

Intricate dynamics of a deterministic walk confined in a strip

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PACS 02.50.-r – Probability theory, stochastic processes, and statistics

PACS 05.40.Fb – Random walks and Lévy flights

PACS 89.75.-k – Complex systems

Abstract. - We study the dynamics of a deterministic walk confined in a narrow two-dimensional space randomly filled with point-like targets. At each step, the walker visits the nearest target not previously visited. Complex dynamics is observed at some intermediate values of the domain width, when, while drifting, the walk performs long intermittent backward excursions. As the width is increased, evidence of a transition from ballistic motion to a weakly non-ergodic regime is shown, characterized by sudden inversions of the drift velocity with a probability slowly decaying with time, as $1/t$ at leading order. Excursion durations, first-passage times and the dynamics of unvisited targets follow power-law distributions. For parameter values below this scaling regime, precursory patterns in the form of “wild” outliers are observed, in close relation with the presence of log-oscillations in the probability distributions. We discuss the connections between this model and several evolving biological systems.

Introduction. – Deterministic walks in disordered environments have received an increasing attention over the past years. They describe diffusion processes following non-random rules and have applications, among others, to the study of the displacements of individuals in complex landscapes. Examples are human travels [1, 2], human displacements in a city [3], movement patterns of hunter-gatherer [4] or foraging animals [5].

From a given position, the next site visited by a purely deterministic walker is assigned from a given set of rules and not stochastically. These walks still have probabilistic and fluctuating features if the environment is random or heterogeneous. Interesting dynamics have been observed, such as normal [6] or anomalous diffusion [7], behaviors analogous to that of the Lorentz gas [8], cycles with power-law distributed periods [1, 9] or Lévy-like step length distributions [10, 11]. Complex behavior can emerge from very simple rules, *e.g.* when each individual step optimizes a given cost function. Some properties of deterministic walks have also been used as tools to process large data sets in galaxy surveys [12], thesaurus graphs [13] or for pattern recognition [14].

In many situations, in particular biological, the deterministic walker itself changes the medium, which introduces memory [8]. An important case is the self-avoiding walk (SAW), which can be implemented to model biological systems with negative feedbacks that tend to avoid

past behaviors. A simple example is that of a foraging animal relying on mental maps to navigate an environment composed of food patches that are not revisited after they have been depleted [5]. In a different context, the brain activity has been modeled by random walks keeping memory of their complete history in order to avoid persistent patterns; recent memory loss producing pathological repetitions, like in the Alzheimer’s disease [15]. In evolutionary ecology, the well-known Red-Queen principle assumes that any organism must constantly evolve in order to prevent its predators or preys to adapt to an otherwise predictable behavior. Similar considerations can apply to the dynamics of technological innovations [16]. Freund and Grassberger introduced some time ago a self-avoiding deterministic walk model in disordered two-dimensional domains, mimicking evolving organisms in phenotype landscapes [7]. These kinds of models are very difficult to handle analytically; they are firstly dynamical and usually not equivalent to canonical SAWs [17].

Here, we study a minimal model of a deterministic walk with a SAW constraint (in the infinite memory limit) and confined in a nearly one dimensional random medium. In an evolutionary context, whereas SAWs can be justified by natural selection, organisms also have developmental constraints due to limited phenotypic variability [18]. Similarly, the development of human artifacts is restricted by design limits. We model this important constraint by a

narrow random medium where the walker can evolve without bounds only in one direction. In narrow landscapes, the model exhibits very rich dynamical features not observed in unbounded ones, such as intermittent behavior, scaling laws, discrete scale invariance and very large events (outliers). A discussion of these results is then presented.

Model description. – Consider a two-dimensional strip of width l and infinite length along the horizontal direction. The strip is randomly filled with fixed point-like targets with uniform number density ρ_0 , representing, say, food patches for a foraging animal or phenotypes for an evolving species. The only control parameter is the reduced domain width, defined as $\delta = l/l_0$, with $l_0 = \rho_0^{-1/2}$ the characteristic distance between neighboring targets. At time $t = 0$, a walker is located at some target with coordinates (x_0, y_0) , taken as the origin. Two rules of motion are then recursively applied: the walker (*i*) moves to the nearest available target, (*ii*) does not visit a previously visited target. When the new target is reached, t is updated to $t + 1$.

The medium can be made one(two)-dimensional in the limit $\delta \ll 1$ ($\delta \gg 1$), respectively. We will focus here on values of δ of $O(1)$, typically in the range $(2, 5)$, such that the walker has a some vertical degree of freedom but a practically one-dimensional motion on large scales, described by its horizontal coordinate $x(t)$. In the simulations, the medium is a rectangle of area unity containing N targets and of width $l = \delta/\sqrt{N}$. Each run starts near the middle of the domain and is stopped before the walker reaches the lateral vertical walls.

Trajectories. – In the one dimensional case ($\delta \ll 1$), the targets are randomly distributed on a line and the motion is simply ballistic. After a possible short transient, the walker breaks the right-left symmetry and always moves to the nearest target to its right (or left) so that $x(t)$ is a sum of same-sign independent random variables with Poisson distribution. The $2d$ case ($\delta \gg 1$) is sometime called the “tourist walk” [1, 2]: the trajectories are not very different from $2d$ random walks, although slightly superdiffusive [19].

For the cases $\delta = O(1)$ of interest here, the situation is quite different and trajectories exhibit a rich structure. As shown in fig. 1 at $\delta = 4.1$, the motion is on average ballistic due to the confining effect of the horizontal walls. The numerically calculated root-mean-square displacement $\langle x(t)^2 \rangle^{1/2}$ follows a linear behavior with time (not shown here). Note that the walker horizontal velocity $x(t) - x(t - 1)$ often changes sign: the walker performs many “backward excursions” while drifting along the strip. These excursions, that were observed in a preliminary study of the model [10], can be explained qualitatively. A walker drifting, say, toward the left does not necessarily visit all the targets of a given neighborhood on its way and may ignore some targets. From time to time, rules (*i*) and (*ii*) make the walker turn back and

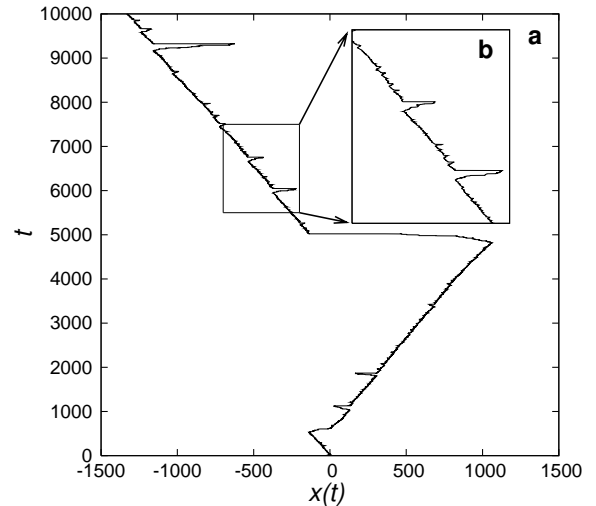


Fig. 1: Space-time diagram of a trajectory with $\delta = 4.1$ (x is in unit of l_0).

visit these unvisited targets toward the right, until it ends up in a region depleted of available targets. In that case, a single step can bring the walker back to the unexplored region located to the left.

Unexpectedly, backward excursions of all sizes can be observed in fig. 1a. Whereas most excursions are short, some can be of order $10^2 - 10^3 \times l_0$, e.g. near $t = 9300$. A close up of fig. 1a (inset b) reveals further details and suggests that the trajectory is fractal. In [10, 11], it was found that these intermittent backward excursions can lead to “Lévy-like” distributions for the distance separating successively visited targets, of the form $\ell^{-(1+\mu)}$, with $\mu \simeq 1$ at $\delta = 4$ [11].

Additionally, the sign of the drift velocity can change suddenly at large t (e.g. at $t \approx 600$ and $t \approx 5000$ in fig. 1a). Such inversions happen during a backward excursion, at some point when the closest unvisited target is located, say, to the right of x_0 for a trajectory that was previously drifting toward the left. Obviously, inversions can not occur in the $1d$ ballistic limit of the model. We investigate below the possible existence of a transition between different dynamical regimes as δ is varied.

Inversion probability and first-passage times.

– We define the explored interval at time t as $[x_{min}(t), x_{max}(t)]$, where $x_{min}(t)$ ($x_{max}(t)$) is the coordinate of the leftmost (rightmost) visited target after t steps, respectively. An inversion (say, from right to left) occurs during the t^{th} step if $x_{min}(t) - x_{min}(t - 1) < 0$ and if there exists a time $t' < t$ such that $x_{max}(t') - x_{max}(t' - 1) > 0$ and such that $x_{min}(t'') - x_{min}(t'' - 1) = x_{max}(t'') - x_{max}(t'' - 1) = 0$ for $t' < t'' < t$.

We then define $P_{inv}(t)$ as the probability that an inversion (to the left or right) occurs during the t^{th} step, and $P_0(t)$ as the probability that the walker crosses x_0 during the t^{th} step. In the random walk language, P_0 is analo-

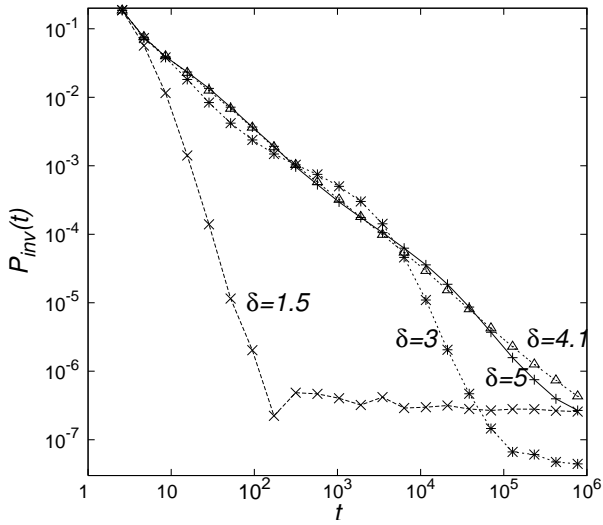


Fig. 2: Probability that the drift velocity changes its sign at time t , as a function of t and for various strip widths. The probabilities are calculated from $8 \cdot 10^4$ independent random media.

gous to the probability of presence at the origin. If motion is essentially ballistic between two inversions, a trajectory crosses its origin at large times only during an inversion: P_{inv} and P_0 have the same asymptotic behavior.

In Figure 2, the numerically computed $P_{inv}(t)$ decays very slowly with time for several $O(1)$ values of the strip width δ . In domains as narrow as $\delta = 1.5$, after an initial steep decay, $P_{inv}(t)$ exhibits a surprising fat tail. At the larger value $\delta = 4.1$, $P_{inv}(t)$ can be well fitted by the simple inverse power-law c/t , with c a constant. The same curves are replotted in Figure 3 as $tP_{inv}(t)$ versus t : for $\delta = 4.1$, the curve remains remarkably constant during almost 6 decades, while strong corrections to scaling are present below and above that parameter value ($\delta = 3$ and 5). Very similar results are obtained for $P_0(t)$.

Despite that the walker crosses less frequently the origin than a $1d$ random walker (where $P_0(t) \sim t^{-1/2}$), the return probability at large times in narrow strips ($\delta \ll t$) remains very high instead of being exponentially small as for usual ballistic motion (*e.g.*, a $1d$ random walker with a bias).

Contrary to random walks, sign changes in $x(t)$ are abrupt and not strongly correlated to the evolution of $x(t)$ during the preceding steps (see Fig. 1). It is therefore useful to make a connection between this result and a simpler two-state stochastic problem consisting of a walker moving ballistically on a line with two possible velocities, v and $-v$. Starting in one state, the walker change its velocity in the time interval $[t, t + 1]$ with probability $p_{inv}(t)$, that is given. It is well known that if $p_{inv}(t)$ decays faster than $1/t$, there is a finite probability that the walker remains indefinitely in a same state (v or $-v$) after reaching this state. If $p_{inv}(t)$ decays as $1/t$ or slower, the probability that the walker remains in a same state forever is zero. The behavior of the system is non-ergodic in the former

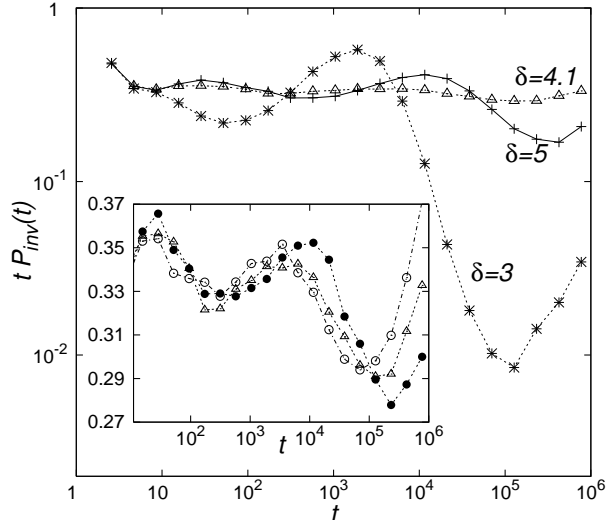


Fig. 3: Same data as in fig. 2, replotted as $tP_{inv}(t)$ vs. t . Inset: details of $tP_{inv}(t)$ for $\delta = 4$ (\circ), 4.1 (\triangle), 4.2 (\bullet).

case, as the left-right symmetry is asymptotically broken, while it is weakly non-ergodic¹ or ergodic in the latter case. Obviously, the above two-state problem only provides an approximate description of our model and makes sense only in the regime where inversions are abrupt.

We investigate more in details the possibility of a non-ergodic/weakly non-ergodic transition as the strip width δ is increased across some critical value δ_c where $P_{inv}(t) \simeq c/t$. Making an analogy between the behavior of P_{inv} (or P_0) and that of a correlation function near a critical point, for δ slightly below δ_c one may look for a standard scaling form: $P_{inv}(t) \simeq t^{-1}g(t/\tau(\delta))$, with $g(x)$ a scaling function rapidly decaying to zero at large x and $\tau(\delta)$ a diverging timescale as $\delta \rightarrow \delta_c$. The inset of Fig. 3 displays $tP_{inv}(t)$ vs. t for different values of δ near 4.1 and shows that the above ansatz does not hold.

Interestingly, the probability exhibits an unusual behavior instead. First, the different curves can not be rescaled onto a single curve. Second, a pure power-law behavior was never obtained for $P_{inv}(t)$ (nor $P_0(t)$) for the values of δ considered in this study. Intricate corrections to scaling in the form of logarithmic oscillations are observed. Log-oscillations have been observed in a variety of systems and are a manifestation of the phenomenon of discrete scale invariance [21]. The log-oscillations have a large period, of order $2 \ln 10$, which complicates the observation of several periods: we can not conclude whether they converge toward a finite amplitude or are amplified. However, the amplitude of the oscillations is minimum at $\delta_c \simeq 4.1$.

The leading $1/t$ decay of P_{inv} at δ_c is probably not a coincidence. From a renormalization group (RG) perspective, the model has a trivial attracting fixed point, $\delta^* = 0$, corresponding to simple ballistic motion in very narrow

¹In this context, weakly non-ergodic means that both states always remain accessible to the walker, although the time interval between two visits diverges asymptotically [20].

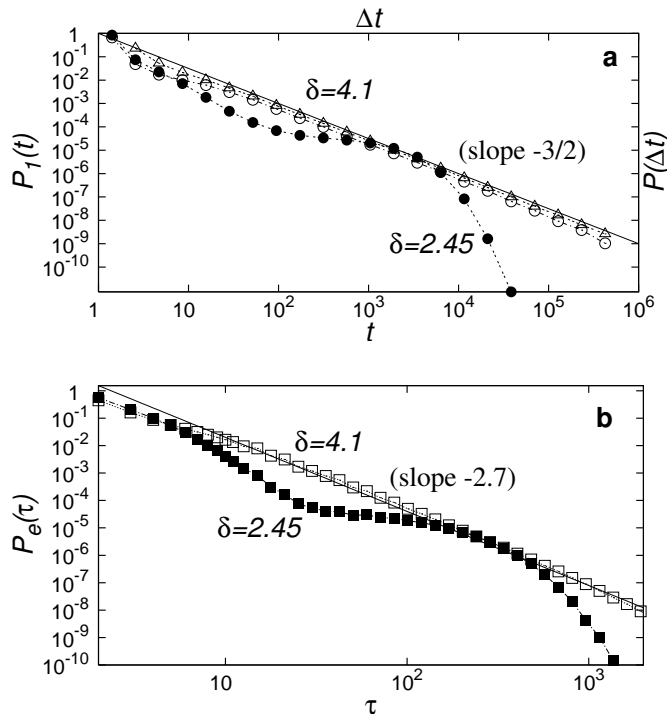


Fig. 4: a) Probability distribution $P_1(t)$ of the first-passage time at $\delta = 4.1$ (Δ). Opposite side: probability distribution $P(\Delta t)$ of the time intervals between two consecutive zeros of $N_u(t)$ for $\delta = 4.1$ (\circ) and $\delta = 2.45$ (\bullet). $P_1(t)$ and $P(\Delta t)$ are obtained from $8 \cdot 10^4$ and 10^3 independent runs respectively. b) Probability distribution of the excursion duration ($\tau \geq 2$). Lines are guides to the eye.

strips. The $1/t$ law indicates that δ_c should lie at the boundary of the basin of attraction of that fixed point. Besides, the increasing corrections from power-law behavior for δ slightly above δ_c (see Fig. 3 at $\delta = 5$) suggest that RG trajectories above δ_c flow towards an other attracting fixed point (that could be $\delta^* = \infty$). This argument supports the idea that the transition is not a cross-over and that δ_c might be a non trivial repelling fixed point.

An other possibility is that the walk may become asymptotically ballistic without inversions after extremely large times, unreachable with standard numerical methods. In this case, the results above would describe a very long transient preceding an asymptotic regime of limited practical relevance.

Other insights into inversion processes can be gained from the distribution $P_1(t)$ of first-passage times. The first-passage time is defined here as the step number when the walker crosses for the first time x_0 . As shown in figure 4a, a $t^{-\alpha}$ law with $\alpha \simeq 3/2$ holds remarkably well over nearly 6 decades in the vicinity of δ_c . As for P_{inv} (and P_0), log-oscillations were detected in P_1 at (and near) δ_c .

This exponent value can be qualitatively explained with the help of the simple two-state approximation described above, where the probability that the velocity changes its sign for the first time at time t reads: $p_1(t) =$

$p_{inv}(t) \exp[-\int_0^t p_{inv}(t') dt']$. If $p_{inv} \simeq c/t$ at large times, then $p_1(t) \sim t^{-\alpha}$ with $\alpha = 1 + c$. The numerical value of c calculated from P_{inv} at $\delta = 4.1$ yields $\alpha \simeq 1.33$. This value is close to, but lower than the observed $3/2$. Therefore, inversion events are not independent but probably long-range correlated. It is actually surprising (and most likely coincidental) that the first-passage exponent is close to the simple value of the $1d$ random walk [22].

Backward excursions and unvisited sites. – We now come back to the description of backward excursions, that are much more frequent than inversion events. The probability distribution of excursion durations, $P_e(\tau)$, can be obtained from the sizes of the time intervals when $x_{min}(t)$ or $x_{max}(t)$ remains constant. As shown in figure 4b, in the vicinity of δ_c this distribution is also well fitted by a power-law behavior, $P_e(\tau) \sim \tau^{-\beta}$, with $\beta \simeq 2.7$. This distribution has finite first moment but infinite variance. On average, the walker remains “trapped” in an excursion during a finite number of steps, but its progression is quite intermittent. For strip widths well below δ_c , excursions are still observed and $P_e(\tau)$ remains fairly broad, although it can no longer be fitted with a power-law. Generally speaking, backward excursions tend to restore the right-left symmetry of the system. For this reason they are reminiscent of the effect of thermal fluctuations on a broken symmetry phase in equilibrium.

An other quantity of interest related to excursion dynamics is the number of unvisited sites in the explored interval $[x_{min}(t), x_{max}(t)]$, denoted as $N_u(t)$. As shown in figure 5a, for values of δ below δ_c , $N_u(t)$ displays cycles of irregular durations analogous to oscillations in excitable systems. The cycles are composed of (i) a slowly increasing part on average and (ii) a fast decay down to zero. This behavior reflects the fact that a small fraction of sites are left as unvisited while the walker is drifting in the disordered medium, leading to an increase in $N_u(t)$. These sites can be visited later, in a long backward excursion, leading to an “avalanche-type” relaxation of $N_u(t)$. (Note that many smaller excursions also occur during the ascending part of $N_u(t)$.) The distribution of the time intervals Δt between two successive zeros of $N_u(t)$ is displayed in figure 4a. One expects $P(\Delta t)$ and the first-passage time distribution $P_1(t)$ to have the same asymptotic behavior, as observed. At $\delta = 4.1$, one finds $P(\Delta t) \sim (\Delta t)^{-3/2}$, implying that $\langle \Delta t \rangle = \infty$: $N_u(t)$ grows asymptotically unbounded. In the transition region, the walker is therefore unable to visit all the targets of the explored interval at large time (it is “inefficient”). Below δ_c , the distribution $P(\Delta t)$ decays faster (figure 4a) and the evolution of $N_u(t)$ seems to have a characteristic cut-off period. In figure 5a, where δ is well below δ_c , this characteristic time is still very long ($\sim 10^4$). Large avalanches, where N_u drops from about 600 to 0, are present.

The walker is *a priori* “efficient” below δ_c , since it regularly leaves no sites as unvisited (fig. 5a). However, this behavior is not persistent on very large time-scales,

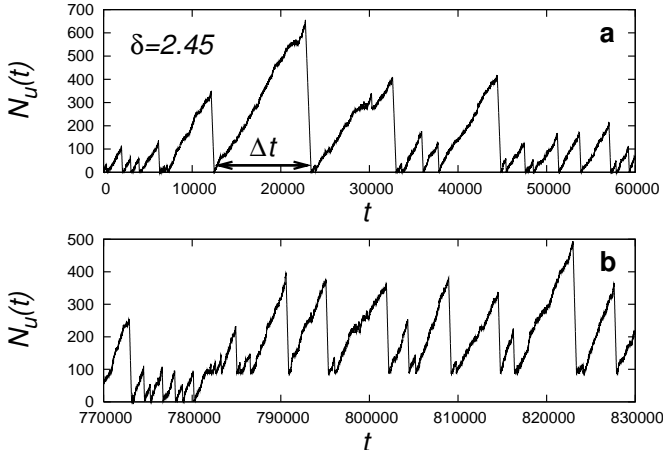


Fig. 5: Number of unvisited targets $N_u(t)$ as a function of t , at early (a) and late (b) times, for a same trajectory below the transition region ($\delta = 2.45$).

as shown in figure 5b. Surprisingly, at a given time that can be of order 10^6 or more, $N_u(t)$ does not come back to zero and starts to oscillate above a finite value. This happens when a large excursion fails to visit some of the unvisited targets left behind. After such an incomplete excursion, $\langle \Delta t \rangle$ obviously starts to grow with time. This behavior is observed in a whole range of parameter values below δ_c , down to about $\delta \sim 1.30$. The walk is therefore efficient during a finite time, until a “catastrophic” event with very large Δt occurs.

One can draw an analogy between these “outliers” and very large earthquakes or financial crisis in other contexts [23,24]: their magnitude can not be explained from a simple extrapolation of the distribution $P(\Delta t)$, as displayed in fig. 4a, to larger arguments. Here, outliers are unusual precursory patterns of the power-law distribution (in sharp contrast with common critical phenomena) and can be attributed to the presence of log-oscillations in $P(\Delta t)$: As noticeable in figure 4a, after an initial steeper decay, the distribution for $\delta = 2.45 < \delta_c$ very closely approaches the critical curve for a range of values around $\Delta t_1 \sim 5 \cdot 10^3$, before decaying fast again. Although computational limitations do not allow to observe more log-periods, it is very likely that the next oscillation can reach (or even cross) the critical distribution, for some values $\Delta t_2 \gg \Delta t_1$ (see figure 3 for a similar behavior at $\delta = 3$). Therefore, the emergence of very large intervals of order Δt_2 between the zeros of N_u becomes as probable as at δ_c . This explains qualitatively the time series of figures 5a-b, composed of many intervals of order $10^3 - 10^4$ followed by a single one of much larger size, precursor of the critical regime with diverging $\langle \Delta t \rangle$.

Discussion. – We have shown that a simple self-avoiding process taking place in a confined Poissonian random medium can display complex dynamics and broad distributions in a wide parameter range. Quenched disorder

introduces randomness in the model, that otherwise follows simple deterministic rules. Similar results as reported here should be observed in a semi-infinite strip with the walker initially located at one end, with the difference that the walker would drift without inversions.

In ref. [7], the trajectories generated by the Red Queen rules can be similar to random walks after time scales that depend strongly on the lattice size and geometry [7]. In contrast with the Red Queen Walk, where sites can be revisited after a very long time, our model has infinite memory, leading to intermittent and complex behavior.

The evolution of single species is known to be intermittent and not gradual, long period of stasis being “punctuated” by burst of rapid biological changes [25]. Such active periods might be driven by the internal dynamics of evolution. According to the fossil record, the number of genera with a lifetime τ follow a power-law $N \sim 1/\tau^\beta$, with $\beta \simeq 2$ [26]. The Bak-Sneppen model [27] considers interacting species with high mutation barriers, leading to self-organized critical states with $\beta = 1.1$ [26]. This evolution is slower than observed because it occurs by collective modes, or avalanches. Changes are easier in our model (where explicit interactions are ignored), but still intermittent. The distribution of time-intervals between successive changes in x_{max} , for instance, is fitted with an exponent $\beta = 2.7$. One may speculate that phenotypic restrictions could play a role on the punctuated dynamics of evolution, in addition to species interactions.

Our system does not become critical in an ordinary way. At a critical width δ_c , the inversion probability of the drift velocity decays as a power law with small log-periodic corrections. In a first harmonic approximation [21], $P_{inv}(t) \simeq ct^{-1}[1 + a_1 \cos(2\pi \ln t / \ln \lambda)]$, with $a_1 \ll 1$ and $\lambda \approx 100$. This asymptotic regime is numerically hard to reach, as observed in other problems with log-oscillations [15,30]. The leading term above precisely represents the law that separates, in analogy with a two-state stochastic process, asymptotically ballistic (non-ergodic) trajectories and walks that keep changing direction indefinitely.

The log-oscillations present in various distribution functions indicate the presence of a hierarchy of time-scales related to each other by a particular scaling factor λ , such that $P(\lambda t) \simeq \lambda^\alpha P(t)$. These oscillations are often displayed by cooperative phenomena taking place on hierarchical structures (spin models near criticality [21], contact processes [28]), or by random walk models with memory [15,29], among other examples. In ref. [30] log-oscillations appear in a simple biased 1d random walk model in a disordered medium containing a small fraction of “slow” sites, where the walker jumps in the direction opposite to the bias with a probability close to one. These slow sites are somehow analogous to our (dynamically generated) backward excursions. A crucial ingredient leading to log-oscillations in [30] is spatial discreteness, where clusters of slow sites trap the walker during a time that increases exponentially with the cluster size. In our model, no such

discreteness is apparent. Instead, inversion events are correlated in a complicated way: if the strip is sufficiently narrow, an inversion can not occur in the interval $[t_1, 2t_1]$ if an inversion occurred at time t_1 . The time intervals separating inversions might introduce a particular scaling factor, although its precise origin is unclear.

Well below the critical region, the distributions exhibit log-oscillations of irregular amplitudes whose maxima can be identified with outliers, that are “wild” precursors of critical fluctuations. From the above discussion, the characteristic size of these events can be roughly extrapolated as being of order λ^n ($n = 1, 2, \dots$), and as probable as at criticality. At δ_c , these specific scales are mixed with all the others ($a_1 \ll 1$), in a practically scale invariant distribution. A detailed study the behavior near δ_c remains to be done. Correlation functions (*e.g.* velocity) other than P_{inv} might exhibit clearer scaling relations.

In this scenario, it is however clear that outliers exist in a wide parameter range and do not even require that the bulk of the distribution follows a power-law. On the contrary, they are off-critical events by nature. This property has to be contrasted with more common views in seismology, for instance, where outliers are either considered as coming from the tail of power-law distributions [31], or, in a more refined way, as coming from a bump at large sizes in a otherwise power-law distribution of bulk events [32].

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Fruitful discussions with H. Larralde, F. Leyvraz, M.G.E da Luz, O. Miramontes, A. Robledo and G.M. Viswanathan are gratefully acknowledged.

REFERENCES

- [1] LIMA G.F., MARTINEZ A.S. and KINOUCHE O., *Phys. Rev. Lett.*, **87** (2001) 010603.
- [2] STANLEY H.G. and BULDYREV S.V., *Nature*, **413** (2001) 373.
- [3] CHOWELL G., HYMAN J.M., EUBANK S. and CASTILLO-CHAVEZ C., *Phys. Rev. E*, **68** (2003) 066102.
- [4] BROWN C.T., LIEBOVITCH L.S. and GLENDON R., *Hum. Ecol.*, **35** (2007) 129.
- [5] BOYER D., RAMOS-FERNANDEZ G., MIRAMONTES O., MATEOS J.L., COCHO G., LARRALDE H., RAMOS H. and ROJAS F., *Proc. R. Soc. B*, **273** (2006) 1743.
- [6] GROSFILS P., BOON J.P., COHEN E.G.D. and BUNIMOVICH L.A., *J. Stat. Phys.*, **97** (1999) 575.
- [7] FREUND H. and GRASSBERGER P., *Physica A*, **190** (1992) 218.
- [8] BUNIMOVICH L.A., *Physica D*, **187** (2004) 20.
- [9] DERRIDA B. and FLYVBJERG H., *J. Physique*, **48** (1987) 971.
- [10] BOYER D., MIRAMONTES O., RAMOS-FERNANDEZ G., MATEOS J.L. and COCHO G., *Physica A*, **342** (2004) 329.
- [11] SANTOS M.C., BOYER D., MIRAMONTES O., VISWANATHAN G.M., RAPOSO E.P., MATEOS J.L. and DA LUZ M.G.E., *Phys. Rev. E*, **75** (2007) 061114.
- [12] ELSON E.C., BASSETT B.A., VAN DER HEYDEN K. and VILAKAZI Z.Z., *Astron. & Astrophys.*, **464** (2007) 1167.
- [13] KINOUCHE O., MARTINEZ A.S., LIMA G.F., LOURENÇO G.M. and RISAU-GUSMAN S., *Physica A*, **315** (2002) 665.
- [14] CAMPITELI M.G., BATISTA P.D., KINOUCHE O. and MARTINEZ A.S., *Phys. Rev. E*, **74** (2006) 026703.
- [15] CRESSONI J.C., DA SILVA M.A.A. and VISWANATHAN G.M., *Phys. Rev. Lett.*, **98** (2007) 070603.
- [16] KAUFFMAN S.A., *At home in the universe* (Oxford University Press, Oxford) 1995.
- [17] PONMURUGAN M., NARASIMHAN S.L. and MURTHY K.P.N., *Physica A*, **371** (2006) 171.
- [18] ARNOLS S.J., *In: Behavioral Mechanisms in evolutionary ecology*, edited by REAL L.A. (University of Chicago Press, Chicago) 1994, p. 258.
- [19] LOPEZ-CORONA O. and BOYER D., *unpublished*.
- [20] BOUCHAUD J.P., *J. Phys. I France*, **2** (1992) 1705.
- [21] SORNETTE D., *Phys. Rep.*, **297** (1998) 239.
- [22] REDNER S., *A guide to first-passage processes* (Cambridge University Press, Cambridge) 2001.
- [23] SORNETTE D., arXiv:0707.2194v1.
- [24] SORNETTE D., *In: Extreme Events in Nature and Society*, edited by ALBEVERIO S., JENTSCH V., KANTZ H. (Springer, Berlin) 2006, p. 95.
- [25] GOULD S.J. and ELDREDGE N., *Paleobiol.*, **3** (1977) 115.
- [26] SNEPPEN K., BAK P., FLYVBJERG H. and JENSEN M.H., *Proc. Natl. Acad. Sci. USA*, **92** (1995) 5209.
- [27] BAK P. and SNEPPEN K., *Phys. Rev. Lett.*, **71** (1993) 4083.
- [28] BAB M.A. and ALBANO E.V., *J. Phys. A-Math. Theor.*, **41** (2008) 045001.
- [29] KENKRE V.M., arXiv:0708.0034v2.
- [30] BERNASCONI J. and SCHNEIDER W.R., *J. Phys. A: Math. Gen.*, **15** (1982) L729.
- [31] BAK P., TANG C. and WIESENFELD K., *Phys. Rev. Lett.*, **59** (1987) 381.
- [32] GIL L. and SORNETTE D., *Phys. Rev. Lett.*, **76** (1996) 3991.