-4}), \\ a_{rr} &= \frac{2m\lambda^2}{r^3 A} B^{-\frac{5}{2}} + O(r^{-4}), \\ a_{r\theta} &= \frac{2ma^2\lambda^2}{r^4 A^3} \sin \theta \cos \theta B^{-\frac{5}{2}} + O(r^{-5}), \\ a_{r\psi} &= \frac{2m\lambda a \sin^2 \theta}{r^2 A} B^{-\frac{5}{2}} + O(r^{-3}), \\ a_{\theta\theta} &= \frac{2ma^4 \sin^2 \theta \cos^2 \theta}{r^3 A^3} B^{-\frac{5}{2}} + O(r^{-4}), \\ a_{\psi\psi} &= \frac{2ma^2 \sin^4 \theta}{r A} B^{-\frac{5}{2}} + O(r^{-2}) \end{aligned}$$

where  $B = 1 + \frac{a^2}{\lambda^2} \sin^2 \theta$ . The second fundamental form of the  $t$ -slice can be computed in terms of the formula

$$K_{ij} = \frac{1}{2N} (\nabla_i N_j + \nabla_j N_i - \partial_t \tilde{g}_{ij})$$

and the tensor  $\bar{h}$  has the following asymptotic behaviors

$$\begin{aligned} \bar{h}_{rr} &= \frac{2m\lambda^2 - ma^2 \sin^2 \theta}{A^2 B^{\frac{5}{2}} \lambda r^3} + O(r^{-4}), \\ \bar{h}_{r\theta} &= O(r^{-4}), \\ \bar{h}_{r\psi} &= \frac{3ma \sin^2 \theta}{A^2 B^{\frac{5}{2}} r^2} + O(r^{-4}), \\ \bar{h}_{\theta\theta} &= -\frac{m\lambda}{A^2 B^{\frac{3}{2}} r} + O(r^{-3}), \\ \bar{h}_{\theta\psi} &= O(r^{-6}), \\ \bar{h}_{\psi\psi} &= \frac{(-m\lambda^2 + 2ma^2 \sin^2 \theta) \sin^2 \theta}{A^2 B^{\frac{5}{2}} \lambda r} + O(r^{-3}). \end{aligned}$$

Denote the frame  $e_1 = \partial_r$ ,  $e_2 = \frac{\partial_\theta}{r}$ ,  $e_3 = \frac{\partial_\psi}{r \sin \theta}$  and  $\{\check{e}^i\}$  the coframe. Using the asymptotic expansion  $\rho^2 = r^2 + O(r)$ , we find the components of the

local angular momentum density  $\tilde{h}^z$

$$\begin{aligned}\tilde{h}^z(\check{e}_2, \check{e}_1) &= -\frac{3ma \sin \theta}{A^2 B^{\frac{5}{2}} r^2} + O(r^{-3}), \\ \tilde{h}^z(\check{e}_2, \check{e}_3) &= \frac{m\lambda^2 - 2ma^2 \sin^2 \theta}{A^2 B^{\frac{5}{2}} \lambda r^2} + O(r^{-3}), \\ \tilde{h}^z(\check{e}_3, \check{e}_2) &= \frac{m\lambda}{A^2 B^{\frac{3}{2}} r^2} + O(r^{-3}), \\ \tilde{h}^z(\check{e}_i, \check{e}_j) &= O(r^{-3}).\end{aligned}$$

Thus, in natural coordinates,

$$\begin{aligned}\tilde{h}_{1r}^z &= \tilde{h}^z(\check{e}_1 \sin \theta \cos \psi + \check{e}_2 \cos \theta \cos \psi - \check{e}_3 \sin \psi, \check{e}_1) \\ &= -\frac{3ma \sin \theta \cos \theta \cos \psi}{A^2 B^{\frac{5}{2}} r^2} + O(r^{-3}), \\ \tilde{h}_{2r}^z &= \tilde{h}^z(\check{e}_1 \sin \theta \sin \psi + \check{e}_2 \cos \theta \sin \psi + \check{e}_3 \sin \psi, \check{e}_1) \\ &= -\frac{3ma \sin \theta \cos \theta \sin \psi}{A^2 B^{\frac{5}{2}} r^2} + O(r^{-3}), \\ \tilde{h}_{3r}^z &= \tilde{h}^z(\check{e}_1 \cos \theta - \check{e}_2 \sin \theta, \check{e}_1) + O(r^{-3}) \\ &= \frac{3ma \sin^2 \theta}{A^2 B^{\frac{5}{2}} r^2} + O(r^{-3}).\end{aligned}$$

Note that the range of  $\bar{\psi}$  from 0 to  $2\pi$  gives that

$$-\frac{a}{\lambda^2} \bar{t} \leq \psi \leq 2\left(1 + \frac{a^2}{\lambda^2}\right)\pi - \frac{a}{\lambda^2} \bar{t},$$

we obtain,

$$\begin{aligned}J_1(z) &= \frac{A^2}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} \tilde{h}_{1r}^z * dr \\ &= -\frac{3ma}{8\pi} \int_{-\frac{a}{\lambda^2} \bar{t}}^{2\left(1 + \frac{a^2}{\lambda^2}\right)\pi - \frac{a}{\lambda^2} \bar{t}} \int_0^\pi \frac{\sin^2 \theta \cos \theta \cos \psi}{B^{\frac{5}{2}}} d\theta d\psi = 0, \\ J_2(z) &= \frac{A^2}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} \tilde{h}_{2r}^z * dr \\ &= -\frac{3ma}{8\pi} \int_{-\frac{a}{\lambda^2} \bar{t}}^{2\left(1 + \frac{a^2}{\lambda^2}\right)\pi - \frac{a}{\lambda^2} \bar{t}} \int_0^\pi \frac{\sin^2 \theta \cos \theta \sin \psi}{B^{\frac{5}{2}}} d\theta d\psi = 0, \\ J_3(z) &= \frac{A^2}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} \tilde{h}_{3r}^z * dr \\ &= \frac{3ma}{8\pi} \int_{-\frac{a}{\lambda^2} \bar{t}}^{2\left(1 + \frac{a^2}{\lambda^2}\right)\pi - \frac{a}{\lambda^2} \bar{t}} \int_0^\pi \frac{\sin^3 \theta}{B^{\frac{5}{2}}} d\theta d\psi = \frac{ma}{1 + \frac{a^2}{\lambda^2}}.\end{aligned}$$

**Remark 7.1.** *If we choose the range of  $\psi$  from 0 to  $2\pi$ , we obtain*

$$J_1(z) = 0, \quad J_2(z) = 0, \quad J_3(z) = \frac{ma}{(1 + \frac{a^2}{\lambda^2})^2}.$$

*It is interesting that  $J_3(z)$  conjugates to  $J_{23}$  of [15] by replacing  $\lambda$  to  $\sqrt{-1}\lambda$ , i.e., the positive cosmological constant to the negative cosmological constant.*

**Remark 7.2.** *Note that the total energy and the total linear momentum vanish for this  $t$ -slice. However, it does not contradict to the positive mass theorem as it does not hold on the  $t$ -slice due to the singularity. The situation is similar to the Schwarzschild spacetime with negative mass. The positive mass theorem for black holes can not apply to this spacetime as naked singularity occurs.*

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