Higher radiative corrections in HQET

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After a brief introduction to Heavy Quark Effective Theory, we discuss α representation in HQET and methods of calculation of some kinds of HQET diagrams up to three loops.

1 Introduction

Effective field theories are very useful for describing physics at low energies $\ll M$, or large distances $\gg 1/M$, where M is a high energy scale where some new particles or interactions become important. Effective Lagrangians are constructed as series in 1/M. Coefficients in them are obtained by matching scattering amplitudes in the full theory and in the effective one up to some order in 1/M. These matching coefficients are the only quantities which depend on M. All calculations inside the effective theory involve only characteristic energy scales (they are $\ll M$) of processes under consideration. The case when there is one such scale is especially simple. We can choose the renormalization scale μ of order of this characteristic energy scale. Then there will be no large logarithms in perturbative series, and truncating such series will produce small errors. If we try to consider the same process in the full theory, there is a second scale M, and no choice of μ allows us to get rid of large logarithms. Also, each extra scale in Feynman diagrams (with loops) makes their calculation much more difficult technically.

Heavy Quark Effective Theory (HQET) is an effective low-energy field theory for some problems in QCD. In many widely-known effective field theories (Heisenberg–Euler theory of low-energy photon interactions, Fermi 4-fermion theory of weak interactions at low energies) the heavy particle (electron or W in these examples) does not appear. In HQET, the heavy quark appears in initial and final states, but is always nearly on-shell and non-relativistic (in some reference frame).

HQET is discussed in textbooks [1, 2] in detail. Here we shall concentrate on methods and results of calculations of multiloop Feynman diagrams in HQET, see e.g. [3]. Various methods of multiloop calculations are presented in the excellent book [4] in great detail, most of these methods are used in HQET.

2 Heavy Quark Effective Theory

2.1 Lagrangian and Feynman rules

We are going to consider a class of QCD problems involving a single heavy quark with (on-shell) mass $m \gg \Lambda_{\text{QCD}}$. Namely, we require that there exists a reference frame where it stays nearly

at rest all the time. In other words, there exists a 4-velocity v ($v^2 = 1$) such that

$$p = mv + \tilde{p}, \qquad (2.1)$$

and the characteristic residual momentum $\tilde{p}^{\mu} \ll m$. Light quarks and gluons also have characteristic momenta $p_i^{\mu} \ll m$. Such problems can be described, instead of QCD, by a simpler effective field theory called HQET. Its Lagrangian is a series in 1/m. At the leading order,

$$L = \overline{\tilde{Q}}_v iv \cdot D\tilde{Q}_v + \mathcal{O}\left(\frac{1}{m}\right) + (\text{light fields}).$$
(2.2)

The HQET heavy-quark field satisfies $\oint \tilde{Q}_v = \tilde{Q}_v$. All light fields are described as in QCD. In the v rest frame,

$$L = \tilde{Q}^{+} i D_0 \tilde{Q} + \mathcal{O}\left(\frac{1}{m}\right) + (\text{light fields}), \qquad (2.3)$$

where \tilde{Q} is a 2-component spinor.

The mass shell of the heavy quark, i.e., the dependence of its residual energy \tilde{p}_0 on its momentum \vec{p} , is

$$\tilde{p}_0 = p_0 - m = \frac{\vec{p}^2}{2m}.$$

At the leading order in 1/m, it becomes $\tilde{p}_0 = 0$. This is exactly what follows from the Lagrangian (2.3).

The HQET Lagrangian (2.2) is not Lorentz-invariant, because it contains a fixed vector v. However, v is not uniquely defined. It can be changed by $\sim \tilde{p}/m$ (see (2.1)). Lagrangians with such different choices of v must produce identical physical predictions. This requirement is called reparametrization invariance, and it restricts $1/m^n$ corrections in the Lagrangian.

The heavy-quark chromomagnetic moment is, by dimensionality, $\sim 1/m$. Therefore, at the leading order the heavy-quark spin does not interact with the gluon field. We may rotate the spin at will without changing physics — heavy-quark spin symmetry. In particular, B and B^* are degenerate and have identical properties, because they can be transformed into each other by rotating the b spin. We can even change the magnitude of the heavy-quark spin (e.g., to switch it off) without changing physics — this supersymmetry group is called superflavour symmetry.

It is difficult to simulate a QCD heavy quark on the lattice because the lattice spacing a must be much less than the minimum characteristic distance of the problem, 1/m. The HQET Lagrangian does not contain m, and the only applicability condition of its discretization is $a \ll 1/\tilde{p}$. When we investigate the structure of heavy–light hadrons, $\tilde{p} \sim \Lambda_{\rm QCD}$, and the condition $a \ll 1/\Lambda_{\rm QCD}$ is the same as for light hadrons.

The Lagrangian (2.2) gives the Feynman rules

$$\overbrace{\tilde{p}}^{a} = i \frac{1 + \psi}{2} \frac{1}{\tilde{p} \cdot v + i0}, \qquad \Longrightarrow = igt^{a}v^{\mu}. \qquad (2.4)$$

In the v rest frame, the propagator is (the unit 2×2 spin matrix assumed)

$$i \frac{1}{\tilde{p}_0 + i0}$$
. (2.5)

In the coordinate space, the heavy quark does not move:

$$= -i\theta(x_0)\delta(\vec{x}) .$$
 (2.6)

HQET-quark loops vanish because the heavy quark propagates only forward in time. We can also see this in the momentum space: all poles of the propagators in such a loop are in the lower \tilde{p}_0 half-plane, and closing the integration contour upwards, we get 0.

These Feynman rules can be also obtained from QCD at $m \to \infty$. The QCD massive-quark propagator gives the HQET one:

$$\begin{array}{c} & & & \\ & & & \\ \hline mv + \tilde{p} \end{array} = \underbrace{\qquad} & & \\ \hline \tilde{p} \end{array} + \mathcal{O}\left(\frac{\tilde{p}}{m}\right), \\ \\ & & \\ \hline \frac{m + m\psi + \tilde{p}}{(mv + \tilde{p})^2 - m^2 + i0} = \frac{1 + \psi}{2} \frac{1}{\tilde{p} \cdot v + i0} + \mathcal{O}\left(\frac{\tilde{p}}{m}\right). \end{array}$$
(2.7)

The QCD vertex, when sandwiched between two projectors, becomes the HQET one:

$$\frac{1+\not p}{2}\gamma^{\mu}\frac{1+\not p}{2} = \frac{1+\not p}{2}v^{\mu}\frac{1+\not p}{2}.$$
(2.8)

When there is an external leg near a vertex, there is no projector; but we can insert it, and the argument holds.

We have thus proved that at the tree level any QCD diagram is equal to the corresponding HQET diagram up to $\mathcal{O}(\tilde{p}/m)$ corrections. This is not true at loops, because loop momenta can be arbitrarily large. Renormalization properties of HQET (anomalous dimensions, etc.) differ from those in QCD.

2.2 One-loop propagator diagram

Let's calculate the simplest one-loop diagram (Fig. 1)

$$\frac{1}{i\pi^{d/2}} \int \frac{d^d k}{\left[-2\left(k+\tilde{p}\right)\cdot v-i0\right]^{n_1} \left[-k^2-i0\right]^{n_2}} = \frac{1}{i\pi^{d/2}} \int \frac{dk_0 \, d^{d-1}\vec{k}}{\left[-2\left(k_0+\omega\right)-i0\right]^{n_1} \left[-k_0^2+\vec{k}\,^2-i0\right]^{n_2}} = (-2\omega)^{d-n_1-2n_2} I(n_1,n_2) \,. \tag{2.9}$$

It depends only on the residual energy $\omega = \tilde{p}_0$, not \vec{p} ; the power of -2ω is clear from dimensional counting.

If $\omega > 0$, real pair production is possible, and we are on a cut. We shall consider the case $\omega < 0$, when the integral is an analytic function of ω . We'll set $-2\omega = 1$. If n_1 is integer and $n_1 \leq 0$, $I(n_1, n_2) = 0$ because this is a massless vacuum diagram. If n_2 is integer and $n_2 \leq 0$, $I(n_1, n_2) = 0$ because the diagram contains an HQET loop.

At $\omega < 0$, all poles in the k_0 plane are below the real axis at $k_0 > 0$ and above the real axis at $k_0 < 0$, and we can rotate the integration contour counterclockwise without crossing poles (if $\omega > 0$, we cross the pole at $k_0 = -\omega - i0$). This Wick rotation

$$k_0 = ik_{E0} \tag{2.10}$$

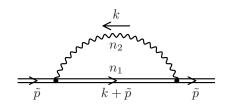


Figure 1: One-loop propagator diagram

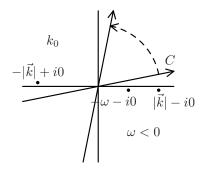


Figure 2: Wick rotation

(Fig. 2) brings us into Euclidean momentum space $(k^2 = -k_E^2)$. In the Euclidean space,

$$I(n_1, n_2) = \frac{1}{\pi^{d/2}} \int \frac{dk_{E0}}{\left(1 - 2ik_{E0}\right)^{n_1}} \int \frac{d^{d-1}\vec{k}}{\left(\vec{k}^2 + k_{E0}^2\right)^{n_2}}.$$
 (2.11)

Using the well-known formula

$$\int \frac{d^d k_E}{(k_E^2 + m^2)^n} = \pi^{d/2} (m^2)^{d/2 - n} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)}$$
(2.12)

with $d \to d-1$, $m^2 \to k_{E0}^2$, $n \to n_2$, we obtain

$$I(n_1, n_2) = \frac{\Gamma\left(n_2 - \frac{d-1}{2}\right)}{\pi^{1/2} \Gamma(n_2)} \int \frac{\left(k_{E0}^2\right)^{(d-1)/2 - n_2} dk_{E0}}{\left(1 - 2ik_{E0}\right)^{n_1}}.$$
(2.13)

The integrand is even in k_{E0} , and has cuts at $k_{E0}^2 < 0$ (it would be wrong to write it as $k_{E0}^{d-1-2n_2}$). Deforming the integration contour around the upper cut (Fig. 3), we can express the integral via the discontinuity at this cut:

$$I(n_1, n_2) = 2 \frac{\Gamma\left(n_2 - \frac{d-1}{2}\right)}{\pi^{1/2} \Gamma(n_2)} \cos\left[\pi\left(\frac{d}{2} - n_2\right)\right] \int_0^\infty \frac{k^{d-1-2n_2} dk}{(2k+1)^{n_1}}.$$
 (2.14)

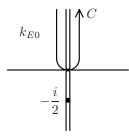


Figure 3: Integration contour

This integral can be easily calculated in Γ functions:

$$I(n_1, n_2) = \frac{2^{2n_2 - d + 1}}{\pi^{1/2}} \cos\left[\pi \left(\frac{d}{2} - n_2\right)\right] \frac{\Gamma(d - 2n_2)\Gamma(n_1 + 2n_2 - d)\Gamma\left(n_2 - \frac{d - 1}{2}\right)}{\Gamma(n_1)\Gamma(n_2)}.$$
 (2.15)

Using the well-known properties of the Γ function

$$\Gamma(2x) = \pi^{-1/2} 2^{2x-1} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right), \qquad \Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin \pi x}, \qquad (2.16)$$

we can simplify this result:

$$I(n_1, n_2) = \frac{\Gamma(n_1 + 2n_2 - d)\Gamma(\frac{d}{2} - n_2)}{\Gamma(n_1)\Gamma(n_2)}.$$
(2.17)

It is also easy to derive this result in coordinate space [3]. HQET propagators in momentum and coordinate space are related by

$$\int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{(-2\omega - i0)^n} \frac{d\omega}{2\pi} = \frac{i}{2\Gamma(n)} \left(\frac{it}{2}\right)^{n-1} e^{-0t} \theta(t) , \qquad (2.18)$$

$$\int_{0}^{\infty} e^{(i\omega-0)t} \left(\frac{it}{2}\right)^{n-1} dt = -\frac{2i\Gamma(n)}{(-2\omega-i0)^{n}};$$
(2.19)

massless propagators — by

$$\int \frac{e^{-ip \cdot x}}{(-p^2 - i0)^n} \frac{d^d p}{(2\pi)^d} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(d/2 - n)}{\Gamma(n)} \left(\frac{4}{-x^2 + i0}\right)^{d/2 - n},$$
(2.20)

$$\int \left(\frac{4}{-x^2+i0}\right)^n e^{ip \cdot x} d^d x = -i(4\pi)^{d/2} \frac{\Gamma(d/2-n)}{\Gamma(n)} \frac{1}{(-p^2-i0)^{d/2-n}}.$$
(2.21)

Our diagram in coordinate space (Fig. 4, x = vt) is just the product of the heavy propagator (2.18) and the light one (2.20) (where $-x^2/4 = -t^2/4 = (it/2)^2$):

$$-\frac{1}{2}\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)}\left(\frac{it}{2}\right)^{n_1+2n_2-d-1}\theta(t)\,.$$

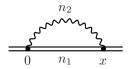


Figure 4: One-loop propagator diagram in coordinate space

The inverse Fourier transform (2.19) gives our diagram (2.9) in momentum space

$$\frac{i}{(4\pi)^{d/2}}I(n_1,n_2)(-2\omega)^{d-n_1-2n_2}$$

where $I(n_1, n_2)$ is given by (2.17).

2.3 Renormalization

The Lagrangian contains bare fields and parameters:

$$L = \overline{\tilde{Q}}_{v0} iv \cdot D_0 \tilde{Q}_{v0} \qquad D_{0\mu} = \partial_\mu - ig_0 A^a_{0\mu} t^a \,. \tag{2.22}$$

They are related to the renormalized ones by the renormalization constants:

$$\tilde{Q}_{v0} = \tilde{Z}_Q^{1/2} \tilde{Q}_v, \quad A_0 = Z_A^{1/2} A, \quad a_0 = Z_A a, \quad g_0 = Z_\alpha^{1/2} g,$$
(2.23)

where a_0 is the gauge-fixing parameter. The ghost field, the light-quark fields (and their masses) are renormalized as in QCD (not written here). Minimal renormalization constants have the structure

$$Z_i = 1 + \frac{Z_{11}}{\varepsilon} \frac{\alpha_s}{4\pi} + \left(\frac{Z_{22}}{\varepsilon^2} + \frac{Z_{21}}{\varepsilon}\right) \left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots$$
(2.24)

They don't contain ε^0 and ε^n (n > 0) terms, only negative powers needed to remove divergences, and hence are called minimal. We have to define α_s to be exactly dimensionless. In the $\overline{\text{MS}}$ scheme α_s depends on the renormalization scale μ :

$$\frac{g_0^2}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\alpha_s(\mu)}{4\pi} Z_\alpha(\alpha_s(\mu)) e^{\gamma_E \varepsilon} , \qquad (2.25)$$

where γ_E is the Euler constant.

Let's calculate the HQET propagator with one-loop accuracy:

$$\longleftrightarrow + \longleftrightarrow + \cdots$$

$$i\tilde{S}(\omega) = i\tilde{S}_{0}(\omega) + i\tilde{S}_{0}(\omega)(-i)\tilde{\Sigma}(\omega)i\tilde{S}_{0}(\omega) + \cdots$$
(2.26)

where

$$\tilde{S}_0(\omega) = \frac{1}{\omega} \,. \tag{2.27}$$

The one-loop heavy-quark self-energy (Fig. 5) is

$$\tilde{\Sigma}(\omega) = iC_F \int \frac{d^d k}{(2\pi)^d} ig_0 v^{\mu} \frac{1}{k_0 + \omega} ig_0 v^{\nu} \frac{-i}{k^2} \left(g_{\mu\nu} - \xi \frac{k_{\mu}k_{\nu}}{k^2}\right) , \qquad (2.28)$$

where $\xi = 1 - a_0$. In the numerator, we may replace $(k \cdot v)^2 = (k_0 + \omega - \omega)^2 \rightarrow \omega^2$, because if we cancel $k_0 + \omega$ in the denominator the integral vanishes. Using (2.17), we obtain

$$\tilde{\Sigma}(\omega) = C_F \frac{g_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \left[2I(1,1) + \frac{\xi}{2}I(1,2) \right] = C_F \frac{g_0^2 (-2\omega)^{1-2\varepsilon}}{(4\pi)^{d/2}} \frac{\Gamma(1+2\varepsilon)\Gamma(1-\varepsilon)}{d-4} \left(\xi + \frac{2}{d-3}\right).$$
(2.29)

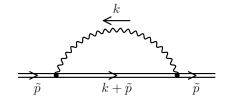


Figure 5: One-loop heavy-quark self-energy

The propagator expressed via renormalized quantities is

$$\omega \tilde{S}(\omega) = 1 + C_F \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-2L\varepsilon} \left[3 - a(\mu) + 4\varepsilon + \cdots \right], \qquad (2.30)$$

where

$$L = \log \frac{-2\omega}{\mu} \,.$$

It should be equal $\tilde{S}(\omega) = \tilde{Z}_Q \tilde{S}_r(\omega)$, where the renormalized propagator $\tilde{S}_r(\omega)$ is finite at $\varepsilon \to 0$. Therefore,

$$\tilde{Z}_Q = 1 + C_F (3-a) \frac{\alpha_s}{4\pi\varepsilon}$$
(2.31)

(it is also easy to write $\tilde{S}_r(\omega)$).

The HQET field does not renormalized in the Yennie gauge a = 3. This is exactly the reason why this gauge has been introduced in the theory of electrons interacting with soft photons (Bloch–Nordsieck model), which is the Abelian HQET. In the Abelian case, this non-renormalization holds to all orders due to the exponentiation theorem. In HQET, this is only true at one loop.

Now we shall discuss the renormalization of g. Due to the gauge invariance, all g's in the Lagrangian are equal. The coupling of the HQET quark field to gluon is thus identical to the usual QCD coupling, where the HQET heavy flavour is not counted in the number of flavours n_f . In order to find Z_{α} we need to renormalize the heavy-quark – gluon vertex and all propagators attached to it. We have already calculated \tilde{Z}_Q . The renormalization of the gluon propagator is well known (Fig. 6):

$$Z_A = 1 - \left[\frac{C_A}{2}\left(a - \frac{13}{3}\right) + \frac{4}{3}T_F n_f\right]\frac{\alpha_s}{4\pi\varepsilon}.$$
(2.32)

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Figure 6: One-loop gluon self-energy

Let's introduce the vertex

where $\tilde{\Lambda}^{\mu}$ starts from one loop. When expressed via renormalized quantities, the vertex should be $\tilde{\Gamma} = \tilde{Z}_{\Gamma}\tilde{\Gamma}_{r}$, where the renormalized vertex $\tilde{\Gamma}_{r}$ is finite at $\varepsilon \to 0$. A physical matrix element is obtained from the corresponding vertex by multiplying it by the wave-function renormalization constant $Z_{i}^{1/2}$ for each external leg. In our case,

$$g_0 \tilde{\Gamma} \tilde{Z}_Q Z_A^{1/2} = g \tilde{\Gamma}_r Z_\alpha^{1/2} \tilde{Z}_\Gamma \tilde{Z}_Q Z_A^{1/2} = \text{finite.}$$

Therefore, $Z_{\alpha}^{1/2} \tilde{Z}_{\Gamma} \tilde{Z}_Q Z_A^{1/2}$ must be finite. But the only minimal (2.24) renormalization constant finite at $\varepsilon \to 0$ is 1:

$$Z_{\alpha} = \left(\tilde{Z}_{\Gamma}\tilde{Z}_{Q}\right)^{-2}Z_{A}^{-1}.$$
(2.34)

At one loop the HQET vertex is given by two diagrams (Fig. 7). It is very easy to calculate the first one. It contains two heavy denominators which can be replaced by a difference:

$$\frac{1}{(k_0 + \omega)(k_0 + \omega')} = \frac{1}{\omega' - \omega} \left(\frac{1}{k_0 + \omega} - \frac{1}{k_0 + \omega'} \right) .$$
(2.35)

We get the difference of the self-energies:

$$\tilde{\Lambda}_{1}^{\mu} = -\left(1 - \frac{C_{A}}{2C_{F}}\right) \frac{\tilde{\Sigma}(\omega') - \tilde{\Sigma}(\omega)}{\omega' - \omega} v^{\mu} \,.$$
(2.36)

This result can also be obtained from the Ward identity. The UV divergence of this contribution is

$$\tilde{\Lambda}_{1}^{\mu} = \left(C_{F} - \frac{C_{A}}{2}\right) (a-3) \frac{\alpha_{s}}{4\pi\varepsilon} v^{\mu} \,. \tag{2.37}$$



Figure 7: One-loop HQET vertex

The second diagram is more difficult. It has been calculated in [5]. Now we only need its UV divergence, and it should be $\sim v^{\mu}$. We may nullify all external momenta. After that, the diagram will contain no scale and hence vanish. It will contain both UV and IR divergences which cancel. Therefore, we'll have to introduce some IR regularization to get the UV divergence. We have

$$ig_0 \tilde{\Lambda}_2^{\mu} = \frac{C_A}{2} \int \frac{d^d k}{(2\pi)^d} ig_0 v^{\alpha'} \frac{i}{k \cdot v} ig_0 v^{\beta'} \\ \times \frac{-i}{k^2} \left(g_{\alpha\alpha'} - \xi \frac{k_\alpha k_{\alpha'}}{k^2} \right) \frac{-i}{k^2} \left(g_{\beta\beta'} - \xi \frac{k_\beta k_{\beta'}}{k^2} \right) ig_0 V^{\alpha\beta\mu}(k, -k, 0) , \qquad (2.38)$$

where the three-gluon vertex is

$$V^{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = (p_3 - p_2)^{\mu_1} g^{\mu_2\mu_3} + (p_1 - p_3)^{\mu_2} g^{\mu_3\mu_1} + (p_2 - p_1)^{\mu_3} g^{\mu_1\mu_2} \,.$$
(2.39)

It vanishes when contracted with the same vector in all three indices:

$$V^{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)v_{\mu_1}v_{\mu_2}v_{\mu_3} = 0$$

and when contracted in two indices with the corresponding momenta:

$$V^{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)p_{1\,\mu_1}p_{2\,\mu_2} = 0.$$

Therefore, ξ^0 and ξ^2 terms vanish:

$$\tilde{\Lambda}_{2}^{\mu}v_{\mu} = iC_{A}g_{0}^{2}\xi \int \frac{d^{d}k}{(2\pi)^{d}} \frac{k^{2} - (k \cdot v)^{2}}{(k^{2})^{3}}$$

Averaging over k directions $(k \cdot v)^2 \rightarrow k^2/d$, we get

$$\tilde{\Lambda}_{2}^{\mu}v_{\mu} = iC_{A}g_{0}^{2}\xi\left(1 - \frac{1}{d}\right)\int \frac{d^{d}k}{(2\pi)^{d}}\frac{1}{(k^{2})^{2}}$$

The UV divergence of this integral can be obtained by introducing any IR regularization, e.g., an IR cut-off in the Euclidean momentum integral or a small mass:

$$\int \left. \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2)^2} \right|_{\rm UV} = \frac{i}{(4\pi)^{d/2} \varepsilon} \,. \tag{2.40}$$

We arrive at the UV divergence of the second vertex diagram:

$$\tilde{\Lambda}_{2}^{\mu} = -\frac{3}{4}C_{A}(1-a)\frac{\alpha_{s}}{4\pi\varepsilon}v^{\mu}.$$
(2.41)

From (2.37) and (2.41) we obtain

$$\tilde{Z}_{\Gamma} = 1 + \left[C_F(a-3) + C_A \frac{a+3}{4} \right] \frac{\alpha_s}{4\pi\varepsilon} \,. \tag{2.42}$$

The product which appears in (2.34) is

$$\tilde{Z}_{\Gamma}\tilde{Z}_Q = 1 + C_A \frac{a+3}{4} \frac{\alpha_s}{4\pi\varepsilon} \,. \tag{2.43}$$

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In the Abelian case $\tilde{Z}_{\Gamma}\tilde{Z}_Q = 1$ to all orders, due to the Ward identity. This is why we only got the non-abelian colour structure C_A in (2.43). Finally, combining this with (2.32), we see that the *a* dependence cancels, and

$$Z_{\alpha} = 1 - \beta_0 \frac{\alpha_s}{4\pi\varepsilon}, \quad \beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F n_f.$$
 (2.44)

Thus we have derived the one-loop β function of QCD (n_f does not include the HQET heavy flavour Q). Most textbooks use the massless-quark – gluon vertex or the ghost – gluon one (see, e.g., [3]). In the later case, calculations are a little shorter. The HQET derivation presented here is as short as the ghost one.

3 α parametrization

3.1 General formulae

 α parametrization of Feynman integrals (including those containing numerators) is discussed in many textbooks, see e.g. [6]. Here we shall discuss HQET integrals; all rules can be trivially obtained from [6], though they were not yet explicitly stated in the literature.

First let's calculate the one-loop diagram (2.9) (Fig. 1) using α parametrization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha \, \alpha^{n-1} e^{-a\alpha} \,. \tag{3.1}$$

We get

$$\frac{1}{\Gamma(n_1)\Gamma(n_2)} \int d\alpha \, \alpha^{n_2-1} \, d\beta \, \beta^{n_1-1} \, d^d k \, e^X \,, \quad X = \alpha k^2 + 2\beta (k+\tilde{p}) \cdot v$$

We shift the integration momentum

$$k = k' - \frac{\beta}{\alpha}v$$

to eliminate the linear term in the exponent:

$$X = \alpha k'^2 - \frac{\beta^2}{\alpha} + 2\beta\omega \,.$$

Now it is easy to calculate the momentum integral:

$$\int d^d k \, e^{\alpha k^2} = i \int d^d k_E \, e^{-\alpha k_E^2} = i \left(\frac{\pi}{\alpha}\right)^{d/2} \,. \tag{3.2}$$

Therefore,

$$(-2\omega)^{d-n_1-2n_2}I(n_1,n_2) = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int d\alpha \,\alpha^{n_2-1} \,d\beta \,\beta^{n_1-1} \,\alpha^{-d/2} \exp\left(-\frac{\beta^2}{\alpha} + 2\beta\omega\right) \,.$$

Now we make the substitution $\beta = \alpha y$ and integrate in α :

$$\frac{\Gamma(n_1+n_2-\frac{d}{2})}{\Gamma(n_1)\Gamma(n_2)}\int_0^\infty dy\, y^{n_1-1} \big[y(y-2\omega)\big]^{d/2-n_1-n_2}\,.$$

The HQET Feynman parameter y has the dimensionality of energy and varies from 0 to ∞ . The y integral can be easily calculated in Γ functions, and we again obtain (2.17).

Now we shall consider the most general HQET Feynman integral without numerators. Any HQET diagram contains a single heavy line and has the form

$$I = \int \prod \frac{d^d k_i}{i\pi^{d/2}} \frac{1}{\prod L_a^{n_a} \prod H_c^{n_c}},$$
(3.3)

where k_i are loop momenta $(i, j \in [1, L])$,

$$L_a = m_a^2 - q_a^2 - i0, \quad H_c = -2q_c \cdot v - i0$$

are light denominators $(a, b \in [1, N_l])$ and heavy ones $(c, d \in [1, N_h])$:

$$q_a = \sum N_{ai}k_i + \sum N_{an}p_n, \quad q_c = \sum N_{ci}k_i + \sum N_{cn}p_n,$$

 p_n are external momenta $(n, m \in [1, N_e - 1])$, and the coefficients N express momenta of propagators via the loop and external momenta (these coefficients are equal to 0 or ± 1). Using the α representation (3.1) for all lines, we obtain

$$I = \frac{1}{\prod \Gamma(n_a) \prod \Gamma(n_c)} \int \prod d\alpha_a \, \alpha_a^{n_a - 1} \prod d\beta_c \, \beta_c^{n_c - 1} \prod \frac{d^d k_i}{i\pi^{d/2}} e^X ,$$

$$X = \sum \alpha_a (q_a^2 - m_a^2) + 2 \sum \beta_c q_c \cdot v .$$
(3.4)

Dimensionalities of the parameters are $\alpha_a \sim 1/M^2$, $\beta_c \sim 1/M$. The exponent is

$$X = \sum M_{ij}k_i \cdot k_j - 2\sum Q_i \cdot k_i + Y, \qquad (3.5)$$

where

$$M_{ij} = \sum \alpha_a N_{ai} N_{aj} ,$$

$$Q_i = -\sum \alpha_a N_{ai} N_{an} p_n - v \sum \beta_c N_{ci} ,$$

$$Y = \sum \alpha_a \left(\sum N_{an} p_n\right)^2 + 2 \sum \beta_c N_{cn} p_n \cdot v - \sum \alpha_a m_a^2 .$$
(3.6)

Now we shift the loop momenta $k_i = k'_i + K_i$ to eliminate linear terms:

$$K_i = \sum M_{ij}^{-1} Q_j \,. \tag{3.7}$$

Then

$$X = \sum M_{ij}k'_{i} \cdot k'_{j} - \sum M_{ij}^{-1}Q_{i} \cdot Q_{j} + Y.$$
(3.8)

Performing the Wick rotation to Euclidean k'_i and integration in the loop momenta, we obtain

$$I = \frac{1}{\prod \Gamma(n_a) \prod \Gamma(n_c)} \int \prod d\alpha_a \, \alpha_a^{n_a - 1} \prod d\beta_c \, \beta_c^{n_c - 1} \left[D(\alpha) \right]^{-d/2} \\ \times \exp\left[-\frac{A(\alpha) + A_1(\alpha, \beta) + A_2(\alpha, \beta)}{D(\alpha)} - \sum \alpha_a m_a^2 \right],$$
(3.9)

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where

$$D(\alpha) = \det M,$$

$$\frac{A(\alpha)}{D(\alpha)} = \sum M_{ij}^{-1} \alpha_a \alpha_b N_{ai} N_{an} N_{bj} N_{bm} p_n \cdot p_m - \sum \alpha_a N_{an} N_{am} p_n \cdot p_m,$$

$$\frac{A_1(\alpha, \beta)}{D(\alpha)} = 2 \sum M_{ij}^{-1} \alpha_a \beta_c N_{ai} N_{an} N_{cj} p_n \cdot v - 2 \sum \beta_c N_{cn} p_n \cdot v,$$

$$\frac{A_2(\alpha, \beta)}{D(\alpha)} = \sum M_{ij}^{-1} \beta_c \beta_d N_{ci} N_{dj}.$$
(3.10)

The polynomials $D(\alpha)$, $A(\alpha)$, $A_1(\alpha, \beta)$, $A_2(\alpha, \beta)$ have dimensionality $1/M^{2L}$. The function $D(\alpha)$ is homogeneous in α_a of degree L. The function $A(\alpha)$ is homogeneous in α_a of degree L + 1 and linear in $p_n \cdot p_m$. The function $A_1(\alpha, \beta)$ is linear in β_c , of degree L in α_a , and linear in $p_n \cdot v$. The function $A_2(\alpha, \beta)$ is quadratic in β_c and of degree L - 1 in α_a ; it does not contain momenta.

It is always possible to calculate (at least) one integration in (3.9). Let's insert $\delta (\sum \alpha_a - \eta) d\eta$ under the integral sign, and make the substitution $\alpha_a = \eta x_a$, $\beta_c = \eta y_c$. Then the η integral is a Γ function:

$$I = \frac{\Gamma\left(\sum n_a + \sum n_c - L\frac{d}{2}\right)}{\prod \Gamma(n_a) \prod \Gamma(n_c)} \int \frac{\prod dx_a \, x_a^{n_a - 1} \prod dy_c \, y_c^{n_c - 1} \, \delta\left(\sum x_a - 1\right)}{[D(x)]^{d/2} \left[\frac{A(x) + A_1(x, y) + A_2(x, y)}{D(x)} + \sum x_a m_a^2\right]^{\sum n_a + \sum n_c - Ld/2}}.$$
(3.11)

The ordinary Feynman parameters x_a are dimensionless and vary from 0 to 1; the HQET Feynman parameters y_c have dimensionality of energy and vary from 0 to ∞ .

3.2 Graph-theoretical rules

The polynomials $D(\alpha)$, $A(\alpha)$, $A_1(\alpha, \beta)$, $A_2(\alpha, \beta)$ can be extracted directly from the diagram, see [6]. We shall formulate the rules and illustrate them by an example shown in Fig. 8.

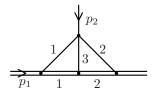
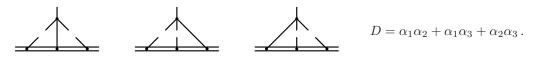


Figure 8: An HQET vertex diagram

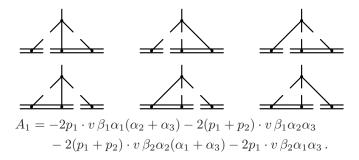
1. Cut a few light lines so as to get a connected tree, form the product of α_a of the cut lines. $D(\alpha)$ is the sum of all such products.



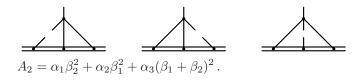
2. Cut a few light lines to get two connected trees (the heavy line is in one of the two parts). Form the product of α_a of the cut lines and multiply it by $(-P^2)$, where P is the momentum flowing from one connected part to the other one. $A(\alpha)$ is the sum of all such terms.

$$A = -p_2^2 \alpha_1 \alpha_2 \alpha_3 \,.$$

3. Cut a single heavy line and a few light ones to get two connected trees (now the heavy line enters one connected part and leaves the other one). Form the product of β_c of the cut heavy line and α_a of the cut light ones and multiply by $(-2P \cdot v)$, where P is the momentum flowing from the first connected part to the second one. $A_1(\alpha, \beta)$ is the sum of all such terms.



4. Cut a few light lines to get a connected diagram with a single loop in such a way that this loop contains at least one heavy line. Sum β_c of the heavy lines belonging to the loop, square the sum, and multiply by α_a of the cut light lines. Sum all terms.



These rules can be simplified a little. Suppose p_1 is the residual momentum of the incoming heavy line, and let's route it along the heavy line. Then the exponent X (3.4) contains $2p_1 \cdot v \sum \beta_c$. We can add this expression to the exponent in (3.9), and then set $p_1 = 0$ while calculating $A_1(\alpha, \beta)$.

There is an analogy between Feynman diagrams in α representation and electrical circuits (Table 1). The average momentum flowing through a propagator corresponds to current. The first Kirchhoff rule is satisfied: the sum of momenta flowing into a vertex vanishes. Light lines are resistors α_a , and heavy lines — voltage sources $\beta_c v$ (batteries with zero internal resistance, the voltage does not depend on the current). The second Kirchhoff rule says that the sum of voltages along a loop (say, loop *i*) must vanish. These equations are nothing but the equations $\sum M_{ij}K_j = Q_i$ which determine the average loop momenta K_i (see (3.7)).

The Joule heat $\sum \alpha_a \bar{q}_a^2$ plus 2 times the energy consumption by the voltage sources $\sum \beta_c \bar{q}_c \cdot v$ gives

$$\frac{A(\alpha) + A_1(\alpha, \beta) + A_2(\alpha, \beta)}{D(\alpha)}$$

| | Current | Voltage |
|-------------------------------|---|----------------------|
| $\bullet \rightarrow \bullet$ | $\bar{q}_a = \sum N_{ai} K_i + \sum N_{an} p_n$ | $\alpha_a \bar{q}_a$ |
| | $\bar{q}_c = \sum N_{ci} K_i + \sum N_{cn} p_n$ | $\beta_c v$ |

Table 1: Analogy with electrical circuits

The case $\alpha_a \to 0$ or $\beta_c \to 0$ corresponds to a short circuit (line shrinks to a point); the case $\alpha_a \to \infty$ — no contact (the line is removed).

Generalization to integrals with numerators is straightforward [6]. Suppose we have a polynomial $\mathcal{P}(q_a, q_c)$ inserted into the numerator of (3.3). Then we can add

$$2\sum q_a \cdot \xi_a + 2\sum q_c \cdot \eta_c$$

to the exponent X (3.4), and apply the differential operator

$$\mathcal{P}\left(\frac{1}{2}\frac{\partial}{\partial\xi_a}, \frac{1}{2}\frac{\partial}{\partial\eta_c}\right)$$

to the result at $\xi_a = 0$, $\eta_c = 0$. Before this step, the only difference is the substitution $\beta_c v \rightarrow \beta_c v + \eta_c$, and the fact that a light line *a* can be also considered "heavy" in A_1 and A_2 calculations, with ξ_a playing the role of βv . Let's formulate the rules to calculate A_1 , A_2 .

3'. Cut a single heavy line (say, c) and a few light ones to get two connected trees, and form the product of $-2(\beta_c v + \eta_c) \cdot P$, where P is the momentum flowing from the first connected part to the second one; multiply this product by α_a of all cut light lines. Or cut a single light line (say, a) and a few light ones to get two connected trees, form the product $-2\xi_a \cdot P$, and multiply it by α_b of these additional cut lines. Here the first and the second connected parts are defined by the direction of the momentum of the first cut line (q_c or q_a for a heavy or light line). $A_1(\alpha, \beta, \xi, \eta)$ is the sum of all such terms.

4'. Cut a few light lines to get a connected diagram with a single loop. Sum $\beta_c v + \eta_c$ or ξ_a of the heavy or light lines belonging to the loop, square the sum, and multiply by α_a of the cut light lines. Sum all terms to get $A_2(\alpha, \beta, \xi, \eta)$.

4 HQET propagator diagrams

4.1 Two loops

4.1.1 Diagram 1

We have calculated the one-loop HQET propagator diagram by three different methods (Sects. 2.2 and 3.1). Now we shall consider two-loop propagator diagrams in HQET. There are two generic topologies of such diagrams (Fig. 9). This means that all other possible topologies can be obtained from these ones by shrinking some lines. The method of calculation of these diagrams has been constructed in [7].

The first diagram (Fig. 10) is

$$-\frac{1}{\pi^d} \int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = (-2\omega)^{2d-n_1-n_2-2(n_3+n_4+n_5)} I(n_1, n_2, n_3, n_4, n_5), \qquad (4.1)$$



Figure 9: Generic topologies of two-loop propagator diagrams

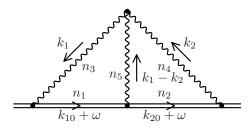


Figure 10: Diagram 1

where

$$D_1 = -2(k_{10} + \omega), \quad D_2 = -2(k_{20} + \omega),$$

$$D_3 = -k_1^2, \quad D_4 = -k_2^2, \quad D_5 = -(k_1 - k_2)^2$$

(the power of -2ω is fixed by dimensionality). It is symmetric with respect to $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$, and vanishes if two adjacent indices are ≤ 0 .

If $n_5 = 0$, the diagram is the product of two one-loop ones:

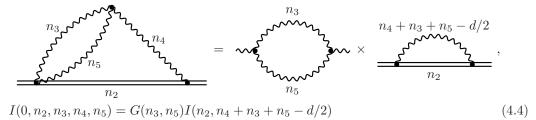
$$I(n_1, n_2, n_3, n_4, 0) = \underbrace{\underbrace{\int_{n_1}^{n_3} \int_{n_4}^{n_4} }_{n_1 n_2}}_{n_1 n_2} = I(n_1, n_3)I(n_2, n_4).$$
(4.2)

If $n_1 = 0$, we first calculate the inner massless loop. The one-loop massless diagram is

$$\frac{1}{i\pi^{d/2}} \int \frac{d^d k}{\left[-(k+p)^2 - i0\right]^{n_1} \left[-k^2 - i0\right]^{n_2}} = (-p^2)^{d/2 - n_1 - n_2} G(n_1, n_2),$$

$$G(n_1, n_2) = \frac{\Gamma(-d/2 + n_1 + n_2)\Gamma(d/2 - n_1)\Gamma(d/2 - n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d - n_1 - n_2)}.$$
(4.3)

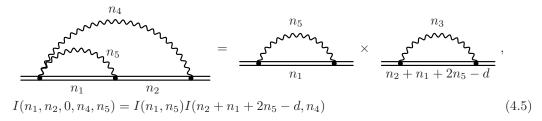
This gives the coefficient $G(n_3, n_5)$, and shifts the power n_4 by $n_3 + n_5 - d/2$:



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(the case $n_2 = 0$ is symmetric). If $n_3 = 0$, we first calculate the inner HQET loop. This gives the coefficient $I(n_1, n_5)$, and shifts the power n_2 by $n_1 + 2n_5 - d$:



(the case $n_4 = 0$ is symmetric).

But what can we do if all 5 powers of denominators are positive? We shall use integration by parts [8]. Integral of any full derivative over the whole space of loop momenta is zero. When applied to the integrand of (4.1), the derivative

$$\frac{\partial}{\partial k_2} \to \frac{n_2}{D_2} 2v + \frac{n_4}{D_4} 2k_2 + \frac{n_5}{D_5} 2(k_2 - k_1).$$

Applying $(\partial/\partial k_2) \cdot k_2$ or $(\partial/\partial k_2) \cdot (k_2 - k_1)$ to the integrand, we obtain zero integral. On the other hand, we can calculate these derivatives explicitly. Using $2k_2 \cdot v = -D_2 - 2\omega$, $2(k_2 - k_1) \cdot k_2 = D_3 - D_4 - D_5$, we see that applying these differential operators is equivalent to inserting

$$d - n_2 - n_5 - 2n_4 - 2\omega \frac{n_2}{D_2} + \frac{n_5}{D_5}(D_3 - D_4),$$

$$d - n_2 - n_4 - 2n_5 + \frac{n_2}{D_2}D_1 + \frac{n_4}{D_4}(D_3 - D_5)$$

under the integral sign. These combinations of integrals vanish. These recurrence relations are usually written as

$$\left[d - n_2 - n_5 - 2n_4 + n_2 \mathbf{2}^+ + n_5 \mathbf{5}^+ (\mathbf{3}^- - \mathbf{4}^-)\right] I = 0, \qquad (4.6)$$

$$\left[d - n_2 - n_4 - 2n_5 + n_2 \mathbf{2}^+ \mathbf{1}^- + n_4 \mathbf{4}^+ (\mathbf{3}^- - \mathbf{5}^-)\right] I = 0, \qquad (4.7)$$

where, for example, 1^- lowers n_1 by 1 and 2^+ raises n_2 by 1. Applying $(\partial/\partial k_2) \cdot v$, we obtain a (less useful) relation

$$\left[-2n_2\mathbf{2}^+ + n_4\mathbf{4}^+(\mathbf{2}^- - 1) + n_5\mathbf{5}^+(\mathbf{2}^- - \mathbf{1}^-)\right]I = 0.$$
(4.8)

A useful relation can be obtained from homogeneity of the integral (4.1) in ω . Applying $\omega(d/d\omega)$, we get the same integral times its dimensionality $2(d - n_3 - n_4 - n_5) - n_1 - n_2$. On the other hand, we can calculate the derivative explicitly:

$$\left[2(d-n_3-n_4-n_5)-n_1-n_2+n_1\mathbf{1}^++n_2\mathbf{2}^+\right]I=0.$$
(4.9)

This homogeneity relation is not independent: it is the sum of the $(\partial/\partial k_2) \cdot k_2$ relation (4.6) and its mirror-symmetric $(\partial/\partial k_1) \cdot k_1$ one.

A particularly useful relation can be obtained by subtracting the 1^- shifted homogeneity relation (4.9) from the $(\partial/\partial k_2) \cdot (k_2 - k_1)$ relation (4.7):

$$[d - n_1 - n_2 - n_4 - 2n_5 + 1 - (2(d - n_3 - n_4 - n_5) - n_1 - n_2 + 1)\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{3}^- - \mathbf{5}^-)]I = 0.$$
(4.10)

Solving it for the I with the unshifted indices, we obtain an expression for $I(n_1, n_2, n_3, n_4, n_5)$ via 3 integrals:

$$I = \frac{(2(d - n_3 - n_4 - n_5) - n_1 - n_2 + 1)\mathbf{1}^- + n_4\mathbf{4}^+(\mathbf{5}^- - \mathbf{3}^-)}{d - n_1 - n_2 - n_4 - 2n_5 + 1}I.$$
 (4.11)

Each of them has $n_1 + n_3 + n_5$ reduced by 1. Each application of (4.11) moves us closer to the origin (Fig. 11). Therefore, after a finite number of steps, any integral $I(n_1, n_2, n_3, n_4, n_5)$ will be reduced to the trivial cases in which one of the indices vanishes.

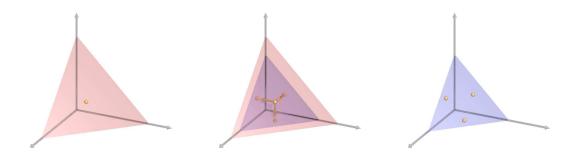


Figure 11: A single step of the integration-by-parts reduction

4.1.2 Diagram 2

The second diagram (Fig. 12) is

$$-\frac{1}{\pi^d} \int \frac{d^d k_1 \, d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}} = (-2\omega)^{2d-n_1-n_2-n_3-2(n_4+n_5)} J(n_1, n_2, n_3, n_4, n_5) \,, \tag{4.12}$$

where

$$D_1 = -2(k_{10} + \omega), \quad D_2 = -2(k_{20} + \omega), \quad D_3 = -2(k_{10} + k_{20} + \omega),$$

$$D_4 = -k_1^2, \quad D_5 = -k_2^2$$

(the power of -2ω is fixed by dimensionality). It is symmetric with respect to $(1 \leftrightarrow 2, 4 \leftrightarrow 5)$, and vanishes if $n_4 \leq 0$ or $n_5 \leq 0$ or two adjacent $n_{1...3}$ are ≤ 0 .

This integral is trivial if $n_3 = 0$ or $n_{1,2} = 0$. In general, it has 3 linear denominators and only 2 loop momenta; therefore, these denominators are linearly dependent:

$$D_1 + D_2 - D_3 = -2\omega \,. \tag{4.13}$$

Inserting this combination under the integral sign, we obtain

$$J = (\mathbf{1}^{-} + \mathbf{2}^{-} - \mathbf{3}^{-})J.$$
(4.14)

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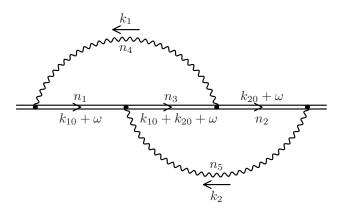


Figure 12: Diagram 2

Each application of this recurrence relation reduces $n_1 + n_2 + n_3$ by 1. Therefore, after a number of such steps any integral will reduce to the trivial cases (Fig. 11).

The integral (4.12) can contain a power of $k_1 \cdot k_2$ in the numerator; this scalar product cannot be expressed via the denominators. However, this is not a serious problem [9].

Let's summarize. All scalar integrals belonging to the two generic topologies of Fig. 9, with any indices n_i (and with any power of $k_1 \cdot k_2$ in the numerator of the second integral) can be reduced to linear combinations of two master integrals

$$\underline{ I}^{\prime}_{1}, \qquad \underline{ I}^{\prime}_{1}, \qquad \underline{ I}^{\prime}_{2}, \qquad \underline{ I}^{\prime}_{2}, \qquad (4.15)$$

with coefficients being rational functions of d. Here the n-loop HQET sunset integral is

$$I_n = \frac{\Gamma(1+2n\varepsilon)\Gamma^n(1-\varepsilon)}{(1-n(d-2))_{2n}}.$$
(4.16)

This reduction can be done using integration by parts [7] (see also [9]).

4.2 Three loops

4.2.1 Reduction

There are 10 generic topologies of three-loop HQET propagator diagrams (Fig. 13).

All these integrals, with any powers of denominators and irreducible numerators, can be

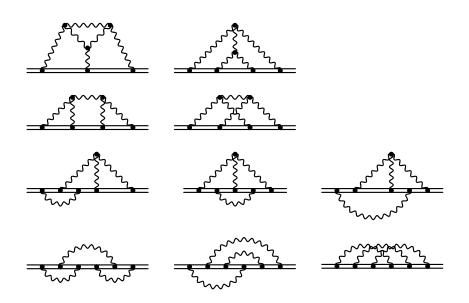


Figure 13: Generic topologies of three-loop propagator diagrams

reduced [9] to 8 master integrals:

$$= I_1^3, \qquad (4.17)$$

$$= I_1 I_2, \qquad (4.18)$$

$$= I_3, \qquad (4.19)$$

$$= \frac{I_1^2 I(6-2d,1)}{I_2 I(5-2d,1)} I_3 = \frac{3d-7}{2d-5} \frac{I_1^2}{I_2} I_3, \qquad (4.20)$$

$$\underbrace{-\frac{G_1^2 I(1,4-d)}{G_2 I(1,3-d)}}_{= \frac{G_1^2 I(1,4-d)}{G_2 I(1,3-d)}} I_3 = -2 \frac{3d-7}{d-3} \frac{G_1^2}{G_2} I_3, \qquad (4.21)$$

$$= G_1 I(1, 1, 1, 1, 2 - d/2), \qquad (4.22)$$

$$= I_1 J(1, 1, 3 - d, 1, 1), \qquad (4.23)$$

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$$= B_8, \qquad (4.24)$$

using integration by parts. Here the *n*-loop HQET sunset I_n is defined by (4.16), and the *n*-loop massless sunset is

This reduction algorithm has been implemented as a REDUCE package Grinder [9]. It is analogous to the massless package Mincer [10]. The first 5 master integrals can be easily expressed via Γ functions, exactly in *d* dimensions. The next two ones reduce to two-loop ones with a single ε -dependent index¹ (Sects. 4.2.2 and 4.2.3). The last one is truly three-loop (Sect. 4.2.4).

4.2.2 J(1, 1, n, 1, 1)

Here we shall calculate the integral J (4.12) (Fig. 12) for arbitrary powers of denominators. To this end, we shall first consider the one-loop diagram with two different residual energies ω_1 and ω_2 (Fig. 14a):

$$I = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{D_1^{n_1} D_2^{n_2} D_3^{n_3}},$$
(4.26)

where

$$D_1 = -2(k_0 + \omega_1), \quad D_2 = -2(k_0 + \omega_2), \quad D_3 = -k^2.$$

If $n_{1,2}$ are integer, this integral can be easily calculated by partial fraction decomposition (Sect. 2.3).

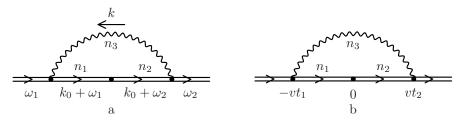


Figure 14: One-loop diagram

Closely following Sect. 2.2, we first integrate in $d^{d-1}\vec{k}$:

$$I = \frac{\Gamma(n_3 - \frac{d-1}{2})}{\pi^{1/2}\Gamma(n_3)} \int \frac{\left(k_{E0}^2\right)^{(d-1)/2 - n_3} dk_{E0}}{\left(-2\omega_1 - 2ik_{E0}\right)^{n_1} \left(-2\omega_2 - 2ik_{E0}\right)^{n_2}}$$
$$= 2\frac{\Gamma(n_3 - \frac{d-1}{2})}{\pi^{1/2}\Gamma(n_3)} \cos\left[\pi \left(\frac{d}{2} - n_3\right)\right] \int_0^\infty \frac{k^{d-1-2n_3} dk}{(2k - 2\omega_1)^{n_1} (2k - 2\omega_2)^{n_2}}$$

¹Grinder uses $B_4 = I_3 I_1^2 / I_2$ and $B_5 = I_3 G_1^2 / G_2$ as elements of its basis instead of (4.20) and (4.21).

We obtain [11]

$$I = I(n_1 + n_2, n_3)_2 F_1 \begin{pmatrix} n_1, n_1 + n_2 + 2n_3 - d \\ n_1 + n_2 \end{pmatrix} \left(1 - \frac{\omega_1}{\omega_2} \right) (-2\omega_2)^{d - n_1 - n_2 - 2n_3}.$$
(4.27)

One can easily check that this result is symmetric with respect to $(\omega_1 \leftrightarrow \omega_2, n_1 \leftrightarrow n_2)$, using properties of hypergeometric function. If $\omega_1 = \omega_2$, it reduces to $I(n_1 + n_2, n_3)$.

Let's also calculate this integral using α representation (Sect. 3.1):

$$I = \frac{1}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} \int d\alpha \, \alpha^{n_3 - 1} \, d\beta_1 \, \beta_1^{n_1 - 1} \, d\beta_2 \, \beta_2^{n_2 - 1} \, \alpha^{-d/2} \\ \times \exp\left[-\frac{(\beta_1 + \beta_2)^2}{\alpha} + 2(\omega_1\beta_1 + \omega_2\beta_2)\right] \,.$$

Now we make the substitution $\beta_{1,2} = \alpha y_{1,2}$ and integrate in α :

$$I = \frac{\Gamma(n_1 + n_2 + n_3 - \frac{d}{2})}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} \int dy_1 \, y_1^{n_1 - 1} \, dy_2 \, y_2^{n_2 - 1} \left[(y_1 + y_2)^2 - 2(\omega_1 y_1 + \omega_2 y_2) \right]^{d/2 - n_1 - n_2 - n_3}$$

After the substitution $y_1 = yx$, $y_2 = y(1 - x)$, the integral in y can be taken:

$$I = \frac{\Gamma(\frac{d}{2} - n_3)\Gamma(n_1 + n_2 + 2n_3 - \frac{d}{2})}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} \times \int_0^1 dx \, x^{n_1 - 1} \, (1 - x)^{n_2 - 1} \, \left[-2\omega_1 x - 2\omega_2(1 - x)\right]^{d - n_1 - n_2 - 2n_3} \,. \tag{4.28}$$

And we again obtain (4.27).

Finally, we shall derive the same result in coordinate space (Fig. 14b, see Sect. 2.2):

$$I = -\frac{1}{4} \frac{\Gamma(\frac{d}{2} - n_3)}{\Gamma(n_1)\Gamma(n_2)\Gamma(n_3)} \int dt_1 \, dt_2 \, e^{i(\omega_1 t_1 + \omega_2 t_2)} \left(\frac{it_1}{2}\right)^{n_1 - 1} \left(\frac{it_2}{2}\right)^{n_2 - 1} \left(\frac{i(t_1 + t_2)}{2}\right)^{2n_3 - d_1} \, dt_2 \, dt_2 \, dt_3 \, dt_4 \, dt_2 \, dt_4 \,$$

The substitution $t_1 = tx$, $t_2 = t(1 - x)$ reduces this expression to (4.28).

Now we return to our main problem — calculating $J = J(n_1, n_2, n_3, n_4, n_5)$ (Fig. 12) with arbitrary indices. We set $-2\omega = 1$; the power of -2ω can be reconstructed by dimensionality. Substituting the one-loop subdiagram (4.27), we have

$$\begin{split} J &= \frac{I(n_1 + n_3, n_4)}{i\pi^{d/2}} \int \frac{d^d k}{(-k^2)^{n_5} (1 - 2k_0)^{n_2}} \\ &\times (1 - 2k_0)^{d - n_1 - n_3 - 2n_4} {}_2F_1 \left(\begin{array}{c} n_1, n_1 + n_3 + 2n_4 - d \\ n_1 + n_3 \end{array} \middle| \frac{-2k_0}{1 - 2k_0} \right) \\ &= \frac{I(n_1 + n_3, n_4) \Gamma\left(n_5 - \frac{d - 1}{2}\right)}{\pi^{d/2} \Gamma(n_5)} \int_{-\infty}^{+\infty} \frac{(k_{E0}^2)^{(d - 1)/2 - n_5} dk_{E0}}{(1 - 2ik_{E0})^{n_1 + n_2 + n_3 + 2n_4 - d}} \\ &\times {}_2F_1 \left(\begin{array}{c} n_1, n_1 + n_3 + 2n_4 - d \\ n_1 + n_3 \end{array} \middle| \frac{-2ik_{E0}}{1 - 2ik_{E0}} \right). \end{split}$$

We can deform the integration contour (Fig. 3, $k_{E0} = iz/2$):

$$J = \frac{I(n_1 + n_3, n_4)\Gamma\left(n_5 - \frac{d-1}{2}\right)}{2^{d-2n_5-1}\pi^{d/2}\Gamma(n_5)} \cos\left[\pi\left(\frac{d}{2} - n_5\right)\right] \\ \times \int_0^\infty \frac{z^{d-2n_5-1}dz}{(z+1)^{n_1+n_2+n_3+2n_4-d}} {}_2F_1\left(\begin{array}{c} n_1, n_1 + n_3 + 2n_4 - d \\ n_1 + n_3 \end{array} \middle| \frac{z}{z+1} \right) \,.$$

Now we substitute the series

$${}_{2}F_{1}\left(\begin{array}{c}a,b\\c\end{array}\middle|x\right) = \frac{\Gamma(c)\Gamma(b)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(n+a)\Gamma(n+b)}{\Gamma(n+1)\Gamma(n+c)}x^{n},$$

and integrate term by term. The result is

$$J = \frac{I(n_1 + n_3, n_4)\Gamma\left(n_5 - \frac{d-1}{2}\right)\Gamma(d - 2n_5)\Gamma(n_1 + n_2 + n_3 + 2n_4 + 2n_5 - 2d)}{2^{d-2n_5 - 1}\pi^{d/2}\Gamma(n_5)\Gamma(n_1 + n_2 + n_3 + 2n_4 - d)} \cos\left[\pi\left(\frac{d}{2} - n_5\right)\right] \times {}_{3}F_2\left(\begin{array}{c}n_1, n_1 + n_3 + 2n_4 - d, d - 2n_5\\n_1 + n_3, n_1 + n_2 + n_3 + 2n_4 - d\end{array}\right| 1\right).$$

Using (2.16), we can simplify this result:

$$J(n_1, n_2, n_3, n_4, n_5) = \frac{\Gamma(\frac{d}{2} - n_4)\Gamma(\frac{d}{2} - n_5)\Gamma(n_1 + n_3 + 2n_4 - d)\Gamma(n_1 + n_2 + n_3 + 2n_4 + 2n_5 - 2d)}{\Gamma(n_4)\Gamma(n_5)\Gamma(n_1 + n_3)\Gamma(n_1 + n_2 + n_3 + 2n_4 - d)} \times {}_{3}F_2 \begin{pmatrix} n_1, n_1 + n_3 + 2n_4 - d, d - 2n_5 \\ n_1 + n_3, n_1 + n_2 + n_3 + 2n_4 - d \\ \end{pmatrix} \left| 1 \right\rangle.$$

$$(4.29)$$

It was first derived in coordinate space [9, 3]. Checking the symmetry $(n_1 \leftrightarrow n_2, n_4 \leftrightarrow n_5)$ requires using some ${}_3F_2$ identities.

4.2.3 *I*(1, 1, 1, 1, *n*)

This diagram has been calculated in [12] using Gegenbauer polynomial technique in coordinate space [13]:

$$= I(1,1,1,1,n) = \frac{\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-n-1\right)}{\Gamma(d-2)}$$

$$\times \left[2\frac{\Gamma(2n-d+3)\Gamma(2n-2d+6)}{(n-d+3)\Gamma(3n-2d+6)}{}_{3}F_{2}\left(\begin{array}{c}n-d+3,n-d+3,2n-2d+6\\n-d+4,3n-2d+6\end{array}\right)1\right)$$

$$-\Gamma(d-n-2)\Gamma^{2}(n-d+3)\right].$$
(4.30)

Some details of this method are discussed in [3].

4.2.4 Inversion

The last three-loop master integral has been calculated [15] using inversion. We shall first consider inversion relations at one and two loops. The one-loop massive on-shell integral defined by

$$\int \frac{d^d k}{\left[m^2 - (k+mv)^2 - i0\right]^{n_1} \left[-k^2 - i0\right]^{n_2}} = i\pi^{d/2} m^{d-2(n_1+n_2)} M(n_1, n_2)$$
(4.31)

can be written in terms of the dimensionless Euclidean momentum $K = k_E/m$:

$$\int \frac{d^d K}{(K^2 - 2iK_0)^{n_1} (K^2)^{n_2}} = \pi^{d/2} M(n_1, n_2) \,.$$

Similarly, the one-loop HQET propagator integral (2.9) expressed via $K = k_E/(-2\omega)$ is

$$\int \frac{d^d K}{(1-2iK_0)^{n_1} (K^2)^{n_2}} = \pi^{d/2} I(n_1, n_2) \,.$$

Inversion $K = K'/K'^2$ transforms the massive on-shell denominator into the HQET one:

$$K^2 - 2iK_0 = \frac{1 - 2iK'_0}{K'^2}$$

Therefore,

$$\frac{n_2}{n_1} = \frac{d - n_1 - n_2}{n_1},$$

$$M(n_1, n_2) = I(n_1, d - n_1 - n_2) = \frac{\Gamma(d - n_1 - 2n_2)\Gamma(-d/2 + n_1 + n_2)}{\Gamma(n_1)\Gamma(d - n_1 - n_2)}.$$
(4.32)

Similarly, at two loops we obtain [14]

This relation is less useful, because the HQET diagram in the right-hand side contains two non-integer indices.

At three loops we have [3]

.

In particular, the HQET ladder diagram with all indices $n_i = 1$ is convergent; its value at d = 4 is related [15] to a massive on-shell diagram

$$\underline{- \zeta_{5} + 12\zeta_{2}\zeta_{3}} = \underline{- \zeta_{5} + 12\zeta_{2}\zeta_{3}}. \quad (4.35)$$

by the second inversion relation. This is one of the on-shell three-loop master integrals, and its value at d = 4 is known [16, 17]. Calculating this ladder diagram with Grinder:

$$=4\frac{(d-3)^2}{(d-4)^2}I_1^3 - \frac{136}{3}\frac{(d-3)(2d-5)(2d-7)}{(d-4)^3}I_1I_2 + 2\frac{(3d-7)(3d-8)(81d^3-891d^2+3266d-3988)}{(d-4)^4(2d-7)}I_3 + 9\frac{(d-3)(3d-7)(3d-8)(3d-10)(3d-11)}{(d-4)^3(2d-5)(2d-7)}\frac{I_1^2}{I_2}I_3 + 8\frac{(d-3)(3d-7)(3d-11)}{(d-4)^3}\frac{G_1^2}{G_2}I_3 - \frac{3}{2}\frac{(d-3)(3d-10)}{(d-4)^2}G_1I(1,1,1,1,2-\frac{d}{2}) - \frac{3d-11}{d-4}B_8,$$
(4.36)

and solving for the most difficult HQET three-loop master integral B_8 (4.24), we obtain the ε expansion of this integral up to $\mathcal{O}(\varepsilon)$. This concludes the investigation of three-loop master integrals, and allows one to solve three-loop propagator problems in HQET up to terms $\mathcal{O}(1)$.

4.2.5 Applications

Using this technique, the HQET heavy-quark propagator has been calculated up to three loops [18], and the heavy-quark field anomalous dimension (obtained earlier by a completely different method [17]) has been confirmed. The anomalous dimension of the HQET heavy–light quark current has been calculated [18]. The correlator of two heavy–light currents has been found, up to three loops, including light-quark mass corrections of order m and m^2 [19]. The quark-condensate contribution to this correlator has been also calculated up to three loops [19]. Its ultraviolet divergence yields the difference of twice the anomalous dimension of the heavyquark current and the that of the quark condensate, thus providing a completely independent confirmation of the result obtained in [18]. The gluon-condensate contribution has been calculated up to two loops [19] (at one loop it vanishes).

5 On-shell HQET propagator diagrams with mass

5.1 Two loops

On-shell HQET propagator diagrams vanish if all flavours (except the HQET one) are considered massless, because loop integrals contain no scale. If there is a massive flavour (c in the b-quark HQET), such diagrams are non-zero. They first appear at two loops. They are used, e.g., to calculate on-shell renormalization constants in HQET.

Let's first consider [20] a class of such integrals (Fig. 15)

$$F(n_1, n_2) = \int \frac{f(k^2) d^d k}{D_1^{n_1} D_2^{n_2}}, \quad \text{where} \quad D_1 = -2k \cdot v - i0, \quad D_2 = -k^2 - i0, \quad (5.1)$$

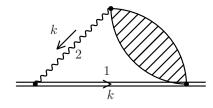


Figure 15: Diagram with a single HQET line

and $f(k^2)$ is an arbitrary function. We can construct an identity in which $f'(k^2)$ terms cancel:

$$\frac{\partial}{\partial k} \cdot \left(k - 2\frac{D_2}{D_1}v\right) \frac{f(k^2)}{D_1^{n_1} D_2^{n_2}} = \left[d - n_1 - 2 - 4(n_1 + 1)\frac{D_2}{D_1^2}\right] \frac{f(k^2)}{D_1^{n_1} D_2^{n_2}}.$$
(5.2)

Integrating it, we obtain an integration-by-parts relation [20]

$$(d - n_1 - 2)F(n_1, n_2) = 4(n_1 + 1)\mathbf{1}^{++}\mathbf{2}^{-}F(n_1, n_2).$$
(5.3)

Let's call integrals with even n_1 apparently even, and with odd n_1 — apparently odd (they would be even and odd in v if we neglected i0 in the denominator). These two classes of integrals are not mixed by the recurrence relation (5.3). We can use this relation to reduce all apparently even integrals to vacuum integrals with $n_1 = 0$ (Fig. 16). Apparently odd integrals with $n_1 < 0$ can be reduced to $n_1 = -1$. Substituting $n_1 = -1$ to (5.3), we see that these integrals vanish, and hence all integrals with odd $n_1 < 0$ vanish too. Apparently odd integrals with $n_1 > 0$ can be reduced to $n_1 = 1$ (Fig. 16); however, they are not related to those with $n_1 = -1$.

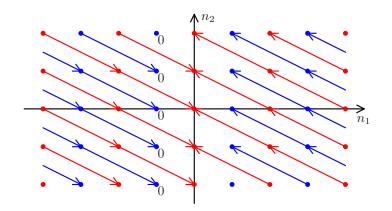


Figure 16: Recurrence relation

The solution of the recurrence relation can thus be written as

$$F(n_1, n_2) = \begin{cases} (-4)^{-n_1/2} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d-n_1}{2}\right)} \frac{\Gamma\left(\frac{1-n_1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} F\left(0, n_2 + \frac{n_1}{2}\right) & \text{even } n_1, \\ 2^{1-n_1} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{n_1+1}{2}\right)\Gamma\left(\frac{d-n_1}{2}\right)} F\left(1, n_2 + \frac{n_1-1}{2}\right) & \text{odd } n_1 > 0, \\ 0 & \text{odd } n_1 < 0. \end{cases}$$
(5.4)

Some of these properties can be understood more directly. If $n_1 < 0$, i0 in $D_1^{-n_1}$ can be safely neglected; averaging this factor over k directions, we obtain 0 for odd n_1 and the upper formula in (5.4) for even n_1 . It was suggested [21] that this last formula can also be used for even $n_1 > 0$, but the proof (presented here) only appeared in [20].

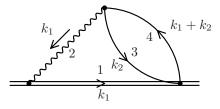


Figure 17: Two-loop diagram

Now let's consider the two-loop diagram (Fig. 17)

$$F(n_1, n_2, n_3, n_4) = \frac{1}{(i\pi^{d/2})^2} \int \frac{d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4}},$$
(5.5)

where

$$D_1 = -2k_1 \cdot v - i0, \quad D_2 = -k_1^2 - i0,$$

$$D_3 = 1 - k_2^2 - i0, \quad D_4 = 1 - (k_1 + k_2)^2 - i0.$$

It is symmetric with respect to $3 \leftrightarrow 4$, and vanishes if n_3 or n_4 is integer and non-positive. It can be calculated using α parametrization [20]:

$$F(n_1, n_2, n_3, n_4) = (5.6)$$

$$\frac{\Gamma(\frac{n_1}{2})\Gamma(\frac{d-n_1}{2} - n_2)\Gamma(\frac{n_1-d}{2} + n_2 + n_3)\Gamma(\frac{n_1-d}{2} + n_2 + n_4)\Gamma(\frac{n_1}{2} + n_2 + n_3 + n_4 - d)}{2\Gamma(n_1)\Gamma(n_3)\Gamma(n_4)\Gamma(\frac{d-n_1}{2})\Gamma(n_1 + 2n_2 + n_3 + n_4 - d)}.$$

In full accordance with (5.4), integrals $F(n_1, n_2, n_3, n_4)$ with even n_1 reduce to $F(0, n_2 + n_1/2, n_3, n_4)$ (this is a well-known two-loop vacuum integral [22]); those with odd $n_1 > 0$ reduce to $F(1, n_2 + (n_1 - 1)/2, n_3, n_4)$; and those with odd $n_1 < 0$ vanish. All apparently even integrals are proportional to the single master integral

$$I_0^2 = \underbrace{} (5.7)$$

and apparently odd ones — to

$$J_0 = - = 2^{4d-9} \pi^2 \frac{\Gamma(5-2d)}{\Gamma^2(2-d/2)}.$$
 (5.8)

Integrals (5.5) can also contain powers of $(2k_2 + k_1) \cdot v$ in the numerator; see [20] for details of their evaluation.

5.2 Three loops

5.2.1 Reduction

There are two generic topologies of three-loop on-shell HQET propagator diagrams with a massive loop (Fig. 18). Algorithms of their reduction to master integrals, using integration by parts identities, have been constructed [20] by Gröbner bases technique [23].

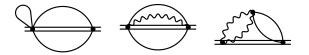


Figure 18: Topologies of three-loop on-shell HQET propagator diagrams with mass

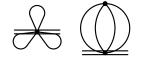
All apparently even integrals of the first topology reduce to



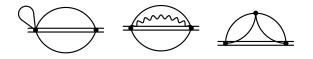
while apparently odd ones to



All apparently even integrals of the second topology reduce to



while apparently odd ones to



The master integrals



can be easily expressed via Γ functions. The master integral



has been investigated in detail [24, 25].

5.2.2 A master integral

Now we shall discuss the integrals

$$I_{n_1 n_2 n_3} = \underbrace{n_2 \quad n_2}_{n_1 \quad n_1} \tag{5.9}$$

 $(I_{111} \text{ is one of the master integrals})$. Several approaches have been tried [20, 26]. The best result was obtained [26] using a method similar to [24].

First we consider (following [28]) the one-loop subdiagram

$$I_{n_1 n_2}(p_0) = \frac{n_2}{n_1} = \frac{1}{i\pi^{d/2}} \int \frac{dk_0 \, d^{d-1} \vec{k}}{[-2(k_0 + p_0) - i0]^{n_1} [1 - k^2 - i0]^{n_2}} \,. \tag{5.10}$$

After the Wick rotation, we integrate in $d^{d-1}\vec{k}$:

$$I_{n_1n_2}(p_0) = \frac{\Gamma(n_2 - (d-1)/2)}{\pi^{1/2}\Gamma(n_2)} \int_{-\infty}^{+\infty} dk_{E0} \frac{(k_{E0}^2 + 1)^{(d-1)/2 - n_2}}{(-2p_0 - 2ik_{E0})^{n_1}}.$$

If $p_0 < 0$, we can deform the integration contour (Fig. 19):

$$I_{n_1n_2}(p_0) = 2\frac{\Gamma(n_2 - (d-1)/2)}{\pi^{1/2}\Gamma(n_2)} \cos\left[\pi\left(\frac{d}{2} - n_2\right)\right] \int_1^\infty dk \frac{(k^2 - 1)^{(d-1)/2 - n_2}}{(2k - 2p_0)^{n_1}}$$

This integral is

$$I_{n_1n_2}(p_0) = \frac{\Gamma(n_1 + n_2 - 2 + \varepsilon)\Gamma(n_1 + 2n_2 - 4 + 2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1 + n_2 - 2 + \varepsilon))} \times {}_2F_1 \left(\begin{array}{c} n_1, n_1 + 2n_2 - 4 + 2\varepsilon \\ n_1 + n_2 - \frac{3}{2} + \varepsilon \end{array} \middle| \frac{1}{2} (1 + p_0) \right),$$

or, after using a $_2F_1$ identity,

$$I_{n_1n_2}(p_0) = \frac{\Gamma(n_1 + n_2 - 2 + \varepsilon)\Gamma(n_1 + 2n_2 - 4 + 2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1 + n_2 - 2 + \varepsilon))} \times {}_2F_1\left(\begin{array}{c} \frac{1}{2}n_1, \frac{1}{2}n_1 + n_2 - 2 + \varepsilon\\ n_1 + n_2 - \frac{3}{2} + \varepsilon \end{array} \middle| 1 - p_0^2 \right).$$
(5.11)

This result was obtained [26] using the HQET Feynman parametrization:

$$I_{n_1n_2}(p_0) = \frac{\Gamma(n_1 + n_2 - 2 + \varepsilon)}{\Gamma(n_1)\Gamma(n_2)} \int_0^\infty y^{n_1 - 1} (y^2 - 2p_0y + 1)^{2 - n_1 - n_2 - \varepsilon} \, dy$$

This integral at $p_0 < 0$ gives (5.11) (a similar expression has been derived in [27]).

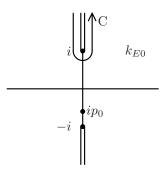


Figure 19: Integration contour

Now we can integrate in $d^{d-1}\vec{p}$ in the three-loop diagram:

$$I_{n_1 n_2 n_3} = \frac{\Gamma(n_3 - 3/2 + \varepsilon)}{\pi^{1/2} \Gamma(n_3)} \int_{-\infty}^{+\infty} I_{n_1 n_2}^2(ip_{E0}) (1 + p_{E0}^2)^{3/2 - n_3 - \varepsilon} dp_{E0} \,. \tag{5.12}$$

The square of $_2F_1$ in (5.11) can be expressed via an $_3F_2$ using the Clausen identity. We analytically continue this $_3F_2$ from $1 + p_{E0}^2 > 1$ to $z = 1/(1 + p_{E0}^2) < 1$ and integrate (5.12) term by term. The result contains, in general, three $_4F_3$ of unit argument.

A convergent integral I_{122} is related to the master integral I_{111} by

$$I_{122} = -\frac{(d-3)^2(d-4)(3d-8)(3d-10)}{8(3d-11)(3d-13)}I_{111}$$

For this integral, we obtain [26]

$$\frac{I_{122}}{\Gamma^{3}(1+\varepsilon)} = -\frac{1}{2\varepsilon^{2}} \left[\frac{1}{1+2\varepsilon} {}_{4}F_{3} \left(\begin{array}{c} 1, \frac{1}{2} - \varepsilon, 1 + \varepsilon, -2\varepsilon \\ \frac{3}{2} + \varepsilon, 1 - \varepsilon, 1 - 2\varepsilon \end{array} \right| 1 \right) \\ - \frac{2}{1+4\varepsilon} \frac{\Gamma^{2}(1-\varepsilon)\Gamma^{3}(1+2\varepsilon)}{\Gamma^{2}(1+\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+4\varepsilon)} {}_{3}F_{2} \left(\begin{array}{c} \frac{1}{2}, 1 + 2\varepsilon, -\varepsilon \\ \frac{3}{2} + 2\varepsilon, 1 - \varepsilon \end{array} \right| 1 \right) \\ + \frac{1}{1+6\varepsilon} \frac{\Gamma^{2}(1-\varepsilon)\Gamma^{4}(1+2\varepsilon)\Gamma(1-2\varepsilon)\Gamma^{2}(1+3\varepsilon)}{\Gamma^{4}(1+\varepsilon)\Gamma(1+4\varepsilon)\Gamma(1-4\varepsilon)\Gamma(1+6\varepsilon)} \right].$$
(5.13)

Expansion of this result up to ε^7 agrees with [26]

$$\frac{I_{122}}{\Gamma^3(1+\varepsilon)} = \frac{\pi^2}{3} \frac{\Gamma^3(1+2\varepsilon)\Gamma^2(1+3\varepsilon)}{\Gamma^6(1+\varepsilon)\Gamma(2+6\varepsilon)}.$$
(5.14)

This equality has also been checked by high precision numerical calculations at some finite ε values. This conjectured hypergeometric identity can also be rewritten in a nice form [28]

$$g_1(\varepsilon)_4 F_3 \left(\begin{array}{c} 1, \frac{1}{2} - \varepsilon, 1 + \varepsilon, -2\varepsilon \\ \frac{3}{2} + \varepsilon, 1 - \varepsilon, 1 - 2\varepsilon \end{array} \middle| 1 \right) - 2g_2(\varepsilon)_3 F_2 \left(\begin{array}{c} \frac{1}{2}, 1 + 2\varepsilon, -\varepsilon \\ \frac{3}{2} + 2\varepsilon, 1 - \varepsilon \end{array} \middle| 1 \right) + g_3(\varepsilon) = 0, \quad (5.15)$$

where

$$b(\varepsilon) = \frac{\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)}, \quad g_n(\varepsilon) = \frac{b^n(\varepsilon)}{b(n\varepsilon)(1+2n\varepsilon)}$$

We have no analytical proof.

5.2.3 Other master integrals

Other master integrals were calculated [20] using Mellin–Barnes representation (see, e.g., [4]). Now we shall discuss a simple example of this technique. Let's consider the one-loop propagator diagram with two massive lines:

$$\frac{1}{i\pi^{d/2}}\int \frac{d^dk}{[m^2-k^2]^{n_1}[m^2-(k+p)^2]^{n_2}}\,.$$

Using Feynman parametrization,

$$= \frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{i\pi^{d/2}} \int \frac{dx \, x^{n_2 - 1} (1 - x)^{n_1 - 1} \, d^d k}{[(m^2 - k^2)(1 - x) + (m^2 - (k + p)^2)x]^{n_1 + n_2}}$$

$$= \frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{i\pi^{d/2}} \int \frac{dx \, x^{n_2 - 1} (1 - x)^{n_1 - 1} \, d^d k}{[-k^2 - 2xp \cdot k - xp^2 + m^2]^{n_1 + n_2}}.$$

After the shift k = k' - xp:

$$= \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{i\pi^{d/2}} \int \frac{dx \, x^{n_2-1}(1-x)^{n_1-1} \, d^d k}{[m^2+x(1-x)(-p^2)-k'^2]^{n_1+n_2}} \,,$$

we can integrate in k':

$$= \frac{\Gamma\left(n_1 + n_2 - \frac{d}{2}\right)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 \frac{dx \, x^{n_2 - 1} (1 - x)^{n_1 - 1}}{[m^2 + x(1 - x)(-p^2)]^{n_1 + n_2 - d/2}} \,.$$

Now we shall use Mellin–Barnes representation

$$\frac{1}{(a+b)^n} = \frac{a^{-n}}{\Gamma(n)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \, \Gamma(-z) \Gamma(n+z) \left(\frac{b}{a}\right)^z \,. \tag{5.16}$$

Here the integration contour is chosen in such a way that all poles of $\Gamma(\dots + z)$ (they are called *left poles*) are to the left of the contour, and all poles of $\Gamma(\dots - z)$ (they are called *right poles*)

are to the right of it. It is easy to check (5.16): closing the contour to the right we get the expansion of the left-hand side in b/a; closing it to the left — the expansion in a/b.

We continue our calculation:

$$= \frac{m^{d-2(n_1+n_2)}}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(-z)\Gamma(n_1+n_2+z) \left(\frac{-p^2}{m^2}\right)^z \int_0^1 dx \, x^{n_2+z-1} (1-x)^{n_1+z-1} dx = \frac{m^{d-2(n_1+n_2)}}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{\Gamma(-z)\Gamma(n_1+z)\Gamma(n_2+z)\Gamma(n_1+n_2-\frac{d}{2}+z)}{\Gamma(n_1+n_2+2z)} \left(\frac{-p^2}{m^2}\right)^z.$$

This means that two massive lines can be replaced by one massless one (raised to the power -z) at the price of one extra integration in z:

$$\underbrace{\sum_{n_2}^{n_1}}_{n_2} = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{\left[m^2 - k^2 - i0\right]^{n_1} \left[m^2 - (k+p)^2 - i0\right]^{n_2}} = \frac{m^{d-2(n_1+n_2)}}{\Gamma(n_1)\Gamma(n_2)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{\Gamma(-z)\Gamma(n_1+z)\Gamma(n_2+z)\Gamma(n_1+n_2-d/2+z)}{\Gamma(n_1+n_2+2z)}$$

$$m^{-2z} \bullet \cdots \bullet$$
(5.17)

This trick allows us to reduce this master integral to a single Mellin–Barnes integral. Using integration by parts, we can kill one of three lines in the left (integer) triangle, and calculate the integrand in Γ functions. This allows us to find several terms of its ε expansion:

$$= \frac{\Gamma^2(2\varepsilon)\Gamma(3\varepsilon-1)}{4\Gamma(4\varepsilon)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz$$

$$\frac{\Gamma(1+z)\Gamma(1/2+\varepsilon+z)\Gamma(1+\varepsilon+z)\Gamma(-2\varepsilon-z)\Gamma(-\varepsilon-z)\Gamma(-\varepsilon)}{\Gamma(3/2+\varepsilon+z)\Gamma(1-2\varepsilon-z)}$$

$$= -\Gamma^3(1+\varepsilon) \left[\frac{\pi^2}{9\varepsilon^2} - \frac{6\zeta_3 - 5\pi^2}{9\varepsilon} + \frac{11}{270}\pi^4 - \frac{10}{3}\zeta_3 + \frac{19}{9}\pi^2 + \left(-\frac{8}{3}\zeta_5 + \frac{8}{9}\pi^2\zeta_3 + \frac{11}{54}\pi^4 - \frac{38}{3}\zeta_3 + \frac{65}{9}\pi^2 \right)\varepsilon + \cdots \right].$$
(5.18)

This master integral has been evaluated in a closed form using Mellin–Barnes in α representation:

$$= \frac{\Gamma(1/2 - \varepsilon)\Gamma(-\varepsilon)\Gamma^2(2\varepsilon)\Gamma(1 + \varepsilon)\Gamma(3\varepsilon - 1)}{4\Gamma(3/2 - \varepsilon)\Gamma(4\varepsilon)} \times \left[\psi\left(\frac{1}{2} - \varepsilon\right) + \psi(1 - \varepsilon) - 2\log 2 + 2\gamma_E\right].$$
(5.19)

This master integral can be written as a double Mellin–Barnes integral using (5.17). It

appears possible to calculate one integral:

$$\int_{-i\infty}^{+i\infty} dz \frac{\Gamma(1+z)\Gamma\left(\frac{3}{2}-\varepsilon+z\right)\Gamma(\varepsilon+z)\Gamma\left(-\frac{1}{2}+\varepsilon-z\right)\Gamma\left(-\frac{3}{2}+2\varepsilon-z\right)\Gamma(-z)}{\Gamma\left(\frac{3}{2}+z\right)\Gamma(\varepsilon-z)}$$
$$= \Gamma^{3}(1+\varepsilon)\frac{32}{3}\pi^{2}\left[-1+2\left(4\log 2-\pi-7\right)\varepsilon+\cdots\right].$$
(5.20)

5.2.4 Applications

Feynman integrals considered here were used [20] for calculating the matching coefficients for the HQET heavy-quark field and the heavy–light quark current between the *b*-quark HQET with dynamic *c*-quark loops and without such loops (the later theory is the low-energy approximation for the former one at scales below m_c). Another recent application — the effect of $m_c \neq 0$ on $b \rightarrow c$ plus lepton pair at three loops [29]. The method of regions was used; the purely soft region (loop momenta ~ m_c) gives integrals of this type. Two extra terms of ε expansion of the master integral of Sect. 5.2.2 were required for this calculation which were not obtained in [20]. This was the initial motivation for [26].

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