Parameter estimation for fractional Ornstein-Uhlenbeck processes

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Abstract

We study a least squares estimator $\widehat{\theta}_T$ for the Ornstein-Uhlenbeck process, $dX_t = \theta X_t dt + \sigma dB_t^H$, driven by fractional Brownian motion B^H with Hurst parameter $H \geq \frac{1}{2}$. We prove the strong consistence of $\widehat{\theta}_T$ (the almost surely convergence of $\widehat{\theta}_T$ to the true parameter θ). We also obtain the rate of this convergence when $1/2 \leq H < 3/4$, applying a central limit theorem for multiple Wiener integrals. This least squares estimator can be used to study other more simulation friendly estimators such as the estimator $\tilde{\theta}_T$ defined by (4.1).

1 Introduction

The Ornstein-Uhlenbeck process X_t driven by a certain type of noise Z_t is described by the Langevin equation

$$X_t = X_0 - \theta \int_0^t X_s ds + \sigma Z_t.$$

If the parameter θ is unknown and if the process $(X_t, 0 \leq t \leq T)$ can be observed continuously, then an important problem is to estimate the parameter θ based on the (single path) observation $(X_t, 0 \leq t \leq T)$. When Z_t is the standard Brownian motion, this problem has been extensively studied (see for example [9], [10] and the references therein). The most popular approaches are either the maximum likelihood estimators or the least squares estimators, and in this case they coincide. Other type of noise processes have also been studied. For example, when Z_t is an α-stable process maximum likelihood estimators do not exist and other approaches are proposed in [5] and [6].

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In this paper we study the parameter estimation problem for the Ornstein-Uhlenbeck process driven by fractional Brownian motion with Hurst parameter H

$$X_t = X_0 - \theta \int_0^t X_s ds + \sigma B_t^H, \tag{1.1}$$

where $\theta > 0$ is an unknown parameter. Although the Ornstein-Uhlenbeck process is defined for all $H \in (0,1)$, we assume $H > 1/2$ in this paper. In [8], the the maximum likelihood estimator $\bar{\theta}_T$ for the parameter θ is obtained and has the following expression

$$\bar{\theta}_T = -\left\{ \int_0^T Q^2(s) dw_s^H \right\}^{-1} \int_0^T Q(s) dZ_s \,,$$

where

$$\begin{aligned}
k_H(t,s) &= \kappa_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\,, \quad \kappa_H = 2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(H+\frac{1}{2}\right)\,; \\
w_t^H &= \lambda_H^{-1} t^{2-2H}\,; \quad \lambda_H = \frac{2H\Gamma(3-2H)\Gamma\left(H+\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}-H\right)}\,; \\
Q(t) &= \frac{d}{dw_t^H}\int_0^t k_H(t,s) X_s ds\,, \quad 0 \le t \le T\,; \\
Z_t &= \int_0^t k_H(t,s) dX_s\,.
\end{aligned}$$

It is proved that $\lim_{T\to\infty} \bar{\theta}_T = \theta$ almost surely.

In this paper we propose two different estimators for the parameter θ and we study their asymptotic behavior. First we introduce an estimator of the form

$$\widehat{\theta}_T = \theta - \sigma \frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt}, \tag{1.2}$$

where $\int_0^T X_t dB_t^H$ is a divergence-type integral (see [1], [3], [4], [7] and the references therein), and we call it the *least squares estimator*. This is motivated by the following heuristic argument. The least square estimator aims to minimize

$$\int_0^T |\dot{X}_t + \theta X_t|^2 dt\,,$$

and this leads to the solution

$$\widehat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}. \tag{1.3}$$

If $H = \frac{1}{2}$, then the integral $\int_0^T X_t dX_t$ is an Itô stochastic integral which can be approximated by forward Riemann sums. However, for $H > \frac{1}{2}$ the numerical simulation of the estimator $\widehat{\theta}_T$ seems extremely difficult. For this reason, in this case we introduce and study a second estimator $\widetilde{\theta}_T$, defined in (4.1).

We prove the almost sure convergence of the estimator $\widehat{\theta}_T$ to θ, as T tends to infinity, and derive the rate of convergence, obtaining a central limit theorem in the case $H \in \left[\frac{1}{2}, \frac{3}{4}\right)$. The proof of the central limit theorem is based on the characterization of the convergence in law for multiple stochastic integrals using the techniques of Malliavin calculus, established recently by Nualart and Ortiz-Latorre in [12]. Finally, we derive the rate of convergence of the estimator $\widetilde{\theta}_T$ from the rate of convergence of $\widehat{\theta}_T$.

2 Preliminaries

In this section we first introduce some basic facts on the Malliavin calculus for the fractional Brownian motion and recall the main result in [12] concerning the central limit theorem for multiple stochastic integrals.

The fractional Brownian motion with Hurst parameter $H \in (0,1)$, $(B_t^H, t \in \mathbb{R})$ is a zero mean Gaussian process with covariance

$$\mathbb{E}(B_t^H B_s^H) = R_H(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) . \tag{2.1}$$

We assume that B^H is defined on a complete probability space $(\Omega, \mathcal{A}, P)$ such that $\mathcal{A}$ is generated by B^H. Fix a time interval $[0,T]$. Denote by $\mathcal{E}$ the set of real valued step functions on $[0,T]$ and let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s),$$

where R_H is the covariance function of the fBm, given in (2.1). The mapping $\mathbf{1}_{[0,t]} \longmapsto B_t^H$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space $\mathcal{H}_1$ spanned by B^H. We denote this isometry by $\varphi \longmapsto B^H(\varphi)$. For $H = \frac{1}{2}$ we have $\mathcal{H} = L^2([0,T])$, whereas for $H > \frac{1}{2}$ we have $L^{\frac{1}{H}}([0,T]) \subset \mathcal{H}$ and for $\varphi, \psi \in L^{\frac{1}{H}}([0,T])$ we have

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T \varphi_s \psi_t |t-s|^{2H-2} ds dt, \tag{2.2}$$

where $\alpha_H = H(2H-1)$.

Let $\mathcal{S}$ be the space of smooth and cylindrical random variables of the form

$$F = f(B^H(\varphi_1), \dots, B^H(\varphi_n)), \tag{2.3}$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded). For a random variable F of the form (2.3) we define its Malliavin derivative as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\varphi_1), \dots, B^H(\varphi_n))\varphi_i.$$

By iteration, one can define the mth derivative $D^m F$, which is an element of $L^2(\Omega; \mathcal{H}^{\otimes m})$, for every $m \geq 2$. For $m \geq 1$, $\mathbb{D}^{m,2}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{m,2}$, defined by the relation

$$\|F\|_{m,2}^2 \;=\; \mathbb{E}\left[|F|^2\right] + \sum_{i=1}^{m} \mathbb{E}\left(\|D^i F\|_{\mathcal{H}^{\otimes i}}^2\right).$$

Let δ be the adjoint of the operator D, also called the *divergence operator*. A random element $u \in L^2(\Omega, \mathcal{H})$ belongs to the domain of δ, denoted $\mathrm{Dom}(\delta)$, if and only if it verifies

$$|E\langle DF, u\rangle_{\mathcal{H}}| \leq c_u \, \|F\|_{L^2},$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \mathrm{Dom}(\delta)$, then the random variable $\delta(u)$ is defined by the duality relationship

$$E(F\delta(u)) = E\langle DF, u\rangle_{\mathcal{H}}, \tag{2.4}$$

which holds for every $F \in \mathbb{D}^{1,2}$. The divergence operator δ is also called the Skorohod integral because in the case of the Brownian motion it coincides with the anticipating stochastic integral introduced by Skorohod in [15]. We will make use of the notation $\delta(u) = \int_0^T u_t dB_t^H$.

For every $n \geq 1$, let $\mathcal{H}_n$ be the nth Wiener chaos of B, that is, the closed linear subspace of $L^2(\Omega, \mathcal{A}, P)$ generated by the random variables $\{H_n\left(B^H(h)\right), h \in H, \|h\|_{\mathcal{H}} = 1\}$, where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = n! H_n\left(B^H(h)\right)$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot n}$ and $\mathcal{H}_n$. For $H = \frac{1}{2}$, I_n coincides with the multiple Itô stochastic integral. On the other hand, $I_n(h^{\otimes n})$ coincides with the iterated divergence $\delta^n(h^{\otimes n})$.

We will make use of the following central limit theorem for multiple stochastic integrals (see [12]).

Theorem 2.1 *Let $\{F_n, n \geq 1\}$ be a sequence of random variables in the p-th Wiener chaos, $p \geq 2$, such that $\lim_{n\to\infty} \mathbb{E}(F_n^2) = \sigma^2$. Then the following conditions are equivalent:*

(i) F_n converges in law to $N(0, \sigma^2)$ as n tends to infinity.

(ii) $\|DF_n\|_{\mathcal{H}}^2$ converges in L^2 to a constant as n tends to infinity.

Remark. In [12] it is proved that (i) is equivalent to the fact that $\|DF_n\|_{\mathcal{H}}^2$ converges in L^2 to $p\sigma^2$ as n tends to infinity. If we assume (ii), the limit of $\|DF_n\|_{\mathcal{H}}^2$ must be equal to $p\sigma^2$ because

$$\mathbb{E}(\|DF_n\|_{\mathcal{H}}^2) = p\mathbb{E}(F_n^2).$$

3 Asymptotic behavior of the least square estimator

Consider Equation (1.1) driven by a fractional Brownian motion B^H with Hurst parameter $H \geq \frac{1}{2}$. Suppose that $X_0 = 0$ and $\theta > 0$. The solution is given by

$$X_t = \sigma \int_0^t e^{-\theta(t-s)} dB_s^H, \tag{3.1}$$

where the stochastic integral is an Itô integral if $H = \frac{1}{2}$ and a path-wise Riemann-Stieltjes integral if $H > \frac{1}{2}$. Let $\widehat{\theta}_T$ be the least squares estimator defined in (1.2). The next lemma provides a useful alternative expression for $\widehat{\theta}_T$.

Lemma 3.1 *Suppose that $H > \frac{1}{2}$. Then*

$$\widehat{\theta}_T = -\frac{X_T^2}{2\int_0^T X_t^2 dt} + \sigma^2 \frac{\alpha_H \int_0^T \int_0^t \xi^{2H-2} e^{-\theta\xi} d\xi dt}{\int_0^T X_t^2 dt}. \tag{3.2}$$

Proof Using the relation between the divergence integral and the path-wise Riemann-Stieltjes integral (see Theorem 3.12 and Equation (3.6) of [3]) we can write

$$\begin{aligned}
\int_0^T X_t \circ dB_t^H &= \int_0^T X_t dB_t^H + \alpha_H \int_0^T \int_0^t D_s X_t (t-s)^{2H-2} ds dt \\
&= \int_0^T X_t dB_t^H + \sigma\alpha_H \int_0^T \int_0^t e^{-\theta(t-s)} (t-s)^{2H-2} ds dt \\
&= \int_0^T X_t dB_t^H + \sigma\alpha_H \int_0^T \int_0^t \xi^{2H-2} e^{-\theta\xi} d\xi dt\,.
\end{aligned}$$

As a consequence, we obtain

$$\widehat{\theta}_T = \theta - \sigma \frac{\int_0^T X_t \circ dB_t^H}{\int_0^T X_t^2 dt} + \sigma^2 \frac{\alpha_H \int_0^T \int_0^t \xi^{2H-2} e^{-\theta\xi} d\xi dt}{\int_0^T X_t^2 dt}. \tag{3.3}$$

On the other hand,

$$\sigma \int_0^T X_t \circ dB_t^H = \int_0^T X_t \circ dX_t + \theta \int_0^T X_t^2 dt = \frac{1}{2} X_T^2 + \theta \int_0^T X_t^2 dt. \tag{3.4}$$

Substituting (3.4) into (3.3) yields (3.2). □

The next theorem establishes the strong consistency of this estimator.

Theorem 3.2 *If $H \geq \frac{1}{2}$, then*

$$\widehat{\theta}_T \to \theta \tag{3.5}$$

almost surely, as T tends to infinity.

In order to prove this theorem we make use of the following technical result.

Lemma 3.3 *Assume $H \geq \frac{1}{2}$. Then,*

$$\frac{1}{T}\int_0^T X_t^2 dt \to \sigma^2 \theta^{-2H} H\Gamma(2H), \tag{3.6}$$

almost surely and in L^2, as T tends to infinity.

Proof For every $t \geq 0$ define

$$Y_t = \sigma \int_{-\infty}^t e^{-\theta(t-s)} dB_s^H = X_t + e^{-\theta t}\xi, \tag{3.7}$$

where $\xi = \sigma \int_{-\infty}^0 e^{\theta s} dB_s^H$. The stochastic process $(Y_t, t \geq 0)$ is Gaussian, stationary and ergodic. For $H = \frac{1}{2}$ this is well-known and for $H > \frac{1}{2}$ this is proved in [2]. Then, the ergodic theorem implies that

$$\frac{1}{T}\int_0^T Y_t^2 dt \to \mathbb{E}(Y_0^2),$$

as T tends to infinity, almost surely and in L^2. This implies that

$$\frac{1}{T}\int_0^T X_t^2 dt \to \mathbb{E}(Y_0^2),$$

as T tends to infinity, almost surely and in L^2. If $H = \frac{1}{2}$, we know that $\mathbb{E}(Y_0^2) = \frac{\sigma^2}{2\theta}$, which implies (3.6). If $H > \frac{1}{2}$, using (2.2) yields

$$\mathbb{E}(X_0^2) = \alpha_H \sigma^2 \int_0^\infty \int_0^\infty e^{-\theta(s+u)} |u-s|^{2H-2} du ds,$$

and (3.6) follows from Lemma 5.1. □

Proof of Theorem 3.2 In the case $H = \frac{1}{2}$, taking into account that the process $\left(\int_0^t X_s dB_s, t \geq 0\right)$ is a martingale with quadratic variation $\int_0^t X_s^2 ds$ it follows that $\widehat{\theta}_T \to \theta$ almost surely, as T tends to infinity.

Now let $H > 1/2$. From Lemma 5.2 we deduce that almost surely

$$\lim_{T\to\infty} \frac{X_T^2}{T} = 0. \tag{3.8}$$

It is easy to check that this convergence also holds in L^2. Then we conclude the proof using Lemma 3.3, (3.8), and

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_0^t \xi^{2H-2} e^{-\theta\xi} d\xi dt = \theta^{1-2H}\Gamma(2H-1).$$

□

The next theorem provides the convergence in distribution to a Gaussian law of the fluctuations in the almost sure convergence (3.5).

Theorem 3.4 *Suppose $H \in \left[\frac{1}{2}, \frac{3}{4}\right)$. Let $(X_t, t \in [0,T])$ be given by (3.1), then*

$$\sqrt{T}\left[\widehat{\theta}_T - \theta\right] \xrightarrow{\mathcal{L}} N(0, \theta\sigma_H^2) \ , \tag{3.9}$$

as T tends to infinity, where

$$\sigma_H^2 = (4H-1)\left(1 + \frac{\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)\Gamma(2H)}\right). \tag{3.10}$$

Proof We have

$$\widehat{\theta}_T - \theta = -\sigma \frac{\int_0^T X_t dB_t^H}{\int_0^T X_t^2 dt} = -\frac{\sigma^2 \int_0^T \left(\int_0^t e^{-\theta(t-s)} dB_s^H\right) dB_t^H}{\int_0^T X_t^2 dt} = -\frac{\sqrt{T} F_T}{\int_0^T X_t^2 dt}, \tag{3.11}$$

where F_T is the double stochastic integral

$$F_T = \frac{\sigma^2}{2\sqrt{T}} I_2\left(e^{-\theta|t-s|}\right). \tag{3.12}$$

From Lemma 3.3 we know that $\frac{1}{T}\int_0^T X_t^2 dt$ converges almost surely and in L^2, as T tends to infinity to $\sigma^2\theta^{-2H} H\Gamma(2H)$. Then, it suffices to show that F_T converges in law as T tends to infinity to a centered normal distribution. In order to show this convergence we will apply Theorem 2.1 to a given sequence of random variables in the second chaos F_{T_k}, where $T_k \uparrow \infty$ as k tends to infinity. To simplify we assume that $T = 1, 2, \ldots$. The proof then follows from the following facts:

(i) $\mathbb{E}(F_T^2)$ converges to $\theta^{1-4H}\sigma^4\delta_H$, where

$$\delta_H = H^2(4H-1)(\Gamma(2H)^2 + \frac{\Gamma(2H)\Gamma(3-4H)\Gamma(4H-1)}{\Gamma(2-2H)}),$$

as T tends to infinity.

(ii) $\|DF_T\|_{\mathcal{H}}^2$ converges in L^2 to a constant as T tends to infinity.

Step 1: Proof of (i) Suppose first that $H = \frac{1}{2}$. In this case, by the isometry of the Itô integral we obtain

$$\mathbb{E}\left(F_T^2\right) = \frac{\sigma^4}{T}\int_0^T \int_0^t e^{-2\theta(t-s)} ds dt = \frac{\sigma^4}{T}\left(\frac{T}{2\theta} + \frac{e^{-2\theta T} - 1}{4\theta^2}\right),$$

which implies that

$$\lim_{T\to\infty} \mathbb{E}(F_T^2) = \frac{\sigma^4}{2\theta}.$$

This implies the desired result because $\delta_{\frac{1}{2}} = \frac{1}{2}$.

Now, let $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$. In this case, by the isometry property of the double stochastic integral I_2, the variance of F_T is given by

$$\mathbb{E}\left(F_T^2\right) = \frac{\sigma^4 \alpha_H^2}{2T} I_T, \tag{3.13}$$

where

$$I_T = \int_{[0,T]^4} e^{-\theta|s_2-u_2|-\theta|s_1-u_1|}|s_2-s_1|^{2H-2}|u_2-u_1|^{2H-2} duds. \tag{3.14}$$

By Lemma 5.3 in the Appendix we obtain that

$$\lim_{T\to\infty} \mathbb{E}(F_T^2) = \theta^{1-4H}\sigma^4\delta_H.$$

Step 2: Proof of (ii) For $s \le T$ we have

$$D_s F_T = \frac{\sigma X_s}{\sqrt{T}} + \frac{\sigma^2}{\sqrt{T}} \int_s^T e^{-\theta(t-s)} dB_t^H.$$

Suppose first that $H = \frac{1}{2}$. In this case,

$$\begin{aligned} \|DF_T\|_{\mathcal{H}}^2 &= \frac{\sigma^2}{T}\int_0^T \left(X_s^2 + 2\sigma X_s \int_s^T e^{-\theta(t-s)} dB_t + \sigma^2 \left(\int_s^T e^{-\theta(t-s)} dB_t \right)^2 \right) ds \\ &= A_T^{(1)} + A_T^{(2)} + A_T^{(3)}. \end{aligned}$$

We already know from (3.6) that $A_T^{(1)}$ converges in L^2 to $\frac{\sigma^4}{2\theta}$ as T tends to infinity. The third term can be written as

$$A_T^{(3)} = \frac{\sigma^4}{T}\int_0^T \left(\int_s^T e^{-\theta(t-s)} dB_t\right)^2 ds = \frac{\sigma^4}{T}\int_0^T \left(\int_0^u e^{-\theta(u-x)} dB_x\right)^2 du,$$

so it also converges in L^2 to $\frac{\sigma^4}{2\theta}$ a T tends to infinity. Finally we can show that

$$\lim_{T\to\infty} \mathbb{E}\left((A_T^{(2)})^2\right) = 0\,.$$

In fact, we have

$$\begin{aligned} \mathbb{E}\left((A_T^{(2)})^2\right) &= \frac{8\sigma^6}{T^2}\int_{\{s<u\le T\}} \mathbb{E}\left(X_s X_u \left(\int_s^T e^{-\theta(t-s)} dB_t\right)\left(\int_u^T e^{-\theta(t-u)} dB_t\right)\right) dsdu \\ &= \frac{8\sigma^8}{T^2}\int_{\{s<u\le T\}} \left(\int_0^s e^{-\theta(s+u-2r)} dr\right)\left(\int_u^T e^{-\theta(2t-s-u)} dt\right) dsdu \\ &= \frac{8\sigma^8}{4\theta^2 T^2}\int_{\{s<u\le T\}} (e^{2\theta s}-1)(e^{-2\theta u}-e^{-2\theta T}) dsdu, \end{aligned}$$

which clearly converges to zero as T tends to infinity. Therefore, $\|DF_T\|^2_{\mathcal{H}}$ converges to $\frac{\sigma^4}{\theta}$ in L^2.

Suppose now that $H > \frac{1}{2}$. From (2.2) we have

$$\begin{aligned}\|DF_T\|^2_{\mathcal{H}} &= \frac{\alpha_H \sigma^2}{T} \int_0^T \int_0^T \left(X_s + \sigma \int_s^T e^{-\theta(t-s)} dB^H_t \right) \\ &\quad \times \left(X_u + \sigma \int_u^T e^{-\theta(t-u)} dB^H_t \right) .|u-s|^{2H-2} duds\end{aligned}$$

We have to prove that $\|DF_T\|^2_{\mathcal{H}}$ converges to a constant in L^2 as T tends to infinity. In fact,

$$\begin{aligned}\|DF_T\|^2_{\mathcal{H}} &= \frac{\alpha_H \sigma^2}{T} \int_0^T \int_0^T \Bigg(X_s X_u + 2\sigma X_u \int_s^T e^{-\theta(t-s)} dB^H_t \\ &\quad + \sigma^2 \int_s^T e^{-\theta(t-s)} dB^H_t \int_u^T e^{-\theta(t-u)} dB^H_t \Bigg) |u-s|^{2H-2} duds \\ &= \frac{\alpha_H \sigma^2}{T} (C^{(1)}_T + C^{(2)}_T + C^{(3)}_T).\end{aligned}$$

For the term $C^{(1)}_T$, since X_t is Gaussian we can write

$$\begin{aligned}\mathbb{E}\left(|C^{(1)}_T - \mathbb{E}(C^{(1)}_T)|^2 \right) &= 2 \int_{[0,T]^4} \mathbb{E}\,(X_s X_t)\; \mathbb{E}\,(X_u X_v) \\ &\quad \times |u-s|^{2H-2} |v-t|^{2H-2} dudvdsdt.\end{aligned}$$

By Lemma 5.4

$$\begin{aligned}&\frac{1}{T} \int_{[0,T]^3} \mathbb{E}\,(X_T X_t)\; \mathbb{E}\,(X_u X_v) (T-u)^{2H-2} |v-t|^{2H-2} dudvdt \\ &\le \frac{1}{T} \int_{[0,T]^3} (T-t)^{2H-2} |u-v|^{2H-2} (T-u)^{2H-2} |v-t|^{2H-2} dudvdt \\ &\le C^2_{\theta,H} T^{8H-6} \int_{[0,1]^3} (1-t)^{2H-2} |u-v|^{2H-2} (1-u)^{2H-2} |v-t|^{2H-2} dudvdt,\end{aligned}$$

which converges to 0 as T tends to infinity when $H < \frac{3}{4}$. Hence, by l'Hôpital rule, $\mathbb{E}\left(|C^{(1)}_T - \mathbb{E}(C^{(1)}_T)|^2 \right)$ converges to 0 as T tends to infinity. In the same way we can prove that $\mathbb{E}\left(|C^{(i)}_T - \mathbb{E}(C^{(i)}_T)|^2 \right)$ converges to zero as T tends to infinity, for $i = 2, 3$ when $H < 3/4$. By triangular inequality, we see that

$$\begin{aligned} & \mathbb{E}\left[\left(\|DF_T\|_{\mathcal{H}}^2 - \mathbb{E}(\|DF_T\|_{\mathcal{H}}^2)\right)^2\right] \\ = \;& \mathbb{E}\left(|C_T^{(1)} + C_T^{(2)} + C_T^{(3)} - \mathbb{E}(C_T^{(1)} + C_T^{(2)} + C_T^{(3)})|^2\right) \\ \leq \;& 9\sum_{i=1}^{3} E\left(|C_T^{(i)} - \mathbb{E}(C_T^{(i)})|^2\right) \\ \to \;& 0\,. \end{aligned}$$

Taking into account that

$$\lim_{T\to\infty} \mathbb{E}(\|DF_T\|_{\mathcal{H}}^2) = 2\lim_{T\to\infty} \mathbb{E}(F_T^2),$$

we conclude the proof of (ii). This completes the proof of the theorem. □

If one replaces the Itô type integral in (1.2) by the path-wise Riemann-Stieltjes integral, then we can obtain the following estimator

$$\widehat{\theta}'_T = -\frac{\int_0^T X_t \circ dX_t}{\int_0^T X_t^2 dt} = \frac{X_T^2}{2\int_0^T X_t^2 dt},$$

which converges to zero in L^2 as T tends to infinity from Lemma 3.3 and (3.8).

4 An alternative estimator

Suppose in this section that $H > \frac{1}{2}$. We introduce the following estimator

$$\widetilde{\theta}_T := \left(\frac{1}{\sigma^2 H\Gamma(2H)T}\int_0^T X_t^2 dt\right)^{-\frac{1}{2H}}. \tag{4.1}$$

From (3.6), we see that $\tilde{\theta}_T$ converges to θ almost surely as $T \to \infty$. Theorem 3.4 allows us to derive the rate of convergence in the approximation of θ by $\widetilde{\theta}_T$.

Theorem 4.1 *Suppose $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$. Then*

$$\sqrt{T}\left[\widetilde{\theta}_T - \theta\right] \xrightarrow{\mathcal{L}} N\left(0, \frac{\theta}{(2H)^2}\sigma_H^2\right), \tag{4.2}$$

as T tends to infinity, where σ_H is defined in (3.10).

Proof From Equation (3.2), we have

$$\int_0^T X_t^2 dt = \frac{\sigma^2\alpha_H \int_0^T \int_0^t \xi^{2H-2} e^{-\theta\xi} d\xi dt - X_T^2/2}{\widehat{\theta}_T}.$$

Thus

$$\sqrt{T}\left[\widetilde{\theta}_T-\theta\right] = \sqrt{T}\left[\left(\frac{H\Gamma(2H)}{\alpha_H\frac{1}{T}\int_0^T\int_0^t\xi^{2H-2}e^{-\theta\xi}d\xi dt-\frac{X_T^2}{2T}}\right)^{\frac{1}{2H}}\widehat{\theta}_T^{\frac{1}{2H}}-\theta\right]$$

From Lemma 5.2 it follows that

$$\alpha_H\frac{1}{T}\int_0^T\int_0^t\xi^{2H-2}e^{-\theta\xi}d\xi dt-\frac{X_T^2}{2T}=\alpha_H\Gamma(2H-1)\theta^{1-2H}+o(\frac{1}{\sqrt{T}}),$$

where $o(\frac{1}{\sqrt{T}})$ denotes a random variable H_T such that $\sqrt{T}H_T$ converges to zero almost surely as T tends to infinity. Therefore,

$$\begin{aligned}\left(\frac{H\Gamma(2H)}{\alpha_H\frac{1}{T}\int_0^T\int_0^t\xi^{2H-2}e^{-\theta\xi}d\xi dt-\frac{X_T^2}{2T}}\right)^{\frac{1}{2H}} &= \left(\frac{1}{\theta^{1-2H}+o(\frac{1}{\sqrt{T}})}\right)^{\frac{1}{2H}}\\ &= \theta^{1-\frac{1}{2H}}+o(\frac{1}{\sqrt{T}}). \end{aligned}\tag{4.3}$$

On the other hand, we can write

$$\sqrt{T}\left[\widehat{\theta}_T^{\frac{1}{2H}}-\theta^{\frac{1}{2H}}\right]=\sqrt{T}\left[\frac{1}{2H}\theta^{\frac{1}{2H}-1}\left(\widehat{\theta}_T-\theta\right)+\frac{1}{2}\left(\widehat{\theta}_T-\theta\right)^2\theta_T^*\right],$$

where θ_T^* is a random point between θ and $\widehat{\theta}_T$. From Theorem 3.4 we obtain the following convergence in law as T tends to infinity:

$$\sqrt{T}\left[\widehat{\theta}_T^{\frac{1}{2H}}-\theta^{\frac{1}{2H}}\right]\to N\left(0,\frac{1}{(2H)^2}\theta^{\frac{1}{2H}}\sigma_H^2\right). \tag{4.4}$$

Finally, from the decomposition

$$\begin{aligned}\sqrt{T}\left[\widetilde{\theta}_T-\theta\right] &= \sqrt{T}\left[\left(\frac{H\Gamma(2H)}{\alpha_H\frac{1}{T}\int_0^T\int_0^t\xi^{2H-2}e^{-\theta\xi}d\xi dt-\frac{X_T^2}{2T}}\right)^{\frac{1}{2H}}-\theta^{1-\frac{1}{2H}}\right]\widehat{\theta}_T^{\frac{1}{2H}}\\ &\quad+\sqrt{T}\theta^{1-\frac{1}{2H}}\left[\widehat{\theta}_T^{\frac{1}{2H}}-\theta^{\frac{1}{2H}}\right],\end{aligned}$$

and using (4.3) and (4.4) we deduce the desired convergence. □

5 Appendix

In the sequel we present some calculations used in the paper.

Lemma 5.1 *For any* $H\in\left(\frac{1}{2},1\right)$

$$(2H-1)\int_0^\infty\int_0^\infty e^{-(s+u)}|u-s|^{2H-2}duds=\Gamma(2H).$$

Proof We can write, by the change-of-variables $u-s=x$,

$$\int_0^\infty \int_0^\infty e^{-(s+u)}|u-s|^{2H-2}dsdu = 2\int_0^\infty \int_0^u e^{-(s+u)}(u-s)^{2H-2}dsdu$$
$$= 2\int_0^\infty \int_0^u e^{-2u+x}x^{2H-2}dxdu.$$

Integrating first in the variable u and using that $(2H-1)\Gamma(2H-1) = \Gamma(2H)$ we conclude the proof. □

Lemma 5.2 *Let Y_t be the stationary Gaussian process defined in (3.7), where $H > 1/2$. Then, for any $\alpha > 0$, $\frac{Y_T}{T^\alpha}$ converges almost surely to zero as T tends to infinity.*

Proof The covariance of the process is Y_t is, using Lemma 5.1 to compute $\mathrm{Var}(\xi)$,

$$\begin{aligned}
\mathrm{Cov}(Y_0, Y_t) &= e^{-\theta t}\mathbb{E}\left(\xi\left[\xi + \sigma\int_0^t e^{\theta u}dB_u^H\right]\right)\\
&= e^{-\theta t}\left[\mathrm{Var}(\xi) + \sigma^2\alpha_H\int_0^t\int_{-\infty}^0 e^{\theta u+\theta v}|u-v|^{2H-2}dudv\right]\\
&= e^{-\theta t}\left[\mathrm{Var}(\xi) + \sigma^2\alpha_H\int_0^t\int_v^\infty e^{-\theta x+2\theta v}x^{2H-2}dxdv\right]\\
&= e^{-\theta t}\left[\sigma^2\theta^{-2H}H\Gamma(2H) + \sigma^2\{\ \theta^{1-2H}H\Gamma(2H)t - \frac{1}{2}t^{2H} + o(t^{2H})\ \}\right]\\
&= \sigma^2\theta^{-2H}H\Gamma(2H)\left[1 - \frac{\theta^{2H}}{\sigma^2\Gamma(2H+1)}t^{2H} + o(t^{2H})\ \}\right].
\end{aligned}$$

Then the result lemma from Theorem 3.1 of Pickands [14]. □

Lemma 5.3 *Let I_T given by (3.14). When $\frac{1}{2} < H < \frac{3}{4}$, we have*

$$\lim_{T\to\infty}\frac{I_T}{T} = \theta^{1-4H}\gamma_H, \tag{5.1}$$

where

$$\gamma_H = (8H-2)\,\Gamma(2H-1)^2 + (16H-4)\frac{\Gamma(2H-1)\Gamma(3-4H)\Gamma(4H-2)}{\Gamma(2-2H)}.$$

Proof Taking the derivative with respect to T we have

$$\frac{dI_T}{dT} = 4\int_{[0,T]^3} e^{-\theta(T-u_2)-\theta|s_1-u_1|}\,(T-s_1)^{2H-2}\,|u_2-u_1|^{2H-2}du_1du_2ds_1. \tag{5.2}$$

Making the change of variable $T - u_2 = x_1$, $T - u_1 = x_2$, and $T - s_1 = x_3$ yields

$$\frac{dI_T}{dT} = 4 \int_{[0,T]^3} e^{-\theta x_1 - \theta|x_2 - x_3|} x_3^{2H-2} |x_1 - x_2|^{2H-2} dx_1 dx_2 dx_3.$$

As a consequence,

$$\lim_{T\to\infty} \frac{dI_T}{dT} = 4 \int_{[0,\infty)^3} e^{-\theta x_1 - \theta|x_2 - x_3|} x_3^{2H-2} |x_1 - x_2|^{2H-2} dx_1 dx_2 dx_3,$$

and this integral is finite. Indeed, we can decompose this integral into the integrals in the six disjoint regions $\{x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)}\}$, where σ runs over all permutations of the indices $\{1, 2, 3\}$. In the case $x_1 < x_3 < x_2$ making the change of variables $x_1 = a$, $x_3 - x_1 = b$, and $x_2 - x_3 = c$, we obtain

$$\begin{aligned} & \int_{[0,\infty)^3} e^{-\theta(a+c)} (a+b)^{2H-2} (b+c)^{2H-2} da db dc \\ \leq & \int_{[0,\infty)^3} e^{-\theta(a+c)} b^{4H-4} da db dc, \end{aligned}$$

which is finite because $H < \frac{3}{4}$. The other cases are simpler and can be handled in a similar way. We can write

$$\int_{[0,\infty)^3} e^{-\theta x_1 - \theta|x_2 - x_3|} x_3^{2H-2} |x_1 - x_2|^{2H-2} dx_1 dx_2 dx_3 = \theta^{1-4H} d_H,$$

where

$$d_H = \int_{[0,\infty)^3} e^{-x-|y-z|} z^{2H-2} |x-y|^{2H-2} dx dy dz. \tag{5.3}$$

The integral in (5.3) can be simplified as follows. First we make the change of variables $y \mapsto w$, where $w = y - x$, and we obtain

$$\begin{aligned} d_H &= \int_0^\infty \int_0^\infty \int_{-x}^\infty e^{-x-|x+w-z|} z^{2H-2} |w|^{2H-2} dw dx dz \\ &= \int_0^\infty \int_0^\infty \int_{z-x}^\infty e^{-(2x+w-z)} z^{2H-2} |w|^{2H-2} dw dx dz \\ &\quad + \int_0^\infty \int_0^\infty \int_{-x}^{z-x} e^{-(z-w)} z^{2H-2} |w|^{2H-2} dw dx dz. \end{aligned}$$

Integrating in x we get

$$\begin{aligned} d_H &= \frac{1}{2} \int_0^\infty \int_{-\infty}^\infty e^{-2[(z-w)\vee 0]-(w-z)} z^{2H-2} |w|^{2H-2} dw dz \\ &\quad + \int_0^\infty \int_{-\infty}^\infty [(z-w) - ((-w)\vee 0)]_+ \, e^{-(z-w)} z^{2H-2} |w|^{2H-2} dw dz. \end{aligned}$$

Therefore,

$$\begin{aligned} d_H &= \frac{1}{2}\int_0^\infty \int_0^\infty e^{-2[(z-w)\vee 0]-(w-z)} z^{2H-2} w^{2H-2} dw dz \\ &+\frac{1}{2}\int_0^\infty \int_0^\infty e^{-(z+w)} z^{2H-2} w^{2H-2} dw dz \\ &+ \int_0^\infty \int_0^\infty [(z-w)]_+ \; e^{-(z-w)} z^{2H-2} w^{2H-2} dw dz \\ &+ \int_0^\infty \int_0^\infty e^{-(z+w)} z^{2H-1} w^{2H-2} dw dz, \end{aligned}$$

and we obtain

$$d_H = f_H + \left(2H - \frac{1}{2}\right)\Gamma(2H-1)^2, \tag{5.4}$$

where

$$f_H = \int_0^\infty \int_0^z (1+z-w) e^{-(z-w)} z^{2H-2} w^{2H-2} dw dz.$$

Making the change-of-variables $z - w = x$ yields

$$f_H = \int_0^\infty \int_0^\infty (1+x) e^{-x} (w+x)^{2H-2} w^{2H-2} dw dx.$$

Substituting the equality $(w+x)^{2H-2} = \frac{1}{\Gamma(2-2H)}\int_0^\infty e^{-\xi(w+x)}\xi^{1-2H} d\xi$ in f_H we obtain

$$\begin{aligned} f_H &= \frac{1}{\Gamma(2-2H)}\int_0^\infty \int_0^\infty \int_0^\infty (1+x) e^{-x-\xi(w+x)} w^{2H-2} \xi^{1-2H} d\xi dw dx \\ &= \frac{\Gamma(2H-1)}{\Gamma(2-2H)}\int_0^\infty \int_0^\infty (1+x) e^{-x-\xi x} \xi^{2-4H} d\xi dx \\ &= \frac{\Gamma(2H-1)\Gamma(3-4H)}{\Gamma(2-2H)}\int_0^\infty (1+x) e^{-x} x^{4H-3} dx \\ &= (4H-1)\frac{\Gamma(2H-1)\Gamma(3-4H)\Gamma(4H-2)}{\Gamma(2-2H)}. \end{aligned} \tag{5.5}$$

Finally from (5.4) and (5.5) we get the desired result. □

Lemma 5.4 *Let X_t be given by (1.2). We have*

$$\mathbb{E}\left[\int_s^T e^{-\theta(\xi-s)} dB_\xi^H \int_t^T e^{-\theta(\eta-t)} dB_\eta^H\right] \le C_{\theta,H}|t-s|^{2H-2}, \tag{5.6}$$

and

$$\mathbb{E}\left[X_t X_s\right] \le \sigma^2 C_{\theta,H}|t-s|^{2H-2}, \tag{5.7}$$

for some constant $C_{\theta,H} > 0$.

Proof Let us assume that $s < t$. We can write using (2.2)

$$\mathbb{E}\left[\int_s^T e^{-\theta(\xi-s)}dB_\xi^H \int_t^T e^{-\theta(\eta-t)}dB_\eta^H\right]$$
$$= \alpha_H \int_t^T \int_s^T e^{-\theta(\xi-s)}e^{-\theta(\eta-t)}|\xi-\eta|^{2H-2}d\xi d\eta = \alpha_H(B_T^{(1)} + B_T^{(2)}),$$

where

$$B_T^{(1)} = \int_t^T \int_t^T e^{-\theta(\xi-s)}e^{-\theta(\eta-t)}|\xi-\eta|^{2H-2}d\xi d\eta$$

and

$$B_T^{(2)} = \int_t^T \int_s^t e^{-\theta(\xi-s)}e^{-\theta(\eta-t)}|\xi-\eta|^{2H-2}d\xi d\eta.$$

It is easy to see that $B_T^{(1)}$ is bounded by $C_{\theta,H}e^{-\theta|t-s|}$. The second term can be estimated as follows

$$\begin{aligned}
B_T^{(2)} &= \int_s^t \int_{t-\xi}^{T-\xi} e^{-\theta(\xi-s+y+\xi-t)}y^{2H-2}dyd\xi \\
&= \int_0^{T-s} y^{2H-2}dy \int_{(t-y)\vee s}^{(T-y)\wedge t} e^{-\theta(y+2\xi-s-t)}d\xi \\
&\leq \frac{1}{2\theta}\int_0^{T-s} e^{-\theta(y+2\,(t-y)\vee s-s-t)}y^{2H-2}dy \\
&= \frac{1}{2\theta}\int_{t-s}^{T-s} e^{-\theta(y+s-t)}y^{2H-2}dy + \frac{1}{2\theta}\int_0^{t-s} e^{-\theta(y+s-t)}y^{2H-2}dy \\
&\leq C_{\theta,H}|t-s|^{2H-2}\int_{t-s}^{T-s} e^{-\theta(y+s-t)}dy + C_\theta\int_0^{t-s} y^{2H-2}dy \\
&\leq C_{\theta,H}|t-s|^{2H-2}\,.
\end{aligned}$$

This proves (5.6). The inequality (5.7) can be proved in a similar way (see also [2]). □

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