

# The Cohomology of Transitive Lie Algebroids \*

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## Abstract

For a transitive Lie algebroid  $\mathcal{A}$  on a connected manifold  $M$  and its a representation on a vector bundle  $F$ , we study the localization map  $\Upsilon^1: H^1(\mathcal{A}, F) \rightarrow H^1(L_x, F_x)$ , where  $L_x$  is the adjoint algebra at  $x \in M$ . The main result in this paper is that:  $\text{Ker}\Upsilon_x^1 = \text{Ker}(p^{1*}) = H_{deR}^1(M, F_0)$ . Here  $p^{1*}$  is the lift of  $H^1(\mathcal{A}, F)$  to its counterpart over the universal covering space  $\widetilde{M} \xrightarrow{p} M$  and  $H_{deR}^1(M, F_0)$  is the  $F_0 = H^0(L, F)$ -coefficient deRham cohomology. We apply these results to study the associated vector bundles to principal fiber bundles and the structure of transitive Lie bialgebroids.

## 1 Introduction

The theory of Lie algebroids is one of important fields in modern differential geometry, which gives an unified way to study Lie algebras and the tangent bundle of a manifold. Its global version is Lie groupoid. Please see [13] for a detailed introduction to the theory of Lie algebroids and Lie groupoids as well as [18] by Weinstein for their applications in Poisson geometry. Moreover, in [3] Connes also points out that groupoids play an essential role in non-commutative geometry.

The purpose of this paper is to study the cohomology of Lie algebroids, which is a basic topic in this field and has wide applications in physics and other fields of mathematics (e.g., see [5], [6] and [7]), as well as its applications for associated vector bundles with respect to some principal fiber bundle and the structure of transitive Lie bialgebroids. The notion of a Lie bialgebroid was introduced by Mackenzie and Xu in [15] as a natural generalization of that of a Lie bialgebra, as well as the infinitesimal version of Poisson groupoids introduced by Weinstein [17]. It has been shown that much of the theory of Poisson groups and Lie bialgebras can be similarly carried out in this general context. It is therefore a basic task to study the structure of Lie bialgebroids. In particular, it is very interesting to figure out what special features a Lie bialgebroid, in which the Lie algebroid structure is transitive, would have.

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The paper is organized as follows. In Section 2, first we give a brief introduction to basic notion and then, for a transitive Lie algebroid  $\mathcal{A}$  and its representation on a vector bundle  $F$ , we define a morphism of cohomology groups, called the localization map, and prove that the Lie algebroid 1-cohomology is totally determined by the 1-cohomology of its adjoint Lie algebra under some topological condition. In Section 3 we study the kernel of the localization map mentioned above. To do it, the connection and parallelism are used (see [1] for more details). As an equivalent statement of the main theorem, we describe some conditions to make a Lie algebroid 1-cocycle to be coboundary. In section 4, we study some properties of the cohomology of associated vector bundles with respect to some principal fiber bundle. In Section 5, we recall some results on the structure of transitive Lie bialgebroids in [1] as an another application of our localization theory.

## 2 Preliminaries

Throughout the paper we suppose that any smooth manifold  $M$  under consideration is connected.

### 2.1 Lie algebroids and its representations

First we introduce some basic concept used below:

**Definition 2.1.** *A Lie algebroid  $\mathcal{A}$  with base space  $M$ , is a (real) vector bundle over  $M$ , together with a bundle map  $\rho : \mathcal{A} \rightarrow TM$ , called the anchor, and there is a (real) Lie algebra structure  $[\cdot, \cdot]_{\mathcal{A}}$  on  $\Gamma(\mathcal{A})$  satisfying the following conditions:*

1. *The induced map  $\rho : \Gamma(\mathcal{A}) \rightarrow \mathcal{X}(M)$  is a Lie algebra morphism;*
2. *for all  $f \in C^\infty(M)$ ,  $A, B \in \Gamma(\mathcal{A})$ , the Leibnitz law holds. That is,*

$$[A, fB]_{\mathcal{A}} = f[A, B]_{\mathcal{A}} + (\rho(A)f)B.$$

In this paper, by  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  we denote a Lie algebroid  $\mathcal{A}$  with anchor map  $\rho : \mathcal{A} \rightarrow TM$  and Lie bracket  $[\cdot, \cdot]_{\mathcal{A}}$  on  $\Gamma(\mathcal{A})$ . For a transitive Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  over  $M$  (i.e., the anchor  $\rho$  is surjective), the *Artiyah sequence* is as follows:

$$0 \rightarrow L \xrightarrow{i} \mathcal{A} \xrightarrow{\rho} TM \rightarrow 0, \quad (1)$$

where  $L = Ker(\rho)$  is called the *adjoint bundle* of  $\mathcal{A}$ . In fact,  $L$  is a Lie algebra bundle. The fiber type can be taken as the Lie algebra  $\mathfrak{g} = L_p$  at any point  $p \in M$ . By  $[\cdot, \cdot]_L$ , we denote the fiber-wise bracket on  $L$  (see [13]).

Let  $F$  be a vector bundle over  $M$ . By  $\mathcal{CDO}(F)$  we denote the *bundle of covariant differential operators* of  $F$ . (see III in [13] and refs. [14, 10] for more details. In the last text, the authors use  $\mathcal{D}(F)$  instead of  $\mathcal{CDO}(F)$ ). Each element  $D \in \mathcal{CDO}(F)_x$  is an operator  $D_x : \Gamma(F) \rightarrow F_x$ , corresponding to a unique  $X \triangleq \sigma(D_x) \in T_xM$ , such that

$$D_x(f\mu) = X(f)\mu(x) + f(x)D_x\mu, \quad \forall f \in C^\infty(M), \mu \in \Gamma(F). \quad (2)$$

Actually,  $(\mathcal{CD}\mathcal{O}(F), [\cdot, \cdot]_{\mathcal{CD}\mathcal{O}(F)}, \sigma)$  is a transitive Lie algebroid over  $M$ , with the bracket of commutator  $[\cdot, \cdot]_{\mathcal{CD}\mathcal{O}(F)}$ . That is, for two  $D_1, D_2 \in \Gamma(\mathcal{CD}\mathcal{O}(F))$ ,

$$[D_1, D_2]_{\mathcal{CD}\mathcal{O}(F)} \triangleq D_1 \circ D_2 - D_2 \circ D_1,$$

is also an element of  $\Gamma(\mathcal{CD}\mathcal{O}(F))$ . Moreover, the corresponding vector field

$$\sigma[D_1, D_2]_{\mathcal{CD}\mathcal{O}(F)} = [\sigma(D_1), \sigma(D_2)].$$

Obviously, the adjoint bundle of this Lie algebroid is  $End(F) = Ker(\sigma)$ .

**Definition 2.2.** A representation of a Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  on a vector bundle  $F \rightarrow M$  is a vector bundle map  $\mathfrak{L} : \mathcal{A} \rightarrow \mathcal{CD}\mathcal{O}(F)$ , which is also a Lie algebroid morphism. That is,

- 1)  $\forall A \in \mathcal{A}_x, x \in M, \rho(A) = \sigma(\mathfrak{L}(A))$ ;
- 2)  $\forall A, B \in \Gamma(\mathcal{A}), \mathfrak{L}[A, B]_{\mathcal{A}} = [\mathfrak{L}(A), \mathfrak{L}(B)] = \mathfrak{L}(A) \circ \mathfrak{L}(B) - \mathfrak{L}(B) \circ \mathfrak{L}(A)$ .

We also denote  $\mathfrak{L}(A)$  by  $\mathfrak{L}_A$ . For example, when  $\mathcal{A}$  is transitive, one can define the adjoint representation of  $\mathcal{A}$  on  $L$ : for each  $A \in \mathcal{A}_x$ , define

$$ad_A : \Gamma(L) \rightarrow L_x, \quad \mu \mapsto [A, \mu]_{\mathcal{A}}(x), \quad \forall \mu \in \Gamma(L). \quad (3)$$

Here, we need to extend  $A$  to be a local section  $\tilde{A}$  of  $\mathcal{A}$  near  $x$ , and then the value of  $[A, \mu]_{\mathcal{A}}(x)$  is defined to be  $[\tilde{A}, \mu]_{\mathcal{A}}(x)$ . Note that, it does not depend on the choice of the extension of  $A$  near  $x$ . Usually  $ad_A$  is written as  $[A, \cdot]_{\mathcal{A}}$ .

As usual, a representation  $\mathfrak{L}$  of  $\mathcal{A}$  on  $F$  defines the Chevalley complex  $(C^k(\mathcal{A}, F), \mathfrak{D})$  and cohomology groups  $H^k(\mathcal{A}, F) = Ker(\mathfrak{D})/Im(\mathfrak{D})$  (see [4]). In detail, we call a series of  $C^\infty(M)$ -modules

$$C^k(\mathcal{A}, F) \triangleq \{\text{bundle maps } \Omega : \wedge^k \mathcal{A} \rightarrow F\}, \quad (k \geq 1),$$

and  $C^0(\mathcal{A}, F) = \Gamma(F)$  the cochain space of  $\mathfrak{L}$ . The coboundary operators (also called the differentials)  $\mathfrak{D} = \mathfrak{D}^k : C^k(\mathcal{A}, F) \rightarrow C^{k+1}(\mathcal{A}, F)$  are defined in the traditional way and satisfy  $\mathfrak{D}^2 = 0$ . For this complex  $(C^k(\mathcal{A}, F), \mathfrak{D})$ , the corresponding cohomology groups are

$$H^k(\mathcal{A}, F) = Ker(\mathfrak{D}^k)/Im(\mathfrak{D}^{k-1}). \quad (k \geq 0)$$

We adopt the convention that  $H^0(\mathcal{A}, F) = Ker(\mathfrak{D}^0)$ . In particular, a closed 0-cochain is a smooth section of  $\Gamma(F)$ , say  $\nu$ , satisfying  $\mathfrak{L}_A \nu = 0, \forall A \in \mathcal{A}$ . The group  $H^0(\mathcal{A}, F)$  is the collection of all such closed 0-cochains. A 1-cochain  $\Omega \in C^1(\mathcal{A}, F)$  is a bundle map from  $\mathcal{A}$  to  $F$ . It is called closed, or a 1-cocycle, denoted by  $\Omega \in D^1(\mathcal{A}, F)$ , if

$$\Omega[A, B]_{\mathcal{A}} = \mathfrak{L}_A(\Omega(B)) - \mathfrak{L}_B(\Omega(A)), \quad \forall A, B \in \Gamma(\mathcal{A}). \quad (4)$$

Especially, we call  $\Omega$  a coboundary, denoted by  $\Omega \in B^1(\mathcal{A}, F)$ , if  $\Omega = \mathfrak{D}\mu$ , i.e.,

$$\Omega(A) = \mathfrak{L}_A(\mu), \quad \forall A \in \Gamma(\mathcal{A})$$

for some  $\mu \in \Gamma(F)$ . In what follows, cochains are simply called chains. Two 1-cocycles are called homologic, if their subtraction is a coboundary. The group  $H^1(\mathcal{A}, F)$  are the quotient group of all 1-cocycles in sense of homological equivalence:

$$H^1(\mathcal{A}, F) \triangleq D^1(\mathcal{A}, F)/B^1(\mathcal{A}, F).$$

We usually write the equivalence class of a 1-cocycle  $\Omega$  by  $[\Omega]$ .

If the Lie algebroid degenerates to a Lie algebra, then the above construction of cohomology groups returns to that of the Lie algebras.

## 2.2 The localization of 1-cohomology of transitive Lie algebroids

We choose an arbitrary point  $x \in M$ . For the Lie algebra  $(L_x, [\cdot, \cdot]_{L_x})$ ,  $\mathfrak{L}$  induces a representation  $\underline{\mathfrak{L}}$  of  $L_x$  on  $F_x$ . In fact, for any  $u \in L_x$ , we define  $\underline{\mathfrak{L}}_u(\mu) \triangleq \mathfrak{L}_u(\bar{\mu})$ , here  $\mu \in F_x$ ,  $\bar{\mu} \in \Gamma(F)$  is a locally smooth extension of  $\mu$ . This is well defined, because  $\mathfrak{L}_u(f\mu) = f(x)\mathfrak{L}_u(\mu)$ ,  $\forall f \in C^\infty(M)$ . Since  $\mathfrak{L}$  is a Lie algebra morphism, so is  $\underline{\mathfrak{L}}: L_x \rightarrow \text{End}(F_x)$  and hence  $\underline{\mathfrak{L}}$  is indeed a representation.

Consider the Chevalley complex  $C^\bullet(L_x, F_x)$ , where  $F_x$  is regarded as an  $L_x$ -module via the representation  $\underline{\mathfrak{L}}$ . We denote the set of closed chains in  $C^k(L_x, F_x)$  by  $D^k(L_x, F_x)$ , and the corresponding Chevalley cohomology groups by  $H^k(L_x, F_x)$ .

We need the following theory of localizations and some results in [2].

**Proposition 2.3.** *Let  $\mathfrak{L}: \mathcal{A} \rightarrow \mathcal{CDO}(F)$  be a representation of a Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  on  $F \rightarrow M$ . For any  $x \in M$ , there is a natural morphism  $\Upsilon_x^k: H^k(\mathcal{A}, F) \rightarrow H^k(L_x, F_x)$  defined by*

$$\Upsilon_x^k([\Omega]) \triangleq [\underline{\Omega}], \quad \forall \Omega \in C^k(\mathcal{A}, F).$$

Here  $\underline{\Omega}$  is the limitation of  $\Omega$  on  $\wedge^k L_x \subset \wedge^k \mathcal{A}_x$ .

We call the group morphism  $\Upsilon_x^k$  defined above the **localization** of the homology group  $H^k(\mathcal{A}, F)$  at  $x$ . For this  $\Upsilon$ , we have the following claims.

**Theorem 2.4.** *Given a representation of a transitive Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  on  $F \rightarrow M$ ,  $\mathfrak{L}: \mathcal{A} \rightarrow \mathcal{CDO}(F)$ , then we have*

- 1) *For any  $x, y \in M$ , there exists an isomorphism  $J: H^1(L_y, F_y) \rightarrow H^1(L_x, F_x)$ , such that the following diagram commutes*

$$\begin{array}{ccc} & H^1(L_y, F_y) & \\ & \nearrow \Upsilon_y^1 & \\ H^1(\mathcal{A}, F) & & \\ & \searrow \Upsilon_x^1 & \\ & H^1(L_x, F_x) & \end{array} \quad \begin{array}{c} \\ \\ \\ \downarrow J \\ \\ \end{array} \quad (5)$$

- 2) *If  $M$  is simply connected, or  $H^0(L_x, F_x) = 0$ , then the localization  $\Upsilon_x^1: H^1(\mathcal{A}, F) \rightarrow H^1(L_x, F_x)$  is an injection.*

In general, the isomorphism  $J$  in 1) depends on the choice of a path from  $x$  to  $y$  in  $M$ . It is shown in [2] that, although the isomorphism  $J: H^1(L_y, F_y) \cong H^1(L_x, F_x)$  is not naturally defined, the two subgroups  $Im\Upsilon_y^1$  and  $Im\Upsilon_x^1$  are naturally isomorphic, under the condition of 2) of this theorem. In this paper, we are going to prove in Corollary 3.11 that even without this condition, the conclusion also holds.

### 3 Kernel of the localization map $\Upsilon^1$

An interesting problem is that, if  $M$  is not simply connected, and  $H^0(L_x, F_x) \neq 0$ , then what is the kernel of  $\Upsilon_x^1$ ? In this section, we give the answer to this problem in two different ways. Please see the following two equations (8) and (15).

**Lemma 3.1.** *Let  $p: \widetilde{M} \rightarrow M$  be a covering map. The for any vector bundle  $F \rightarrow M$ , we have*

$$\mathcal{CDO}(p^1F) \cong p^1\mathcal{CDO}(F).$$

**Proof.** For each pair  $(z, D_0) \in p^1\mathcal{CDO}(F)_z$ , where  $z \in \widetilde{M}$ ,  $x = p(z)$ ,  $D_0 \in \mathcal{CDO}(F)_x$ , we define  $\phi(z, D_0) \in \mathcal{CDO}(p^1F)_z$  as follows: given an arbitrary  $\lambda \in \Gamma(p^1F)$ , we find a decomposition near  $z$ :

$$\lambda = \sum_i f_i \mu_i, \quad \text{where } f_i \in C^\infty(\widetilde{M}), \mu_i \in \Gamma(F),$$

and we then set

$$\phi(z, D_0)\lambda \triangleq \sum_i (Z(f_i)\mu_i(x) + f_i(z)D_0(\mu_i)),$$

where  $Z \in T_z\widetilde{M}$  is the unique tangent vector satisfying  $p_*(Z) = \sigma(D_0)$ . It is easy to prove that this definition does not depend on the choice of decompositions of  $\lambda$ . In this way, we obtain  $\phi: p^1\mathcal{CDO}(F) \rightarrow \mathcal{CDO}(p^1F)$ , which is obviously an injection.

Conversely, for any  $z \in \widetilde{M}$ ,  $x = p(z)$ , and  $D \in \mathcal{CDO}(p^1F)_z$ ,  $\sigma(D) = Z \in T_z\widetilde{M}$ ,  $D$  induces  $D_0 \in \mathcal{CDO}(F)_x$ , such that  $\sigma(D_0) = p_*(Z)$ ,  $\phi(z, D_0) = D$ . In fact, each section  $\mu \in \Gamma(F)$  can be naturally regarded as  $\mu \in \Gamma(p^1F)$ . Let  $D(\mu) = \nu \in (p^1F)_z = F_x$  and we define  $D_0(\mu) = \nu$ . Therefore, for the preceding  $\lambda \in \Gamma(p^1F)$ , it is not hard to see

$$D(\lambda) = \sum_i (Z(f_i)\mu_i(x) + f_i(z)D(\mu_i)) = \phi(z, D_0)(\lambda).$$

This shows that  $\phi$  is also surjective. So  $\phi$  is indeed an isomorphism from  $p^1\mathcal{CDO}(F)$  to  $\mathcal{CDO}(p^1F)$ . ■

**Lemma 3.2.** *Suppose that a Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  has a representation on  $F \rightarrow M$ ,  $\mathfrak{L}: \mathcal{A} \rightarrow \mathcal{CDO}(F)$ . Let  $p: \widetilde{M} \rightarrow M$  be a covering map. Then the pull back bundle  $\widetilde{\mathcal{A}} = p^1\mathcal{A}$  is also a Lie algebroid over  $\widetilde{M}$ , and it has an induced representation on  $\widetilde{F} = p^1F$ ,  $\widetilde{\mathfrak{L}}: \widetilde{\mathcal{A}} \rightarrow \mathcal{CDO}(\widetilde{F})$ , such that the following diagram commutes.*

$$\begin{array}{ccc} \widetilde{\mathcal{A}} & \xrightarrow{\widetilde{\mathfrak{L}}} & \mathcal{CDO}(\widetilde{F}) \cong p^1\mathcal{CDO}(F) \\ p^1 \downarrow & & p^1 \downarrow \\ \mathcal{A} & \xrightarrow{\mathfrak{L}} & \mathcal{CDO}(F). \end{array} \quad (6)$$

In other words, for  $z \in \widetilde{M}$ ,  $x = p(z)$ ,  $A \in \mathcal{A}_x$ , one has

$$\widetilde{\mathfrak{L}}(x, A) = (x, \mathfrak{L}(A)).$$

Moreover, if  $\mathcal{A}$  is transitive, then so is  $\widetilde{\mathcal{A}}$ .

We omit the proofs of this lemma and the following two theorems.

**Theorem 3.3.** *With the same assumptions as in Lemma 3.2, we have a group morphism*

$$p^{k*} : H^k(\mathcal{A}, F) \rightarrow H^k(\widetilde{\mathcal{A}}, \widetilde{F}).$$

**Theorem 3.4.** *With the same assumptions as in Lemma 3.2, we have a commute diagram*

$$\begin{array}{ccc} H^k(\mathcal{A}, F) & \xrightarrow{p^{k*}} & H^k(\widetilde{\mathcal{A}}, \widetilde{F}) \\ \Upsilon_x^k \downarrow & & \downarrow \widetilde{\Upsilon}_{\tilde{x}}^k \\ H^k(L_x, F_x) & = & H^k(L_{\tilde{x}}, F_{\tilde{x}}). \end{array} \quad (7)$$

Here  $\widetilde{\Upsilon}_{\tilde{x}}^k$  is the localization of  $H^k(\widetilde{\mathcal{A}}, \widetilde{F})$  at some  $\tilde{x} \in \widetilde{M}$  with  $p(\tilde{x}) = x$ . And one naturally regards  $H^k(L_x, F_x) = H^k(L_{\tilde{x}}, F_{\tilde{x}})$ .

**Corollary 3.5.** *If  $\mathcal{A}$  is a transitive Lie algebroid and  $p: \widetilde{M} \rightarrow M$  is a universal covering, then*

$$\text{Ker}(\Upsilon_x^1) = \text{Ker}(p^{1*}). \quad (8)$$

**Proof.** By Theorem 3.4,  $\Upsilon_x^1 = \widetilde{\Upsilon}_{\tilde{x}}^1 \circ p^{1*}$ . By 2) of Theorem 2.4 and  $\widetilde{M}$  being simply connected, we know  $\widetilde{\Upsilon}_{\tilde{x}}^1$  is an injection. So we get (8). ■

The conclusion of Equation (8) of course describes  $\text{Ker}(\Upsilon_x^1)$ , but it has no relationship with the group  $H^0(L_x, F_x)$ . We now give another description of  $\text{Ker}(\Upsilon_x^1)$ . For the Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  and its representation on  $F \rightarrow M$ ,  $\mathfrak{L}: \mathcal{A} \rightarrow \mathcal{CDO}(F)$ , and any  $x \in M$ , we consider a sub vector space

$$F_{0x} = \{\nu \in F_x | \mathfrak{L}_u(\nu) = 0, \forall u \in L_x\} = H^0(L_x, F_x).$$

Then by Theorem 2.4, when  $\mathcal{A}$  is transitive,

$$F_0 = H^0(L, F) \subset F$$

is a sub vector bundle. Since for each  $u \in \Gamma(L)$ ,  $A \in \Gamma(\mathcal{A})$ ,  $\nu \in \Gamma(F_0)$ , we have

$$\mathfrak{L}_u(\mathfrak{L}_A \nu) = \mathfrak{L}_{[u, A]} \nu - \mathfrak{L}_A(\mathfrak{L}_u \nu) = 0,$$

and hence  $\mathfrak{L}_A \nu \in \Gamma(F_0)$ . So we have an induced representation of  $\mathcal{A}$  on  $F_0$ , also denoted by  $\mathfrak{L}$ .

Meanwhile,  $\mathfrak{L}$  induces a representation of  $TM$  on  $F_0$ , denoted by  $\bar{\mathfrak{L}}: TM \rightarrow \mathcal{CDO}(F_0)$ . In fact, for each  $X \in T_x M$ , and an arbitrary  $A \in \mathcal{A}_x$ ,  $\rho(A) = X$ , we set

$$\bar{\mathfrak{L}}_X \nu \triangleq \mathfrak{L}_A \nu, \quad \forall \nu \in \Gamma(F_0).$$

Obviously this definition does not depend on the choice of  $A$ , and  $\bar{\mathfrak{L}}$  is well defined. A representation of the tangent bundle  $TM$  on  $F_0$  is also referred as a *flat connection* of  $F_0$ .

We will call  $\bar{\mathfrak{L}}$  the **reduced (flat) connection** of  $F_0$  coming from the representation  $\mathfrak{L}$ . Now, elements of

$$C^k(TM, F_0) = \text{Hom}(\wedge^k(TM), F_0) \quad (\text{with } C^0(TM, F_0) = \Gamma(F_0))$$

are also called the  $F_0$ -coefficient  $k$ -forms. With the usual exterior differential operator  $d : C^k(TM, F_0) \rightarrow C^{k+1}(TM, F_0)$ ,  $C^\bullet(TM, F_0)$  is known as the  $F_0$ -coefficient de Rham complex. Especially,  $D^1(TM, F_0)$  is the kernel of  $d : C^1(TM, F_0) \rightarrow C^2(TM, F_0)$  and

$$H_{deR}^1(M, F_0) = D^1(TM, F_0)/d(\Gamma(F_0))$$

is the first  $F_0$ -coefficient **de Rham cohomology** of  $M$ . We are going to prove that this group is just the kernel of  $\Upsilon_x^1$  (Theorem 3.9).

**Lemma 3.6.** *There is a one-one correspondence between*

$$D^1(\mathcal{A}, F_0)_0 = \{\Omega \in D^1(\mathcal{A}, F_0) \mid \Omega(v) = 0, \forall v \in L\}$$

and  $D^1(TM, F_0)$ , denoted by  $\Omega \mapsto \bar{\Omega}$ . For the inversion map, we denote

$$\theta \in D^1(TM, F_0) \mapsto \rho^*\theta \in D^1(\mathcal{A}, F_0)_0.$$

More over, this map induces an injection of  $H_{deR}^1(M, F_0)$  into  $H^1(\mathcal{A}, F)$ :  $[\theta] \mapsto [\rho^*\theta]$ .

**Proof.** Given any  $\Omega \in D^1(\mathcal{A}, F_0)_0$ , we define  $\bar{\Omega}$  to be a map sending  $X \in T_xM$  to  $\bar{\Omega}(X) \triangleq \Omega(A)$ , where  $A \in \mathcal{A}_x$ ,  $\rho(A) = X$ . This is of course well defined. It is also easy to check

$$\bar{\mathfrak{L}}_X \bar{\Omega}(Y) - \bar{\mathfrak{L}}_Y \bar{\Omega}(X) - \bar{\Omega}([X, Y]) = 0, \quad \forall X, Y \in \mathcal{X}(M). \quad (9)$$

I.e.,  $\bar{\Omega} \in D^1(TM, F_0)$ .

On the other hand, given any  $\theta \in D^1(TM, F_0)$  satisfying Equation (9), we can define

$$\rho^*\theta(A) = \theta(\rho(A)), \quad \forall A \in \mathcal{A}.$$

$\rho^*\theta$  naturally satisfies  $\rho^*\theta|_L = 0$ , and it is also a cocycle. It is just by the definitions to see that the map

$$\rho^*(\cdot) : H_{deR}^1(M, F_0) \rightarrow H^1(\mathcal{A}, F_0); \quad [\theta] \mapsto [\rho^*\theta],$$

is an injection of cohomology groups. But as we shall see in the following exact Sequence (12) that  $H^1(\mathcal{A}, F_0)$  is embedded into  $H^1(\mathcal{A}, F)$ . So we conclude that  $H_{deR}^1(M, F_0)$  can be embedded into  $H^1(\mathcal{A}, F)$  via  $[\rho^*(\cdot)]$ . ■

Let  $\bar{F} = F/F_0$  be the quotient bundle. Now one obtains an exact sequence

$$0 \rightarrow F_0 \xrightarrow{i} F \xrightarrow{j} \bar{F} \rightarrow 0. \quad (10)$$

The algebroid  $\mathcal{A}$  has an induced representation on  $\bar{F}$ , in an obvious sense and also denoted by  $\mathfrak{L}$ :

$$\mathfrak{L}_A[\mu] \triangleq [\mathfrak{L}_A\mu], \quad \forall \mu \in \Gamma(F).$$

Now, we get an exact sequence of complexes

$$0 \rightarrow C^k(\mathcal{A}, F_0) \xrightarrow{i} C^k(\mathcal{A}, F) \xrightarrow{j} C^k(\mathcal{A}, \bar{F}) \rightarrow 0. \quad (11)$$

Here  $i, j$  are both cochain maps. So we have the following long exact sequence (the Mayer-Vietoris) of cohomology groups

$$\begin{aligned} 0 &\longrightarrow H^0(\mathcal{A}, F_0) \xrightarrow{i_{*0}} H^0(\mathcal{A}, F) \xrightarrow{j_{*0}} H^0(\mathcal{A}, \overline{F}) (= 0) \\ &\longrightarrow H^1(\mathcal{A}, F_0) \xrightarrow{i_{*1}} H^1(\mathcal{A}, F) \xrightarrow{j_{*1}} H^1(\mathcal{A}, \overline{F}) \longrightarrow \cdots \end{aligned} \quad (12)$$

Similarly, at any  $x \in M$  the Lie algebra  $L_x$  has the trivial representation on  $F_{0x}$  and an induces representation on  $\overline{F}_x$ . So we have another exact sequence of complexes

$$0 \rightarrow C^k(L_x, F_{0x}) \xrightarrow{i} C^k(L_x, F_x) \xrightarrow{j} C^k(L_x, \overline{F}_x) \rightarrow 0.$$

And it induces a long exact sequence,

$$\begin{aligned} 0 &\longrightarrow H^0(L_x, F_{0x}) \xrightarrow{i_{*0}} H^0(L_x, F_x) \xrightarrow{j_{*0}} H^0(L_x, \overline{F}_x) (= 0) \\ &\longrightarrow H^1(L_x, F_{0x}) \xrightarrow{i_{*1}} H^1(L_x, F_x) \xrightarrow{j_{*1}} H^1(L_x, \overline{F}_x) \longrightarrow \cdots \end{aligned} \quad (13)$$

Moreover, there is a series of vertical arrows  $\Upsilon^k$  between (12) and (13), namely the localization maps with respect to the preceding representations, such that the diagram commute. Here we pick out a part of the diagram as follows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathcal{A}, F_0) & \xrightarrow{i_{*1}} & H^1(\mathcal{A}, F) & \xrightarrow{j_{*1}} & H^1(\mathcal{A}, \overline{F}) & \longrightarrow & \cdots \\ & & \downarrow \Upsilon_{0x}^1 & & \downarrow \Upsilon_x^1 & & \downarrow \overline{\Upsilon}_x^1 & & \\ 0 & \longrightarrow & H^1(L_x, F_{0x}) & \xrightarrow{i_{*1}} & H^1(L_x, F_x) & \xrightarrow{j_{*1}} & H^1(L_x, \overline{F}_x) & \longrightarrow & \cdots \end{array} \quad (14)$$

By 2) of Theorem 2.4 and  $H^0(L_x, \overline{F}_x) = 0$ , we know that  $\overline{\Upsilon}_x^1$  is an injection.

**Corollary 3.7.**

$$Ker(\Upsilon_x^1) = Ker(j_{*1}) \cap i_{*1}(Ker(\Upsilon_{0x}^1)) \cong Ker(\Upsilon_{0x}^1).$$

**Proof.** Suppose that  $\omega \in H^1(\mathcal{A}, F)$  satisfies  $\Upsilon_x^1(\omega) = 0$ , then

$$\underline{j}_{*1} \circ \Upsilon_x^1(\omega) = \overline{\Upsilon}_x^1 \circ j_{*1}(\omega) = 0.$$

Since  $\overline{\Upsilon}_x^1$  is an injection, we have  $j_{*1}(\omega) = 0$ . Hence we know that

$$\omega \in Ker(j_{*1}) = Im(i_{*1}) \cong H^1(\mathcal{A}, F_0).$$

Since the left square in (14) commutes, we have  $\omega \in Ker(j_{*1}) \cap i_{*1}(Ker(\Upsilon_{0x}^1))$ . Conversely, given any  $\omega$  as above, there naturally holds  $\Upsilon_x^1(\omega) = 0$ .

Since  $i_{*1}$  is also an injection, we have the isomorphism in the expression. ■

We can of course directly regard  $H^1(\mathcal{A}, F_0) = Ker(j_{*1}) \subset H^1(\mathcal{A}, F)$  and therefore  $\Upsilon_{0x}^1 = \Upsilon_x^1|_{H^1(\mathcal{A}, F_0)}$ . The above corollary says that to know what  $Ker(\Upsilon_x^1)$  really is, it suffices to study  $Ker(\Upsilon_{0x}^1)$ .

**Theorem 3.8.**  $H^1(L_x, F_{0x}) = D^1(L_x, F_{0x})$ , and

$$Ker(\Upsilon_x^1) \cong Ker(\Upsilon_{0x}^1) = \{[\Omega] \mid \Omega \in D^1(\mathcal{A}, F_0), \Omega|_{L_x} = 0\}. \quad (15)$$



**Proof.** By definition of the complex  $C^k(L_x, F_{0x})$ , we have  $\mathfrak{D}F_{0x} = 0$  and hence  $H^1(L_x, F_{0x}) = D^1(L_x, F_{0x})$ .

Given any  $[\Omega] \in H^1(\mathcal{A}, F_0)$ , where  $\Omega \in D^1(\mathcal{A}, F_0)$ , we have

$$\Upsilon_{0x}^1([\Omega]) = [\Omega|_{L_x}] = \Omega|_{L_x}.$$

So we have the expression in (15). ■

Now we have the second way expressing the kernel of the localization map.

**Theorem 3.9.**  $Ker(\Upsilon_x^1) = Im[\rho^*(\cdot)] \cong H_{deR}^1(M, F_0)$ .

**Proof.** We first point out that, in (15), an  $\Omega \in D^1(\mathcal{A}, F_0)$  satisfies  $\Omega|_{L_x} = 0$ , for some  $x \in M$ , if and only if  $\Omega|_{L_y} = 0$  hold for all  $y \in M$ , i.e., or  $\Omega \in D^1(\mathcal{A}, F_0)_0$ . In fact, the first conclusion of Theorem 2.4 claims that the kernel of the localization map does not depend on the choice of the points:  $Ker(\Upsilon_x^1) = Ker(\Upsilon_y^1)$ . So the set described in (15) does not depend on the choice of  $x$ .

Combining Theorem 3.8 with these facts and the correspondence given by Lemma 3.6, we know that each element in the kernel of  $\Upsilon_x^1$  must be the cohomology class of some  $\rho^*\theta \in D^1(\mathcal{A}, F_0)_0$ . ■

We restate the conclusions of Corollary 3.5, Theorem 3.8, and Theorem 3.9 in the following theorem.

**Theorem 3.10.** *Suppose that a transitive Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  has a representation on a vector bundle  $F \rightarrow M$ ,  $\mathfrak{L} : \mathcal{A} \rightarrow \mathcal{CDO}(F)$ . Let  $p : \tilde{M} \rightarrow M$  be a universal covering map. The pull back Lie algebroid  $\tilde{\mathcal{A}} = p^*\mathcal{A}$  has an induced representation on  $\tilde{F} = p^*F$ , denoted by  $\tilde{\mathfrak{L}}$ . Write*

$$F_0 = \{\nu \in F_y | y \in M, \mathfrak{L}_u\nu = 0, \forall u \in L_y\}.$$

*The Lie algebroid  $\mathcal{A}$  has an induced representation on  $F_0$ , also denoted by  $\mathfrak{L}$ . Let  $\tilde{\mathfrak{L}} : TM \rightarrow \mathcal{CDO}(F_0)$  be the reduced (flat) connection of  $F_0$  coming from  $\mathfrak{L}$ . Then for each  $\Omega \in D^1(\mathcal{A}, F)$ ,  $x \in M$ , the following six statements are equivalent.*

1)  $\delta_x \stackrel{\Delta}{=} \Omega|_{L_x}$  is a coboundary, i.e.,  $\exists \tau \in F_x$ , such that

$$\delta_x(u) = \mathfrak{L}_u\tau, \quad \forall u \in L_x.$$

2)  $\delta_y \stackrel{\Delta}{=} \Omega|_{L_y}$  is a coboundary, for every  $y \in M$ .

3) The pull back cochain  $p^{1*}\Omega \in D^1(\tilde{\mathcal{A}}, \tilde{F})$  is a coboundary, i.e., there exists some  $\tilde{\mu} \in \Gamma(\tilde{F})$ , such that

$$\Omega(A) = \tilde{\mathfrak{L}}_{(\tilde{y}, A)}\tilde{\mu}, \quad \forall A \in \mathcal{A}_y.$$

Here  $\tilde{y} \in \tilde{M}$  satisfies  $p(\tilde{y}) = y$ .

4) There exists an  $\Omega_0 \in D^1(\mathcal{A}, F_0)$ ,  $\Omega_0|_{L_x} = 0$ , such that  $\Omega$  and  $\Omega_0$  are homologic.

5) There exists an  $\Omega_0 \in D^1(\mathcal{A}, F_0)$ ,  $\Omega_0|_L = 0$ , such that  $\Omega$  and  $\Omega_0$  are homologic.

6) There exist  $\theta \in D^1(TM, F_0)$  and  $\mu \in \Gamma(F)$ , such that

$$\Omega = \rho^*\theta + \mathfrak{D}\mu.$$

**Corollary 3.11.** *For any  $x, y \in M$ , the image of localizations  $\Upsilon_x^1$  and  $\Upsilon_y^1$  are naturally isomorphic. That is, the isomorphism  $J$  in Diagram (5) naturally defines an isomorphism  $J: H^1(L_y, F_y) \cong H^1(L_x, F_x)$ .*

**Proof.** Consider the commute diagram which follow from Diagram (5),

$$\begin{array}{ccc} & & \text{Im}\Upsilon_y^1 \\ & \nearrow \Upsilon_y^1 & \downarrow J \\ H^1(\mathcal{A}, F) & & \text{Im}\Upsilon_x^1 \\ & \searrow \Upsilon_x^1 & \end{array}$$

Since it is proved in Theorem 3.10 that  $\text{Ker}(\Upsilon_x^1) = \text{Ker}(\Upsilon_y^1)$ , the map  $J$  in the above diagram must be an isomorphism and naturally defined. ■

## 4 Application of the localization theories for principal bundles and their associated bundles

The remaining part of this paper is devoted to apply the preceding localization theories to that of principal bundles and their associated vector bundles. The idea originally appeared in [1] and in what follows, we will recover some results in that text.

We first recall basic facts about principal bundles and the associated bundles. Let  $(P, \overset{\pi}{\rightarrow}, M; G)$  be a principal bundle with structure group  $G$  (a Lie group) on the base manifold  $M$ . We always assume that  $G$  freely acts on  $P$  to the *right*.

The action of  $G$  on  $P$  naturally lifts to an action on  $TP$ . We denote the orbit of  $w \in T_pP$  by  $[w]$  and quotient manifold by  $\frac{TP}{G}$ . Since this action is free,  $\frac{TP}{G}$  admits a vector bundle structure with base  $M$ , and bundle projection  $q: [w] \mapsto \pi(p)$ . Sections of  $\frac{TP}{G}$  can be regarded as vector fields on  $P$  which are  $G$ -invariant:

$$\Gamma\left(\frac{TP}{G}\right) = \{U \in \mathcal{X}(P) \mid U_{p \cdot g} = R_{g*}U_p, \quad \forall p \in P, g \in G\}.$$

It follows that  $\Gamma\left(\frac{TP}{G}\right)$  has an induced Lie bracket structure transferred from  $\mathcal{X}(P)$ . Besides, the tangent map  $\pi_*$  can also be transferred to  $\frac{TP}{G} \rightarrow TM$ . Thus,  $(\frac{TP}{G}, \pi_*, M)$  is a Lie algebroid which is transitive (known as the *gauge algebroid*).

The Artiyah sequence for this algebroid is as follows

$$0 \rightarrow \frac{T^\perp P}{G} \xrightarrow{i} \frac{TP}{G} \xrightarrow{\pi_*} TM \rightarrow 0. \quad (16)$$

Here by  $T^\perp P$  we denote the collection of all vertical vectors, which are in fact the set

$$\left\{ \widehat{\mathfrak{r}}_p \triangleq \frac{d}{dt} \Big|_{t=0} p \cdot \exp t\mathfrak{r}; \quad p \in P, \mathfrak{r} \in \text{Lie}(G) \right\}.$$

If we also have a right action of  $G$  on a vector space  $V$ , then we obtain the associated bundle (by  $G$  acting diagonally on  $P \times V$ ):

$$F = \frac{P \times V}{G}.$$

Elements in  $F$  are of the form  $[p, v]$ , for  $p \in P$ ,  $v \in V$ . Most importantly, sections of  $F$  can be naturally regarded as  $V$ -valued,  $G$ -equivariant functions on  $P$ . I.e.,

$$\Gamma(F) \cong C^{\infty, G}(P, V) = \{\mu \in C^{\infty}(P, V) \mid \mu(p.g) = \mu(p).g, \quad \forall p \in P, g \in G\}.$$

In fact, for any  $\mu \in C^{\infty}(P, V)$ , it can be regarded as a  $V$ -valued function on  $P$  given by

$$\mu(p) \triangleq v, \quad \text{such that } [p, v] = \mu(\pi(p)).$$

And conversely, a  $V$ -valued,  $G$ -equivariant function  $\mu \in C^{\infty, G}(P, V)$  corresponds to the section of  $F$  given by

$$\mu(x) \triangleq [p, \mu(p)], \quad \text{by choosing an arbitrary } p \in \pi^{-1}(x), \quad \forall x \in M.$$

The Lie group  $G$  has the canonical adjoint action on  $\mathfrak{g}$ , and hence  $G$  has a right action on  $\mathfrak{g}$  defined by

$$\mathfrak{r}.g \triangleq Ad_{g^{-1}}\mathfrak{r}, \quad \forall \mathfrak{r} \in \mathfrak{g}, g \in G.$$

The adjoint Lie algebra bundle  $\frac{TP}{G}$  of the gauge Lie algebroid in Sequence (16) is indeed the associated bundle

$$\frac{P \times Lie(G)}{G}.$$

In fact, given any  $\mathfrak{g}$ -valued  $G$ -equivariant function  $\kappa: P \rightarrow \mathfrak{g}$ , which satisfies

$$\kappa(p.g) = \kappa(p).g = Ad_{g^{-1}}\kappa(p), \quad \forall p \in P, g \in G,$$

it corresponds to a vertical vector field on  $P$ :

$$\widehat{\kappa}|_p \triangleq \widehat{\kappa(p)}_p = \frac{d}{dt}\Big|_{t=0} p. \exp t\kappa(p), \quad \forall p \in P. \quad (17)$$

One can directly check that  $\widehat{\kappa}$  is a  $G$ -invariant vector field on  $P$ .

The ring of smooth functions  $C^{\infty}(M)$  can also be regarded as  $G$ -invariant functions on  $P$ :

$$C^{\infty}(M) \cong \pi^*C^{\infty}(M) = \{f \in C^{\infty}(P) \mid f(p.g) = f(p), \quad \forall p \in P, g \in G\}.$$

There is a standard representation of the gauge algebroid  $\frac{TP}{G}$  on  $F$ , defined by

$$\mathfrak{L}_U(\mu) \triangleq U(\mu), \quad \forall U \in \Gamma\left(\frac{TP}{G}\right), \mu \in C^{\infty, G}(P, V).$$

Let  $\Omega^{1, G}(P, V)$  denote the  $V$ -valued,  $G$ -equivariant 1-forms on  $P$ :

$$\begin{aligned} & \Omega^{1, G}(P, V) \\ &= \{\omega : TP \rightarrow V; \omega|_p(w).g = \omega|_{p.g}(R_{g*}w), \forall p \in P, g \in G, w \in T_pP\} \\ &= \left\{ \omega : TP \rightarrow V; \omega(U) \in C^{\infty, G}(P, V), \quad \forall U \in \Gamma\left(\frac{TP}{G}\right) \right\}. \end{aligned}$$

**Lemma 4.1.** *There is a canonical pull back morphism of  $C^\infty(M)$ -modules*

$$\pi^* : \Gamma(\text{Hom}(TM, F)) \rightarrow \Omega^{1,G}(P, V), \quad \vartheta \mapsto \pi^*(\vartheta),$$

where

$$\pi^*(\vartheta) : w \mapsto \vartheta(\pi_* w)(p), \quad \forall w \in T_p P.$$

Moreover,  $\pi^*$  is injective.

**Proof.** To see that  $\pi^*(\vartheta)$  is a  $G$ -equivariant 1-form, we suppose that for  $w \in T_p P$ ,  $R_{g*}w \in T_{p.g}P$ ,  $x = \pi(p) \in M$  and

$$X = \pi_*(w) = \pi_*(R_{g*}w) \in T_x M,$$

one has  $\vartheta(X) = [p, v] = [p.g, v.g]$ . Then,

$$\begin{aligned} \pi^*(\vartheta)|_p(w).g &= \vartheta(X)(p).g \\ &= v.g = \vartheta(X)(p.g) = \pi^*(\vartheta)|_{p.g}(R_{g*}w). \end{aligned}$$

It is easy to see that  $\pi^*(\vartheta)$  being a zero 1-form implies that  $\vartheta$  is zero. ■

We also introduce the notation  $\Omega^{k,G}(P, V)$  ( $k \geq 1$ ) to denote the  $V$ -valued  $G$ -equivariant  $k$ -forms on  $P$ . I.e.,

$$\begin{aligned} &\Omega^{k,G}(P, V) \\ &= \left\{ \omega : \wedge^k TP \rightarrow V; \omega|_p(w).g = \omega|_{p.g}(R_{g*}w), \forall p \in P, g \in G, w \in \wedge^k T_p P \right\} \\ &= \left\{ \omega : \wedge^k TP \rightarrow V; \omega(U) \in C^{\infty,G}(P, V), \quad \forall U \in \Gamma(\wedge^k \frac{TP}{G}) \right\}. \end{aligned}$$

These  $\Omega^{k,G}(P, V)$  together with  $\Omega^{0,G}(P, V) = C^{\infty,G}(P, V)$  become a complex (over the ring  $C^\infty(M)$ ), equipped with the usual exterior differential operator  $d$ . It is in fact isomorphic (as complexes) to  $C^k(\frac{TP}{G}, F)$ . And in turn, we have

$$H_{deR}^{k,G}(P, V) \cong H^k(\frac{TP}{G}, F).$$

**Definition 4.2.** *Given an arbitrary  $p \in P$ , the localization map for the gauge algebroid  $\frac{TP}{G}$  and its associated vector bundle  $F = \frac{P \times V}{G}$  is a group morphism*

$$\Upsilon_p^k : H_{deR}^{k,G}(P, V) \rightarrow H^k(\text{Lie}(G), V).$$

It sends the cohomology class of  $\omega \in \Omega^{k,G}(P, V)$  to the cohomology class of  $\widehat{\omega}|_p \in D^k(G, V)$  which is defined by

$$\widehat{\omega}|_p(\mathfrak{r}) \triangleq \omega|_p(\widehat{\mathfrak{r}}_p), \quad \forall \mathfrak{r} \in \text{Lie}(G).$$

Let  $G_e$  denote the subgroup of  $G$  which is the connected component of  $G$  containing the unit element  $e$ . Consider a sub vector space

$$V_0 = \{v \in V | v.h = v, \quad \forall h \in G_e\}.$$

One is easy to prove that  $V_0$  is  $G$ -invariant. Hence  $G$  also has a right action on  $V_0$ .

**Lemma 4.3.** For the representation of the gauge Lie algebroid  $\frac{TP}{G}$  on  $F = \frac{P \times V}{G}$ ,

$$F_0 = \frac{P \times V_0}{G} = H^0\left(\frac{T^\perp P}{G}, F\right).$$

**Proof.** Let  $\kappa: P \rightarrow \mathfrak{g}$  be a  $\mathfrak{g}$ -valued  $G$ -equivariant function which corresponds to a  $G$ -invariant vertical vector field  $\widehat{\kappa}$  on  $P$  given by (17). Let  $\mu \in C^\infty, G(P, V)$  be a  $V$ -valued,  $G$ -equivariant function which can also be regarded as an element of  $\Gamma(F)$ . At any  $p \in P$ ,  $s \in \mathbb{R}$ , let  $g = \exp s\kappa(p)$ . Then we have

$$\begin{aligned} & \widehat{\lambda}|_{p.g}(\mu) \\ &= \frac{d}{dt}\Big|_{t=0} \mu(p.g. \exp t\kappa(p.g)) = \frac{d}{dt}\Big|_{t=0} \mu(p).g. \exp tAd_{g^{-1}}\kappa(p) \\ &= \frac{d}{dt}\Big|_{t=0} \mu(p). \exp t\kappa(p).g = \frac{d}{dt}\Big|_{t=0} \mu(p). \exp t\kappa(p).g \\ &= \frac{d}{dt}\Big|_{t=0} \mu(p). \exp(t+s)\kappa(p) = \frac{d}{dt}\Big|_{t=s} \mu(p). \exp t\kappa(p). \end{aligned}$$

Hence we know that  $\mu \in \Gamma(F_0)$  implies

$$\frac{d}{dt} \mu(p) \exp t\kappa(p) = 0, \quad \text{i.e., } \mu(p) \exp t\kappa(p) \equiv \mu(p), \quad \forall t \in \mathbb{R}.$$

Since  $\kappa$  is arbitrary and  $G_e$  is generated by elements of the form  $\exp t\mathfrak{r}$ ,  $\mathfrak{r} \in \mathfrak{g}$ ,  $\mu(p)$  is an element of  $V_0$ . This shows that the function  $\mu$  takes values in  $V_0$ . Conversely, if  $\mu \in \Gamma(F_0)$ , then it obviously satisfies  $\widehat{\kappa}(\mu) = 0$ ,  $\forall \widehat{\kappa} \in \frac{T^\perp P}{G}$ . ■

Using Theorem 3.9, we have the following conclusion describing the kernel of the localization map  $\Upsilon_p^1$ .

**Theorem 4.4.**

$$\text{Ker}(\Upsilon_p^1) = \pi^* H_{deR}^1(M, F_0) \cong H_{deR}^1(M, F_0).$$

Here  $\pi^*: H_{deR}^1(M, F_0) \rightarrow H_{deR}^{k,G}(P, V)$  is given by

$$[\theta] \mapsto [\pi^*(\theta)], \quad \forall \theta \in D^1(TM, F_0),$$

and it is an injection.

And one may restate the above theorem into the following form, analogue to that of Theorem 3.10.

**Theorem 4.5.** With the preceding notations, let  $w \in \Omega^{1,G}(P, V)$  be a close 1-form. For any  $p \in P$ , we have five equivalent statements:

1)  $\omega|_{T_p^\perp P}$  is a coboundary, i.e.,  $\exists v \in V$  such that

$$\omega(\widehat{\mathfrak{r}}_p) = \frac{d}{dt}\Big|_{t=0} v. \exp t\mathfrak{r}, \quad \forall \mathfrak{r} \in \text{Lie}(G).$$

2)  $\omega|_{T_q^\perp P}$  is a coboundary, for all  $q \in P$ .

3) There exists a closed 1-form  $\omega_0 \in \Omega^{1,G}(P, V_0)$ ,  $\omega_0|_{T_p^\perp P} = 0$ , such that  $\omega_0$  and  $\omega$  are homologic, i.e.,

$$\omega = \omega_0 + d\mu, \quad \text{for some } \mu \in C^{\infty,G}(P, V).$$

4) There exists a closed 1-form  $\omega_0 \in \Omega^{1,G}(P, V_0)$ ,  $\omega_0|_{T_p^\perp P} = 0$ , such that  $\omega_0$  and  $\omega$  are homologic.

5) For some closed 1-form  $\theta \in D^1(TM, F_0)$  and  $\mu \in C^{\infty,G}(P, V)$ ,

$$\omega = \pi^*\theta + d\mu.$$

In particular, we conclude that:

**Corollary 4.6.** *If  $M$  is simply connected (or  $V_0 = 0$ ), then the localization map  $\Upsilon_p^1$  is an injection. In other words, any one of the five statements in the above theorem implies that  $\omega$  is a coboundary, i.e.,  $\omega = d\mu$  for some  $\mu \in C^{\infty,G}(P, V)$ .*

## 5 Transitive Lie Bialgebroids

In this section, we recall some results on the structure of transitive Lie bialgebroids in [1] as an another application of our localization theory.

• **Lie bialgebroids.** A Lie bialgebroid is a pair of Lie algebroids  $(\mathcal{A}, \mathcal{A}^*)$  satisfying the following compatibility condition

$$d_*[A, B]_{\mathcal{A}} = [d_*A, B]_{\mathcal{A}} + [A, d_*B]_{\mathcal{A}}, \quad \forall A, B \in \Gamma(\mathcal{A}), \quad (18)$$

where the differential  $d_*$  on  $\Gamma(\wedge^\bullet \mathcal{A})$  comes from the Lie algebroid structure on  $\mathcal{A}^*$  (see [9], [15] for more details). Of course, one can also denote a Lie bialgebroid by the pair  $(\mathcal{A}, d_*)$ , since the anchor  $\rho_* : \mathcal{A}^* \rightarrow TM$  and the Lie bracket  $[\cdot, \cdot]_*$  on the dual bundle are defined by  $d_*$  as follows:  $\rho_*(df) = d_*f$ ,  $\forall f \in C^\infty(M)$  and for all  $A \in \Gamma(\mathcal{A})$ ,  $\xi, \eta \in \Gamma(\mathcal{A}^*)$ ,

$$\langle [\xi, \eta]_*, A \rangle = \rho_*(\xi)\langle \eta, A \rangle - \rho_*(\eta)\langle \xi, A \rangle - d_*A(\xi, \eta).$$

Again we suppose that the Lie algebroid  $\mathcal{A}$  is transitive. Recall the adjoint representation of  $\mathcal{A}$  on  $L$  defined in (3). In this paper, we also consider the adjoint representation of  $\mathcal{A}$  on  $L \wedge L$  associated to that on  $L$ , and we write  $L^2$  for  $L \wedge L$ . In this case,  $\Omega \in C^1(\mathcal{A}, L^2)$ , i.e., a bundle map from  $\mathcal{A}$  to  $L^2$ , is a *Lie algebroid 1-cocycle* if and only if

$$\Omega[A, B]_{\mathcal{A}} = [\Omega(A), B]_{\mathcal{A}} + [A, \Omega(B)]_{\mathcal{A}}, \quad \forall A, B \in \Gamma(\mathcal{A}). \quad (19)$$

$\Omega$  is a coboundary if  $\Omega = [\mu, \cdot]_{\mathcal{A}}$  for some  $\mu \in \Gamma(L^2)$ .

The structure of transitive Lie bialgebroids is studied in [1]. We quote directly some of the conclusions in that text.

**Definition 5.1.** *For a transitive Lie algebroid  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$ , given  $\Lambda \in \Gamma(\wedge^2 \mathcal{A})$  and a bundle map  $\Omega : \mathcal{A} \rightarrow L^2$ , the pair  $(\Lambda, \Omega)$  is called  $\mathcal{A}$ -compatible if  $\Omega$  is a 1-cocycle and satisfies*

$$\left[\frac{1}{2}[\Lambda, \Lambda]_{\mathcal{A}} + \Omega(\Lambda), \cdot\right]_{\mathcal{A}} + \Omega^2 = 0, \quad \text{as a map } \Gamma(\mathcal{A}) \rightarrow \Gamma(\wedge^3 \mathcal{A}). \quad (20)$$

Here  $\Omega(\Lambda)$  and  $\Omega^2$  make sense by means of the extension of  $\Omega$  as a derivation of the graded bundle,  $\Omega: \wedge^k \mathcal{A} \rightarrow \wedge^{k+1} \mathcal{A}$ ,  $k \geq 0$ . For  $k = 0$ , it is zero. For  $k \geq 1$ , it is defined by

$$\Omega(A_1 \wedge \cdots \wedge A_k) = \sum_{i=1}^k (-1)^{i+1} A_1 \wedge \cdots \wedge \Omega(A_i) \wedge \cdots \wedge A_k, \quad (21)$$

for all  $A_1 \wedge \cdots \wedge A_k \in \Gamma(\wedge^k \mathcal{A})$ .

It is easy to see that if  $(\Lambda, \Omega)$  is  $\mathcal{A}$ -compatible, then so is the pair  $(\Lambda + \nu, \Omega - [\nu, \cdot]_{\mathcal{A}})$ , for any  $\nu \in \Gamma(L^2)$ . Thus, two  $\mathcal{A}$ -compatible pairs  $(\Lambda, \Omega)$  and  $(\Lambda', \Omega')$  are called equivalent, written  $(\Lambda, \Omega) \sim (\Lambda', \Omega')$ , if  $\exists \nu \in \Gamma(L^2)$ , such that  $\Lambda' = \Lambda + \nu$  and  $\Omega' = \Omega - [\nu, \cdot]_{\mathcal{A}}$ .

**Theorem 5.2.** *Let  $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$  be a transitive Lie algebroid over  $M$ . Then there is a one-to-one correspondence between Lie bialgebroids  $(\mathcal{A}, d_*)$  and equivalence classes of  $\mathcal{A}$ -compatible pairs  $(\Lambda, \Omega)$  such that*

$$d_* = [\Lambda, \cdot]_{\mathcal{A}} + \Omega. \quad (22)$$

For a Lie bialgebra  $(\mathfrak{g}, \mathfrak{g}^*)$ , it is obvious that one can take  $\Lambda = 0$  and  $-\Omega$  as the cobracket of  $\mathfrak{g}$ . Another special case is the following.

**Corollary 5.3.** *If  $\Omega: \mathcal{A} \rightarrow L^2$  is a 1-cocycle and satisfies  $\Omega^2 = 0$ , as a map  $\mathcal{A} \rightarrow \wedge^3 \mathcal{A}$ , then  $(\mathcal{A}, \Omega)$  is a Lie bialgebroid. In this case, the anchor  $\rho_*$  of  $\mathcal{A}^*$  is zero and  $\mathcal{A}^*$  is a bundle of Lie algebras whose bracket is defined by*

$$\langle [\xi, \eta]_*, A \rangle = -\langle \Omega(A), \xi \wedge \eta \rangle,$$

for all  $\xi, \eta \in \mathcal{A}^*$  and  $A \in \mathcal{A}$ .

**Corollary 5.4.** *Let  $(\mathcal{A}, d_*)$  be a transitive Lie algebroid over  $M$  and suppose that  $d_* = [\Lambda, \cdot]_{\mathcal{A}} + \Omega$  is given as in (22). Let  $p: \tilde{M} \rightarrow M$  be a covering. Then the pull back bundle  $\tilde{\mathcal{A}} = p^* \mathcal{A}$  is also a transitive Lie algebroid. For the pull back section  $\tilde{\Lambda} \in \Gamma(\tilde{\mathcal{A}})$  and the pull back bundle map*

$$\tilde{\Omega}: \tilde{\mathcal{A}} \rightarrow \tilde{L}^2,$$

let

$$\tilde{d}_* = [\tilde{\Lambda}, \cdot]_{\tilde{\mathcal{A}}} + \tilde{\Omega}.$$

Then  $(\tilde{\mathcal{A}}, \tilde{d}_*)$  is also a Lie bialgebroid over  $\tilde{M}$ .

It is known that, for any section  $\Lambda \in \Gamma(\wedge^2 \mathcal{A})$ , one can define a bracket on  $\Gamma(\mathcal{A}^*)$  by

$$[\xi, \eta]_{\Lambda} = L_{\Lambda^{\#} \xi} \eta - L_{\Lambda^{\#} \eta} \xi - d \langle \Lambda^{\#} \xi, \eta \rangle.$$

With the bracket defined above and anchor map

$$\rho_* \triangleq \rho \circ \Lambda^{\#}: \mathcal{A}^* \rightarrow TM,$$

the dual bundle  $\mathcal{A}^*$  becomes a Lie algebroid if and only if  $[X, [\Lambda, \Lambda]_{\mathcal{A}}]_{\mathcal{A}} = 0$ ,  $\forall X \in \Gamma(\mathcal{A})$  ([11], Theorem 2.1). In this situation, the induced differential on  $\Gamma(\wedge^{\bullet} \mathcal{A})$  has the form,  $d_* = [\Lambda, \cdot]_{\mathcal{A}}$ , and clearly satisfies compatibility condition (18). The Lie bialgebroid arising

in this way is called a *coboundary (or exact) Lie bialgebroid* [11]. In the particular case where  $[\Lambda, \Lambda]_{\mathcal{A}} = 0$ , the Lie bialgebroid is called *triangular* [16].

By our definition of  $\mathcal{A}$ -compatible pairs, the pair corresponding to a coboundary Lie bialgebroid can be chosen to be  $(\Lambda, 0)$  or, equivalently,  $(\mathcal{A}, \mathcal{A}^*)$  is a coboundary Lie bialgebroid if and only if the second element of the  $\mathcal{A}$ -compatible pair  $\Omega \in C^1(\mathcal{A}, L^2)$  is a coboundary. Therefore, to deal with coboundary Lie bialgebroids, one first needs to study the properties of Lie algebroid 1-cocycles.

**Corollary 5.5.** *With the same assumptions as in Theorem 5.2, if  $\text{Rank}(\mathcal{A}) = 1$ , then  $d_* = 0$ . That is,  $\mathcal{A}^*$  admits trivial Lie algebroid structures.*

**Corollary 5.6.** *With the same assumptions as in Theorem 5.2, if  $\text{Rank}(L) = 1$ , then  $d_* = [\Lambda, \cdot]$ . That is,  $(\mathcal{A}, d_*)$  is coboundary.*

The following theorem follows directly from 2) of Theorem 2.4.

**Theorem 5.7.** *Suppose that a transitive Lie algebroid  $\mathcal{A}$  satisfies **one** of the following conditions*

- 1)  $H^0(\mathfrak{g}, \mathfrak{g}^2) = 0$ , where  $\mathfrak{g} = L_x$ , for some  $x \in M$ ;
- 2)  $M$  is simply connected.

Then  $\Omega \in C^1(\mathcal{A}, L^2)$  is coboundary if and only if  $\delta_x \triangleq \Omega|_{\mathfrak{g}}$  is coboundary.

In particular, if  $H^1(\mathfrak{g}, \mathfrak{g}^2) = 0$ , any Lie bialgebroid  $(\mathcal{A}, \mathcal{A}^*)$  is coboundary.

It is a well known result that for any nontrivial representation of a semi-simple Lie algebra  $\mathfrak{g}$  on some vector space  $V$ , the cohomology groups  $H^0(\mathfrak{g}, V)$  and  $H^1(\mathfrak{g}, V)$  are both zero. So we conclude:

**Corollary 5.8.** *Let  $\mathcal{A}$  be a transitive Lie algebroid and let  $\mathfrak{g} = L_x$  be the fiber type of  $L$ . If  $\mathfrak{g}$  is semi-simple and its adjoint representation on  $\mathfrak{g}^2$  is not trivial, then any Lie bialgebroid  $(\mathcal{A}, \mathcal{A}^*)$  is coboundary.*

We also have the following corollaries which follow from Theorem 3.10. Note that now

$$L_0^2 = \{ \nu \in L_y^2 | y \in M, [u, \nu]_L = 0, \forall u \in L_y \}$$

is a vector bundle over  $M$  which has a natural flat connection.

**Corollary 5.9.** *Suppose that a transitive Lie bialgebroid  $(\mathcal{A}, d_*)$  satisfies  $H^1(\mathfrak{g}, \mathfrak{g}^2) = 0$ , where  $\mathfrak{g} = L_x$  for some  $x \in M$ . Then*

- 1) for a universal covering  $p: \widetilde{M} \rightarrow M$ , the pull back Lie bialgebroid  $(\widetilde{\mathcal{A}}, \widetilde{d}_*)$  given by Corollary 5.4 is coboundary;
- 2) the compatible pair corresponding to  $(\mathcal{A}, d_*)$  can be chosen to be  $(\Lambda, \rho^*\theta)$ , for some closed  $L_0^2$ -coefficient 1-form  $\theta \in D^1(TM, L_0^2)$ , where

$$\rho^*\theta : A \mapsto \theta(\rho(A)), \quad \forall A \in \mathcal{A}.$$

Note that in this case,  $\Omega|_L = \rho^*\theta|_L$  is trivial, and hence  $\Omega^2 = 0$ . So the compatible relation given in Equation (20) becomes

$$\left[ \frac{1}{2}[\Lambda, \Lambda]_{\mathcal{A}} + (\rho^*\theta) \lrcorner \Lambda, \cdot \right]_{\mathcal{A}} = 0, \quad \text{as a map } \Gamma(\mathcal{A}) \rightarrow \Gamma(\wedge^3 \mathcal{A}).$$



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